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HIGHER ORDER INTERVAL ITERATIVE METHODS FOR NONLINEAR EQUATIONS†

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Abstract. In this paper, a fifth order extension of Potra's third order iterative method is proposed for solving nonlinear equations. A convergence theorem along with the error bounds is established. The method takes three functions and one derivative evaluations giving its efficiency index equals to 1.495. Some numerical examples are also solved and the results obtained are compared with some other existing fifth order methods. Next, the interval extension of both third and fifth order Potra's method are developed by using the concepts of interval analysis. Convergence analysis of these methods are discussed to establish their third and fifth orders respectively. A number of numerical examples are worked out using INTLAB in order to demonstrate the efficacy of the methods. The results of the proposed methods are compared with the results of the interval Newton method.

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1. Introduction

One of the most important and challenging problems in scientific computing is to find efficiently the real roots of nonlinear equations. A number of applications such as Kinetic theory of gases, elasticity and other applied areas give rise to boundary value problems which are reduced to solving nonlinear equations. Mathematical modeling of dynamic systems leads to difference or differential equations whose solutions usually represent the equilibrium states of the systems obtained by solving nonlinear equations. Many optimization problems also lead to these equations. For example, the locations of extremal points of a function require finding the zeros of the derivative of that function. A lot of research

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works is carried out for this purpose. Some excellent text books such as Ortega [10], Ostrowski [2], Traub [11], Rall [13] and many others are published for good reviews of the most important methods [1, 3, 4, 7, 14, 17].
We have considered the problem of solving nonlinear equations

$$f(x) = 0 \tag{1}$$

where $f : R \to R$ be a continuously differentiable function. Generally, iterative methods are used to solve these equations. These methods require prior knowledge of one or more initial guesses to the desired root. Once the initial interval is known to contain a root, several classical methods for solving such equations are bisection method, secant method, regula-falsi method, steffensen's method, Newton's method and their variants. Bisection and regula- falsi methods are linearly convergent, secant method super-linearly convergent, whereas Steffensen's method and Newton's method are quadratically convergent. Starting with a suitably chosen x_0 near to the root, the Newton's method is given for $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

The method may fail if the initial guess is far away from the root or the derivative being very small in the vicinity of the root. The theorems and lemmas can be given to show its quadratically convergence. If the efficiency index [22] of an iterative method is defined as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functions evaluations per iteration then the efficiency index of Newton's method is 1.414. Time to time, Newton's method has been derived in several ways and modified in a variety of ways. There also exists several methods that can accelerate the order of convergence from quadratic to cubic and even higher. Some of the well known higher order methods are Potra's, Chebyshev's, Halley's, Euler's, Super-Halley's methods. These methods are also important as many applications require quick convergence of their solutions. Other important higher order method which is a base for many more general methods is the Ostrowski's method [2]. The order of convergence of this method increases at the expanse of additional function evaluations at other point iterated by the Newton's method. However, all numerical computations require the estimation of errors of computation. There are two methods for this.

- Perform the method and a separate error estimation, or
- Determine a sequence of nested intervals such that each interval contains the root. Thus, an approximation to the root along with error bounds are obtained at the same time.

The second approach leads to the concepts of interval analysis which was formally introduced by Moore[18]. In practice, interval analysis provides automatically approximation to the solutions along with the rigorous error bounds on accumulated rounding errors, approximation errors, and propagated uncertainties in initial data during the process of computation. Many researchers (see, [5, 9, 15, 18, 19, 20, 21]) have discussed several interval methods to find the roots of nonlinear equations. Moore [18] established the interval extension of Newton's method to enclose the simple roots of nonlinear equations which converges quadratically. Various forms of interval Newton method are developed by Hansen [5], Alefeld and Herzberger [9] and Krawczyk [16]. Hansen also extended interval Newton method for bounding the solution of system of nonlinear equations. Later, Lofti and Bakhtiari [21] have developed the interval extension of classic King's method using interval analysis to solve nonlinear equations. They have also shown that interval extension of King's method has fourth order of convergence. Bakhtiari et al. [15] established the interval Ostrowski-type methods with guaranteed convergence of order four and six respectively. Interval Ostrowski type methods take less number of iterations to converge to smallest interval containing the root as compared to interval Newton method.

In this paper, a fifth order extension of Potra's third order iterative method is proposed for solving nonlinear equations. A convergence theorem along with the error bounds is established. The method takes three functions and one derivative evaluations giving its efficiency index equals to 1.495. Some numerical examples are also solved and results obtained are compared with some other existing fifth order methods. Next, the interval extension of both third and fifth order Potra's method are developed using the concepts of interval analysis. Convergence analysis of these methods are discussed to establish their third and fifth orders respectively. A number of numerical examples are worked out using INTLAB in order to demonstrate the efficacy of the methods. The results of the proposed methods are compared with the results of the interval Newton method.

