J. Appl. Math. & Informatics Vol. **33***(2015), No. 1 - 2, pp. 13 - 31*
http://dx.doi.org/10.14317/jami.2015.013

STRONG CONVERGENCE OF STRICT PSEUDO-CONTRACTIONS IN Q-UNIFORMLY SMOOTH BANACH SPACES†

YONGGANG PEI*, FUJUN LIU AND QINGHUI GAO

Abstract. In this paper, we introduce a general iterative algorithm for finding a common element of the common fixed point set of an infinite family of λ_i-strict pseudo-contractions and the solution set of a general system of variational inclusions for two inverse strongly accretive operators in q-uniformly smooth Banach spaces. Then, we analyze the strong convergence of the iterative sequence generated by the proposed iterative algorithm under mild conditions.

AMS Mathematics Subject Classification : 47H09, 47H10, 47H17.
Key words and phrases : q-uniformly smooth Banach space, convergence, iterative algorithm, fixed point, accretive operator.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E respectively. Let C be a subset of E and T be a mapping on C. We use $F(T)$ to denote the set of fixed points of T. Let $q > 1$ be a real number. The (generalized) duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\right\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q. Among nonlinear mappings, nonexpansive mappings

Received February 24, 2014. Revised September 20, 2014. Accepted October 12, 2014.
*Corresponding author.
†This work was supported by the Key Scientific and Technological Project of Henan Province (142102210058), Science and Technology Research Program of Education Department of Henan Province (13A110005) and PhD research funds of Henan normal university (QD13041).

and strict pseudo-contractions are two kinds of the most important nonlinear mappings. The study of them has a very long history (see [1-16,19-31] and the references therein). Recall that a mapping $T : C \to E$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

A mapping $T : C \to E$ is λ-strict pseudo-contractive in the terminology of Browder and Petryshyn (see [2, 3, 4]), if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda \|(I - T)x - (I - T)y\|^q \tag{1.1}$$

for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$. It is clear that (1.1) is equivalent to the following inequality

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \lambda \|(I - T)x - (I - T)y\|^q . \tag{1.2}$$

Remark 1.1. The class of strictly pseudo-contractive mappings has been studied by several authors (see, e.g., [2, 3, 4, 20, 22]). However, their iterative methods are far less developed though Browder and Petryshyn [24] initiated their work in 1967. As a matter of fact, strictly pseudo-contractive mappings have more powerful applications in solving inverse problems (see, e.g., [32]). Therefore it is interesting to develop the theory of iterative methods for strictly pseudo-contractive mappings.

In the early sixties, Stampacchia [33] first introduced variational inequality theory, which has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences (see [7, 8, 9, 10, 11, 34] and the references therein). In 1968, Brezis [34] initiated the study of the existence theory of a class of variational inequalities later known as variational inclusions, using proximal-point mappings due to Moreau [35]. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. It can be viewed as innovative and novel extension of the variational principles and thus, has wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Recently, some new and interesting problems, which are called to be system of variational inequalitys/inclusions received many attentions. System of variational inequalitys/inclusions can be viewed as natural and innovative generalizations of the variational inequalities/inclusions and it can provide new insight regarding problems being studied and can stimulate new and innovative ideas for solving problem.

Ceng et al. [26] proposed the following new system of variational inequality problem in a Hilbert space H: find $x^*, y^* \in C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{cases} \tag{1.3}$$

where $\lambda, \mu > 0$ are two constants, $A, B : E \to E$ are two nonlinear mappings. This is called the new system of variational inequalities. If we add up the requirement that $x^* = y^*$ and $A = B$, then problem (1.3) reduces to the classical variational inequality problem: find $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0, \ \forall y \in C. \tag{1.4}$$

In order to find the solutions of the system of variational inequality problem (1.3), Ceng et al. [26] studied the following approximation method. Let the mappings $A, B : C \to H$ be inverse-strongly monotone, $S : C \to C$ be nonexpansive. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n). \end{cases} \tag{1.5}$$

They proved that the iterative sequence defined by the relaxed extragradient method (1.5) converges strongly to a fixed point of S, which is a solution of the system of variational inequality (1.3).

On the other hand, in order to find the common element of the solutions set of a variational inclusion and the set of fixed points of a nonexpansive mapping T, Zhang et al. [6] introduced the following new iterative scheme in a Hilbert space H. Starting with an arbitrary point $x_1 = x \in H$, define sequence $\{x_n\}$ by

$$\begin{cases} y_n = J_{M,\lambda}(x_n - \lambda Ax_n), \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)Sy_n, \end{cases} \tag{1.6}$$

where $A : H \to H$ is an α-cocoercive mapping, $M : H \to 2^H$ is a maximal monotone mapping, $S : H \to H$ is a nonexpansive mapping and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Under mild conditions, they obtained a strong convergence theorem.

Motivated by Zhang et al. [6] and Zeng et al. [26], Qin et al. [8] considered the following new system of variational inclusion problem in a uniformly convex and 2-uniformly smooth Banach space: find $(x^*, y^*) \in E \times E$ such that

$$\begin{cases} \theta \in x^* - y^* + \rho_1(Ay^* + M_1x^*), \\ \theta \in y^* - x^* + \rho_2(Bx^* + M_2y^*). \end{cases} \tag{1.7}$$

The following problems are special cases of problem (1.7).