This paper is organized as follows. In Section 2, we have described the basic concepts of interval analysis and notations used in this paper. In section 3, Potra's third order method is reviewed in brief and its fifth order extension is developed. The convergence analysis and numerical examples for it are also provided. In Section 4, the interval extensions of Potra's third and fifth order methods are developed and numerical examples are also worked out to demonstrate the applicability and efficiency of proposed interval methods. Finally, conclusions are included in Section 5.

2. Preliminaries

In this Section, we will give some definitions, concepts and notations used in this paper.The capital letters denote intervals and small letters denote the reals.

Definition 2.1. An interval X can be defined as

$$X = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}.$$

Let $\mathbb{IR}$ denote the set of all closed intervals. If $X = [\underline{x}, \overline{x}] \in \mathbb{IR}$ then X can be written as $X = [X^c - \Delta X, X^c + \Delta X]$, where, $X^c = \frac{\underline{x}+\overline{x}}{2}$ and $\Delta X = \frac{\overline{x}-\underline{x}}{2}$ represent the midpoint and the radius of X respectively.
For $X = [\underline{x}, \overline{x}],\ Y = [\underline{y}, \overline{y}] \in \mathbb{IR}$, the interval arithmetic operations over $\mathbb{IR}$ are

defined as

$$X + Y = [\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}],$$

$$X - Y = [\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}],$$

$$X \times Y = [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [min(\underline{x}.\underline{y}, \underline{x}.\overline{y}, \overline{x}.\underline{y}, \overline{x}.\overline{y}), max(\underline{x}.\underline{y}, \underline{x}.\overline{y}, \overline{x}.\underline{y}, \overline{x}.\overline{y})],$$

$$\frac{X}{Y} = \frac{[\underline{x}, \overline{x}]}{[\underline{y}, \overline{y}]} = [\underline{x}, \overline{x}]\Big[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\Big], 0 \notin Y.$$

Definition 2.2. The distance between two intervals $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ is defined as

$$d(X, Y) = max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}.$$

Definition 2.3. The width of an interval X is defined as

$$wid(X) = \overline{x} - \underline{x}.$$

The width of an interval posseses the following nice properties.

$$wid(aX + bY) = |a|wid(X) + |b|wid(Y),$$

$$wid(XY) = |X|wid(Y) + |Y|wid(X),$$

$$wid\Big(\frac{X}{Y}\Big) = \frac{1}{\underline{y}\overline{y}}(|Y|wid(X) + |X|wid(Y)), \text{if } 0 \notin Y.$$

Definition 2.4. We say that F is an interval extension of f, if for degenerate interval arguments, F agrees with f,i.e. $F([x, x]) = f(x)$.

Definition 2.5. An interval valued function F of the interval variables $x_1, x_2, \ldots, x_n$ is inclusion monotonic if

$$Y_i \subseteq X_i, i = 1, 2, \ldots, n,$$

implies

$$F(Y_1, Y_2, \ldots, Y_n) \subseteq F(X_1, X_2, \ldots, X_n).$$

Definition 2.6. An interval sequence $\{X^{(k)}\}$ is nested if $X^{(k+1)} \subseteq X^{(k)}$ for all k.

We shall now state a number of Lemmas and Theorems without proofs. For their proofs one can refer to [18].

Lemma 2.7. *Every nested sequence* $\{X^{(k)}\}$ *converges and has the limit* $\cap_{k=1}^{\infty}\{X^{(k)}\}$.

Lemma 2.8. *Suppose* $\{X^{(k)}\}$ *is such that there is a real number* $x \in X^{(k)}$ *for all* k. *Define* $\{Y^{(k)}\}$ *by* $Y^{(1)} = X^{(1)}$ *and* $Y^{(k+1)} = X^{(k+1)} \cap Y^{(k)}$ *for all* $k = 1, 2, \ldots$ *Then* $Y^{(k)}$ *is nested with limit* Y, *and* $x \in Y \subseteq Y^{(k)}\ \forall\ k$.

Lemma 2.9. *If* $f : R \to R$ *is continuously differentiable on the interval* X *and* $0 \notin F'(X)$, *then* X *either contains a simple root* x^* *or no roots.*

Theorem 2.10. *Let $f \in C(X^{(0)})$ with $0 \notin F'(X^{(0)})$ and f has exactly one simple root $x^* \in X^{(0)}$. Then, the sequence $\{X^{(k)}\}$ such that $X^{(k+1)} \subset X^{(k)}$ is said to be of order p if there exists a constant γ such that*

$$wid(X^{(k+1)}) \leq \gamma(wid(X^{(k)}))^p.$$

2.1. Interval Newton method (INM). In this section, we shall briefly review the interval Newton method as given by [19]. Consider the problem of solving nonlinear equation (1), where $f : R \to R$ be a continuously differentiable real valued function of a real variable x. The interval version of Newton's method given by (2) is as follows. Let $F'(X)$ be an inclusion monotonic interval extension of $f'(x)$. Starting with $X^{(0)}$ containing the root, generate the sequence of intervals $\{X^{(k)}\}$ for $k = 0, 1, 2, \ldots$ by

$$X^{(k+1)} = X^{(k)} \cap N(X^{(k)}) \tag{3}$$

where

$$N(X) = mid(X) - \frac{f(mid(X))}{F'(X)}.$$

The computational algorithm for (3) can be given as follows:

Algorithm 1

Require: $X^{(0)}$ that contains exactly one root;
  tolerance TOL;
  maximum number of iteration J;
  functions f, f', ***F'***.
 for i = 0: J-1 **do**
  Compute $\boldsymbol{N}(X^{(k)})$.
  $X^{(k+1)} := \boldsymbol{N}(X^{(k)}) \cap X^{(k)}$.
  if wid$(X^{(k+1)}) \leq$ TOL **then**
   goto Ensure STEP
  end if
 end for
Ensure: (X^{k+1}); (The procedure was successful).

Theorem 2.11. *If an interval $X^{(0)}$ contains a zero x^* of $f(x)$, then so does $X^{(k)}$ for all $k = 0, 1, 2, \ldots$, defined by (3). Furthermore, the intervals $X^{(k)}$ form a nested sequence converging to x^* if $0 \notin F'(X^{(0)})$.*

Proof. The proof is given in [19] and therefore omitted. □

Theorem 2.12. *Given a real rational function f of a single real variable x with rational extensions F, F' of f, f', respectively, such that f has a simple zero x^* in an interval $X^{(0)}$ for which $F(X^{(0)})$ is defined and $F'(X^{(0)})$ is defined and* *does not contain zero i.e.* $0 \notin F'(X^{(0)})$*, then there is an interval* $X \subseteq X^{(0)}$ *containing* x^* *and a positive real number* C *such that*

$$wid(X^{(k+1)}) \leq C(wid(X^{(k)}))^2.$$

Proof. The proof is given in [19] and therefore omitted. □

3. Potra's third and fifth order methods in R

In this section, we shall describe in brief Potra's [8] third order method and its extension to fifth order method for solving nonlinear equations in R. The well known quadratically convergent Newton's method for solving (1) is given by (2). The third order Potra's method is an important and basic method for finding a simple root of (1) and is given for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)}, \end{aligned} \tag{4}$$

for a suitably chosen x_0 near to the root. The order of convergence of this method is equal to 3. For the convergence analysis of the method, one can refer to [8].

The fifth order extension of this method for finding a simple root of (1) can be given for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \Big(\frac{f(x_n)}{f(x_n) - 2f(y_n)}\Big)\frac{f(z_n)}{f'(x_n)}. \end{aligned} \tag{5}$$

for a suitably chosen x_0 near to the root. This method is an improvement of Potra's method known as modified Potra's method (MPM) with fifth order of convergence and convergence theorem is given as follows.

Theorem 3.1. *Let* $f : R \to R$ *has a continuous derivatives up to third order in* R*. If* $f(x)$ *has a simple root* x^* *in* R *and* x_0 *be a initial approximation near to* x^**, then the iterative scheme given by (5) satisfies the following error equation*

$$e_{n+1} = \left(4c_2^4 - 2c_2^2 c_3\right) e_n^5 + O(e_n^6), \tag{6}$$

where $e_n = x_n - x^*$ *and* $c_j = (\frac{f^j(x^*)}{j! f'(x^*)}), j = 2, 3, \ldots$

Proof. Let $e_n = x_n - x^*$ be the error in the iterate x_n . Using Taylor's series expansion, we get

$$\begin{aligned} f(x_n) &= f'(x^*)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)], \\ f'(x_n) &= f'(x^*)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)]. \end{aligned}$$

Substituting $f(x_n)$ and $f'(x_n)$ in $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, we get

$$y_n = x^* + c_2 e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4\right) e_n^4 + O(e_n^5).$$

Now expanding $f(y_n)$ by Taylor's series about x^*, we get

$$\begin{aligned} f(y_n) &= f'(x^*)[(y_n - x^*) + c_2(y_n - x^*)^2 + c_3(y_n - x^*)^3 + c_4(y_n - x^*)^4 + O(e_n^5)], \\ &= f'(x^*)[c_2 e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3 + \left(5c_2^3 - 7c_2c_3 + 3c_4\right) e_n^4 + O(e_n^5)]. \end{aligned}$$

Substituting y_n, $f'(x_n)$ and $f(y_n)$ in expression $z_n = y_n - \frac{f(y_n)}{f'(x_n)}$, we get

$$z_n = [x^* + 2c_2^2 e_n^3 + \left(-9c_2^3 + 7c_2c_3\right) e_n^4 + O(e_k^5)].$$

Now

$$\begin{aligned} \frac{f(x_n)}{f(x_n) - 2f(y_n)} &= [1 + 2c_2 e_n + \big(- 2c_2^2 + 4c_3\big)e_n^2 - 2\big(2c_2c_3 - 3c_4\big)e_n^3 \\ &+ 2\big(2c_2^4 - 3c_2^2c_3 - 2c_2c_4 + 4c_5\big)e_n^4 - 2\big(4c_2^5 - 14c_2^3c_3 \\ &+ 9c_2c_3^2 + 5c_2^2c_4 - 2c_3c_4 + 2c_2c_5 - 5c_6\big)e_n^5]. \end{aligned}$$

Again, expand $f(z_n)$ by Taylor series about x^*, we get

$$\begin{aligned} f(z_n) &= f'(x^*)[(z_n - x^*) + c_2(z_n - x^*)^2 + c_3(z_n - x^*)^3 + c_4(z_n - x^*)^4 + O(e_n^5)], \\ &= f'(x^*)[2c_2^2 e_n^3 + \left(-9c_2^3 + 7c_2c_3\right) e_n^4 + \big(30c_2^4 - 44c_2^2c_3 + 6c_3^2 \\ &+ 10c_2c_4\big)e_n^5 + O(e_n^6)]. \end{aligned}$$

Substituting $f'(x_n)$, $f(z_n)$ and $\frac{f(x_n)}{f(x_n)-2f(y_n)}$ in (5), we get

$$e_{n+1} = \left(4c_2^4 - 2c_2^2c_3\right) e_n^5 + O(e_n^6).$$

□

Therefore, the order of convergence of the method given by (5) is five. It requires three functions and one derivative evaluations, so its efficiency index is 1.495 which is similar to other fifth order methods.

3.1. Numerical Examples. In this subsection, some numerical examples are worked out to show the efficacy of proposed method. We have considered the following test functions with their approximate root.

a) $f_1(x) = x(x^9 - 1) - 1$ $\quad x^* = 1.075766066086837$

b) $f_2(x) = 2xe^{-1} - 2e^{-x} + 1$ $\quad x^* = 0.422477709641236$

c) $f_3(x) = e^{-x} + \cos(x)$ $\quad x^* = 1.746139530408013$

d) $f_4(x) = e^{-5x}(x - 1) + x^5$ $\quad x^* = 0.516153518757934$

e) $f_5(x) = x^3 + 4x^2 - 10$ $\quad x^* = 1.365230013414097$

f) $f_6(x) = (\sin(x))^2 - x^2 + 1$ $\quad x^* = 1.404491648215341$

We have compared the results obtained by proposed method (MPM) with Newton's method (NM), Potra's method (PM) and Fang et al. [12] method (FM). Number of iterations (NN) and number of function evaluations (NFE) taken by these methods are presented in the following table.

Table 1. Comparison of results of MPM with those of NM, PM and FM

Functions	x_0	**NN**				**NFE**			
		NM	**PM**	**FM**	**MPM**	**NM**	**PM**	**FM**	**MPM**
f_1	**2**	**11**	**8**	**5**	**5**	**22**	**24**	**20**	**20**
f_2	**1**	**5**	**4**	**2**	**2**	**10**	**12**	**12**	**8**
f_3	**1**	**4**	**3**	**2**	**2**	**8**	**9**	**8**	**8**
f_4	**1**	**7**	**5**	**3**	**3**	**14**	**15**	**12**	**12**
f_5	**2**	**5**	**4**	**3**	**3**	**10**	**12**	**12**	**12**
f_6	**2**	**5**	**4**	**3**	**3**	**10**	**12**	**12**	**12**

It is observed that modified Potra's method takes less number of iterations as compared to Newton's method and Potra's method but gives similar results with existing fifth order methods. The fifth order method proposed by Fang et al. [12] requires two derivative evaluations but our method requires only one derivative evaluation, thus, leading to less computational work.

4. Proposed interval Potra methods

In this section, the interval extensions of Potra's third and fifth order methods for enclosing the simple roots of nonlinear equations are proposed. The convergence theorems are established for their guaranteed convergence. The error bounds are also derived for them.

4.1. Interval Potra method (IPM). In this subsection, we shall develop an interval extension of Potra's third order method and its convergence analysis to solve (1). Let $X^{(0)}$ be a given interval containing the root x^* of nonlinear equation $f(x) = 0$. Let $F'(X)$ be an inclusion monotonic interval extension of $f'(x)$. Define for $k = 0, 1, 2, \ldots$

$$N(X) = mid(X) - \frac{f(mid(X))}{F'(X)},$$

$$Y^{(k)} = X^{(k)} \cap N(X^{(k)}), \tag{7}$$

$$P(X,Y) = mid(Y) - \frac{f(mid(Y))}{F'(X)}.$$

$$X^{(k+1)} = X^{(k)} \cap P(X^{(k)}, Y^{(k)}), \tag{8}$$

The interval Potra method (IPM) requires two functions evaluations and one interval extension of derivative of the function per step. Its convergence analysis and error bounds can be given as follows. The sequence of intervals $\{X^{(k)}\}$ generated by interval Potra method possesses the following properties.

Theorem 4.1. *Suppose $f \in C(X^{(0)})$ and $0 \notin F'(X^{(k)})$ for $k = 0, 1, 2, \dots$ If x^* is a root of f and $x^* \in X^{(0)}$, then $x^* \in X^{(k)}$ for $k = 1, 2, \dots$ Also, the intervals $X^{(k)}$ form a nested sequence of intervals converging to x^*.*

Proof. Given $f \in C(X^{(0)})$, $0 \notin F'(X^{(k)})$ for $k = 0, 1, 2 \dots$ Since $x^* \in X^{(0)}$ then, by induction $x^* \in X^{(k)}$ for $k = 1, 2 \dots$ Also, by the *Lemma* (2.2), the intervals $X^{(k)}$ generated by the iteration (8), form a nested sequence of intervals. Therefore, we have, $x^* \in X^{(k)}$ then $x^* \in \cap_{k=0}^{\infty} X^{(k)}$. □

Theorem 4.2. *Suppose $f \in C(X^{(0)})$ and $0 \notin F'(X^{(k)})$ for $k = 0, 1, 2 \dots$*

(1) If $x^* \in X^{(0)}$ and $P(X^{(k)}, Y^{(k)}) \subseteq X^{(k)}$, then $X^{(k)}$ contains exactly one root of f.

(2) If $X^{(k)} \cap P(X^{(k)}, Y^{(k)}) = \phi$, then $X^{(k)}$ contains no root of f.

Proof. To prove (1), we have $f \in C(X^{(0)})$, $x^* \in X^{(0)}$. Since $0 \notin F'(X^{(k)})$ for $k = 0, 1, 2 \dots$, we get f is monotonic on $X^{(k)}$. By using *Lemma* (2.3), f has a unique simple root $x^* \in X^{(k)}$ for all k. Also, if $P(X^{(k)}, Y^{(k)}) \subseteq X^{(k)}$, then by *Theorem* 4.1, we have $x^* \in X^{(k)}$. Therefore, $X^{(k)}$ contains exactly one root of f. The proof of (2) follows from the fact that if $P(X^{(k)}, Y^{(k)}) \subseteq X^{(k)}$ then x^* is the only zero of f. If $X^{(k)} \cap P(X^{(k)}, Y^{(k)}) = \phi$, this contradicts our assumptions. Hence $X^{(k)}$ contains no root of f. □

If we take starting interval $X^{(0)}$ such that

$$P(X^{(0)}, Y^{(0)}) \subseteq N(X^{(0)}) \subseteq X^{(0)}$$

then, Theorems 4.1 and 4.2 guarantee a nested sequence of intervals $\{X^{(k)}\}$ convergent to an interval X^* such that $x^* \in X^{(0)}$ and $X^* = P(X^*, Y^*)$ and $X^* \subseteq X^{(k)}$ for all $k = 0, 1, 2, \dots$ Thus, the sequence $\{X^{(k)}\}$ converges to X^* if assumptions of Theorems 4.1 and 4.2 hold. Now, we establish the theorem regarding rate of convergence of the sequences $\{X^{(k)}\}$ generated by (8).

Theorem 4.3. *Suppose $f \in C(X^{(0)})$ with $0 \notin F'(X^{(0)})$, and f has exactly one simple root $x^* \in X^{(0)}$. Then, if $P(X^{(k)}, Y^{(k)}) \subset X^{(k)}$, the sequence* (8) *convergence with rate of convergence equal to three. i.e., there exists a constant γ such that*

$$wid(X^{(k+1)}) \leq \gamma (wid(X^{(k)}))^3.$$

Proof. Given $f \in C(X^{(0)})$, $0 \notin F'(X^{(0)})$.
By using Mean value theorem on $mid(X^{(k)})$ and x^*, we get

$$f(mid(X^{(k)})) = f'(\xi_1)(mid(X^{(k)}) - x^*),$$

where, $\xi_1 \in (mid(X^{(k)}), x^*)$.
Similarly, for the sequence $\{Y^{(k)}\}$, we have

$$f(mid(Y^{(k)})) = f'(\xi_2)(mid(Y^{(k)}) - x^*), \tag{9}$$

where $\xi_2 \in (mid(Y^{(k)}), x^*)$.
Since, $P(X^{(k)}, Y^{(k)}) \subseteq X^{(k)}$ and from (8), we have

$$X^{(k+1)} = mid(Y^{(k)}) - \frac{[mid(Y^{(k)}) - x^*]f'(\xi_2)}{F'(X^{(k)})}, \tag{10}$$

Therefore,

$$wid(X^{(k+1)}) \leq |mid(Y^{(k)}) - x^*||f'(\xi_2)|wid\left(\frac{1}{F'(X^{(k)})}\right) \tag{11}$$

For the sequence $\{Y^{(k)}\}$ given by (7) and using Theorem 2.6, we get

$$|mid(Y^{(k)}) - x^*| \leq wid(Y^{(k)}) \leq (wid(X^{(k)}))^2. \tag{12}$$

Let $|f'(\xi_2)| \leq M_2$. Also,

$$wid\left(\frac{1}{F'(X^{(k)})}\right) \leq wid(X^{(k)}). \tag{13}$$

Substituting (12) and (13) in (11), we get

$$wid(X^{(k+1)}) \leq M_2(wid(X^{(k)}))^3.$$

Taking $M_2 = \gamma$, we get

$$wid(X^{(k+1)}) \leq \gamma(wid(X^{(k)}))^3.$$

□

The computational algorithm for (8) can be given as follows:

Algorithm 2

Require: $X^{(0)}$ that contains exactly one root;
tolerance TOL;
maximum number of iteration J;
functions f, f', ***F'***.
for i = 0: J-1 **do**
Compute $\boldsymbol{N}(X^{(k)})$.
$Y^{(k)} := \boldsymbol{N}(X^{(k)}) \cap X^{(k)}$.
Compute $\boldsymbol{P}(X^{(k)}, Y^{(k)})$.
$X^{(k+1)} := \boldsymbol{P}((X^{(k)}, Y^{(k)})) \cap X^{(k)}$.
if wid$(X^{(k+1)}) \leq$ TOL **then**
goto Ensure STEP
end if
end for
Ensure: (X^{k+1}); (The procedure was successful).

4.2. Interval modified Potra method (IMPM). In this subsection, an interval extension of Potra's fifth order method for enclosing simple roots of nonlinear equations is developed. Let $F'(X)$ be an inclusion monotone interval extension of $f'(x)$. Starting with $X^{(0)}$ containing the root x^*, generate the sequence of intervals $\{X^{(k)}\}$ for $k = 0, 1, 2, \dots$ by

$$N(X) = mid(X) - \frac{f(mid(X))}{F'(X)},$$

$$Y^{(k)} = X^{(k)} \cap N(X^{(k)}), \tag{14}$$

$$P(X,Y) = mid(Y) - \frac{f(mid(Y))}{F'(X)},$$

$$Z^{(k)} = X^{(k)} \cap P(X^{(k)}, Y^{(k)}), \tag{15}$$

$$\mu = \frac{f(mid(X))}{[f(mid(X)) - 2f(mid(Y))]F'(X)}, \tag{16}$$

$$S(X,Y,Z) = mid(Z) - \mu f(mid(Z)).$$

$$X^{(k+1)} = X^{(k)} \cap S(X^{(k)}, Y^{(k)}, Z^{(k)}), \tag{17}$$

This method requires three functions evaluations and one interval extension of derivative of the function per step. Theorems for guaranteed convergence of sequence of intervals generated by interval modified Potra method (17) can be established in a similar manner as given for Theorems 4.1 and 4.2 . The following theorem establishes its fifth order of convergence.

Theorem 4.4. *Suppose $f \in C(X^{(0)})$ with $0 \notin F'(X^{(0)})$, and f has exactly one simple root $x^* \in X^{(0)}$. Then, if $R(X^{(k)}, Y^{(k)}, Z^{(k)}) \subset X^{(k)}$, the sequence* (17) *converges with rate of convergence equal to five. i.e., there exists a constant δ such that*

$$wid(X^{(k+1)}) \le \delta (wid(X^{(k)}))^5.$$

Proof. Given $f \in C(X^{(0)})$, $0 \notin F'(X^{(0)})$.
By using mean value theorem on $mid(X^{(k)})$ and x^*, we get

$$f(mid(X^{(k)})) = f'(\xi_1)(mid(X^{(k)}) - x^*),$$

where, $\xi_1 \in (mid(X^{(k)}), x^*)$.
Similarly, for the sequence $\{Z^{(k)}\}$, we have

$$f(mid(Z^{(k)})) = f'(\xi_2)(mid(Z^{(k)}) - x^*)$$

where $\xi_2 \in (mid(Z^{(k)}), x^*)$.
Since $R(X^{(k)}, Y^{(k)}, Z^{(k)}) \subseteq X^{(k)}$, Thus from (17), we have

$$X^{(k+1)} = mid(Z^{(k)}) - \mu f'(\xi_2)(mid(Z^{(k)}) - x^*), \tag{18}$$

Therefore,

$$wid(X^{(k+1)}) \leq |f'(\xi_2)||mid(Z^{(k)}) - x^*|wid(\mu) \tag{19}$$

For the sequence $\{Z^{(k)}\}$ given by (15) and using Theorem 4.3, we get

$$|mid(Z^{(k)}) - x^*| \leq wid(Z^{(k)}) \leq \gamma(wid(X^{(k)}))^3. \tag{20}$$

Now,

$$wid(\mu) \leq \frac{|f'(\xi_1)||mid(X^{(k)}) - x^*|}{|f(mid(X)) - 2f(mid(Y)|} wid\left(\frac{1}{F'(X^{(k)})}\right) \tag{21}$$

Let $|f'(\xi_1)| \leq M_1$ and $|f(mid(X)) - 2f(mid(Y)| \leq M_3$.
Therefore,

$$wid(\mu) \leq \frac{M_1}{M_3}(wid(X^{(k)}))^2. \tag{22}$$

Substituting (20) and (22) in (19), we get

$$wid(X^{(k+1)}) \leq \frac{M_1\gamma}{M_3}(wid(X^{(k)}))^5.$$

Taking $\delta = \frac{M_1\gamma}{M_3}$, we get

$$wid(X^{(k+1)}) \leq \delta(wid(X^{(k)}))^5.$$

□

The computational algorithm for (17) can be given as follows:

Algorithm 3

Require: $X^{(0)}$ that contains exactly one root;
tolerance TOL;
maximum number of iteration J;
functions f, f', ***F'***.
for i = 0: J-1 **do**
Compute $\boldsymbol{N}(X^{(k)})$.
$Y^{(k)} := \boldsymbol{N}(X^{(k)}) \cap X^{(k)}$.
Compute $\boldsymbol{P}(X^{(k)}, Y^{(k)})$.
$Z^{(k)} := \boldsymbol{P}((X^{(k)}, Y^{(k)})) \cap X^{(k)}$.
Compute $\boldsymbol{S}(X^{(k)}, Y^{(k)}, Z^{(k)})$.
$X^{(k+1)} := \boldsymbol{S}((X^{(k)}, Y^{(k)}, Z^{(k)})) \cap X^{(k)}$.
if wid$(X^{(k+1)}) \leq$ TOL **then**
goto Ensure STEP
end if
end for
Ensure: (X^{k+1}); (The procedure was successful).

4.3. Numerical examples. In this subsection, the same numerical examples used in subsection (3.1) are taken to demonstrate the efficacy and applicability of the proposed interval versions of Potra's third and fifth order methods. All the numerical computations have been performed in INTLAB toolbox. The results obtained by proposed interval methods IPM, IMPM and INM are displayed in Tables 2-7 respectively. It is observed that new methods take less number of iterations to converge to the smallest interval containing the root x^* as compared to interval Newton method [19]. The stopping criteria is used either $X^{(k+1)} = X^{(k)}$ or $wid(X^{(k)}) \leqslant TOL$, $TOL = 10^{-15}$ to obtain narrowest possible interval containing x^*.

Table 2. Comparison of number of iterations and width, $X^{(0)} = [1, 1.5]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[1.00000000000000, 1.23157901169516]	2.3×10^{-1}	[1.01853906531014, 1.11350683166591]	9.4×10^{-2}	[1.06661509063597, 1.09125041424363]	2.4×10^{-2}
2	[1.01853906531014, 1.10215348995452]	8.4×10^{-2}	[1.07468057471099, 1.07618494532801]	1.5×10^{-3}	[1.07576601918926, 1.07576611951165]	1.0×10^{-7}
3	[1.07180976833894, 1.08476244466504]	1.3×10^{-2}	[1.07576606127736, 1.07576606792703]	6.6×10^{-9}	[1.07576606608683, 1.07576606608684]	4.4×10^{-16}
4	[1.07564709432121, 1.07593118087384]	2.8×10^{-4}	[1.07576606608683, 1.07576606608684]	4.4×10^{-16}		
5	[1.07576603950219, 1.07576609732578]	5.7×10^{-8}				
6	[1.07576606608683, 1.07576606608684]	1.5×10^{-15}				
7	[1.07576606608683, 1.07576606608684]	4.4×10^{-16}				

Table 3. Comparison of number of iterations and width, $X^{(0)} = [0, 1]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[0.39479017823530, 0.44340944198504]	4.8×10^{-1}	[0.39479017823530, 0.44340944198504]	2.1×10^{-2}	[0.42241174430913, 0.42255244777448]	1.4×10^{-4}
2	[0.42242163622163, 0.42252683549021]	1.0×10^{-4}	[0.42247752539077, 0.42247785740516]	2.5×10^{-11}	[0.42247770964123, 0.42247770964124]	4.4×10^{-16}
3	[0.42247770952034, 0.42247770975441]	2.3×10^{-8}	[0.42247770964123, 0.42247770964124]	3.3×10^{-16}		
4	[0.42247770964123, 0.42247770964124]	3.8×10^{-16}				

Table 4. Comparison of number of iterations and width, $X^{(0)} = [1, 2]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[1.71483425583505, 1.80084508200837]	8.6×10^{-2}	[1.74398054028649, 1.74794282910305]	3.9×10^{-3}	[1.74609967758845, 1.74615492428432]	5.5×10^{-5}
2	[1.74595214614604, 1.74626541809104]	3.1×10^{-4}	[1.74613953040512, 1.74613953041044]	5.3×10^{-12}	[1.74613953040801, 1.74613953040802]	2.2×10^{-16}
3	[1.74613952881648, 1.74613953171564]	2.8×10^{-9}	[1.74613953040801, 1.74613953040802]	4.4×10^{-16}		
4	[1.74613953040801, 1.74613953040802]	2.2×10^{-16}				

Table 5. Comparison of number of iterations and width, $X^{(0)} = [0, 1]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[0.50089022721017, 1.00000000000000]	4.9×10^{-1}	[0.00000000000000, 0.72934022393758]	7.2×10^{-1}	[0.36485064789316, 0.65940308753237]	2.9×10^{-1}
2	[0.50089022721017, 0.51615351875342]	2.0×10^{-2}	[0.16412597991183, 0.55026610606693]	3.8×10^{-1}	[0.51385574720824, 0.51678114259226]	2.9×10^{-3}
3	[0.50089022721017, 0.56399264308726]	6.3×10^{-2}	[0.48192269271915, 0.55026610606693]	6.8×10^{-2}	[0.51615351874731, 0.51615351876843]	2.1×10^{-11}
4	[0.51229641243988, 0.51960737718507]	7.3×10^{-3}	[0.51615290723934, 0.51615397587196]	1.0×10^{-6}	[0.51615351875793, 0.51615351875794]	2.2×10^{-16}
5	[0.51614832274527, 0.51615895062410]	1.0×10^{-5}	[0.51615351875793, 0.51615351875794]	2.2×10^{-16}		
6	[0.51615351875342, 0.51615351876247]	9.0×10^{-12}				
7	[0.51615351875793, 0.51615351875794]	2.2×10^{-16}				

Table 6. Comparison of number of iterations and width, $X^{(0)} = [1, 2]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[1.28409090909090, 1.41517857142858]	1.3×10^{-1}	[1.28409090909090, 1.41517857142858]	1.4×10^{-2}	[1.36509211758375, 1.36553103587538]	4.4×10^{-4}
2	[1.36438207994412, 1.36642685175846]	2.0×10^{-2}	[1.36522611523345, 1.36523420169871]	2.0×10^{-9}	[1.36523001341409, 1.36523001341410]	6.7×10^{-16}
3	[1.36522985334393, 1.36523020303635]	3.4×10^{-7}	[1.36523001341409, 1.36523001341410]	4.4×10^{-16}		
4	[1.36523001341409, 1.36523001341410]	5.3×10^{-15}				
5	[1.36523001341409, 1.36523001341410]	2.2×10^{-16}				

Table 7. Comparison of number of iterations and width, $X^{(0)} = [1, 2]$

Number of Iterations	INM		IPM		IMPM	
	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)	$\mathbf{X}^k$	wid($\mathbf{X}^k$)
1	[1.22263973155080, 1.44722925199692]	2.2×10^{-1}	[1.36873688097690, 1.51259806673463]	1.4×10^{-1}	[1.39662811444144, 1.43228851622473]	3.5×10^{-2}
2	[1.39627841884613, 1.42731522360090]	3.1×10^{-2}	[1.40445707182822, 1.40457252522344]	1.1×10^{-4}	[1.40449162078164, 1.40449167747577]	5.7×10^{-8}
3	[1.40434894960459, 1.40470828106882]	4.0×10^{-4}	[1.40449164821531, 1.40449164821540]	7.9×10^{-14}	[1.40449164821534, 1.40449164821535]	6.6×10^{-16}
4	[1.40449163858213, 1.40449165998386]	2.1×10^{-8}	[1.40449164821534, 1.40449164821535]	8.9×10^{-16}		
5	[1.40449164821534, 1.40449164821535]	6.7×10^{-16}				

5. Conclusions

A modified fifth order Potra's method extending the third order Potra's method to solve nonlinear equations and its convergence analysis is described. Later, using the techniques of interval analysis, the interval extensions of both third and fifth order Potra's method along with their convergence analysis are developed. A number of numerical examples are worked out to demonstrate the efficacy of the proposed methods. The results obtained by the proposed interval methods are compared with those obtained by the interval Newton method. It is observed that these methods are better in terms of computational speed and accuracy.

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