(1) If $A = B$ and $M_1 = M_2 = M$, then problem (1.7) reduces to the problem: find $(x^*, y^*) \in E \times E$ such that

$$\begin{cases} \theta \in x^* - y^* + \rho_1(Ay^* + Mx^*), \\ \theta \in y^* - x^* + \rho_2(Ax^* + My^*). \end{cases}$$

(2) If $x^* = y^*$, problem (1.7) reduces to the problem: find $x^* \in E$ such that

$$0 \in Ax^* + Mx^*.$$

Qin et al. [8] also introduced the following scheme for finding a common element of the solution set of the general system (1.7) and the fixed point set of a λ-strict pseudo-contraction. Starting with an arbitrary point $x_1 = u \in E$, define sequences $\{x_n\}$ by

$$\begin{cases} z_n = J_{M_2,\rho_2}(x_n - \rho_2 B x_n), \\ y_n = J_{M_1,\rho_1}(z_n - \rho_1 A z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1-\alpha_n-\beta_n)[\mu S x_n + (1-\mu) y_n],\ n \geq 1. \end{cases} \tag{1.8}$$

And they proved a strong convergence theorem under mild conditions.

One question arises naturally: Can we extend Theorem 2.1 of Zhang et al. [6], Theorem 3.1 of Qin et al. [8], Theorem 3.1 of Zeng et al. [26] from Hilbert spaces or 2-uniformly smooth Banach spaces to more broad q-uniformly smooth Banach spaces? We put forth another question: Can we get some more general results even without the condition of uniform convexity of Banach spaces ? However, the condition of uniform convexity of Banach spaces is necessary in Theorem 3.1 of Qin et al. [8], Yao et al. [36] and so on.

The purpose of this article is to give the affirmative answers to these questions mentioned above. Motivated by Zhang et al. [6], Qin et al. [8], Yao et al. [9], Hao [10], J. C. Yao [11], and Takahashi et al. [12], we consider a relaxed extragradient-type method for finding common elements of the solution set of a general system of variational inclusions for inverse-strongly accretive mappings and the common fixed point set of an infinite family of λ_i-strict pseudo-contractions. Furthermore, we obtain strong convergence theorems under mild conditions to improve and extend the corresponding results.

2. Preliminaries

The norm of a Banach space E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. If, for each $y \in S(E)$, the above limit is uniformly attained for $x \in S(E)$, then the norm of E is said to be uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, the above limit is attained uniformly for $y \in S(E)$.

Let $\rho_E : [0,1) \to [0,1)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \|y\| \leq t\right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable. In particular, the norm of E is Fréchet differentiable.

Recall that, a mapping $T : C \to E$ is said to be L-Lipschitz if for all $x, y \in C$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\| \text{ for all } x, y \in C.$$

In particular, if $0 < L < 1$, then T is called contractive and if $L = 1$, then T reduces to a nonexpansive mapping.

For some $\eta > 0, T : C \to E$ is said to be η-strongly accretive, if for all $x, y \in C$, there exists $\eta > 0$, $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \eta \|x - y\|^q.$$

For some $\mu > 0, T : C \to E$ is said to be μ-inverse strongly accretive, if for all $x, y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \mu \|Tx - Ty\|^q.$$

A set-valued mapping $T : D(T) \subseteq E \to 2^E$ is said to be accretive if for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that for all $u \in T(x)$ and $v \in T(y)$,

$$\langle u - v, j(x - y)\rangle \geq 0.$$

A set-valued mapping $T : D(T) \subseteq E \to 2^E$ is said to be m-accretive if T is accretive and $(I + \rho T)(D(T)) = E$ for every (equivalently, for some scalar $\rho > 0$), where I is the identity mapping.

Let $M : D(M) \to 2^E$ be m-accretive. Denote by $J_{M,\rho}$ the resolvent of M for $\rho > 0$:

$$J_{M,\rho} = (I + \rho M)^{-1}.$$

It is known that $J_{M,\rho}$ is a single-valued and nonexpansive mapping from E to $\overline{D(M)}$ which will be assumed convex (this is so provided E is uniformly smooth and uniformly convex).

Let $\{T_n\}$ be a family of mappings from a subset C of a Banach space E into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}$ satisfies the $AKTT$-condition (see [17]), if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{\omega \in B} \|T_{n+1}\omega - T_n\omega\| < \infty.$$

The following proposition supports $\{T_n\}$ satisfying AKTT-condition.

Proposition 2.1. *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E. Assume that $\{S_i : C \to C\}_{i=1}^{\infty}$ is a countable family of λ_i -strict pseudo-contractions with $\{\lambda_i\} \subset (0, 1)$ and $\inf\{\lambda_i : i \geq 1\} > 0$ such that $F = \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_n x = \sum_{k=1}^{n} \beta_n^k S_k x, \ \forall x \in C.$$

Let $\{\beta_n^k\}$ be a family of nonnegative numbers with $k \leq n$ such that

(i) $\sum_{k=1}^{n} \beta_n^k = 1$, *for all $n \in \mathbb{N}$,*

(ii) $\lim_{n\to\infty} \beta_n^k > 0$, *for every* $k \in \mathbb{N}$,
(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| < \infty$.

Then the following results hold:

(1) *Each* T_n *is a* λ *-strict pseudo-contraction.*
(2) $F(\sum_{k=1}^{\infty} \beta_n^k S_k x) = F$.
(3) $\{T_n\}$ *satisfies* $AKTT$*-condition.*
(4) *If* $T : C \to C$ *is defined by*

$$Tx = \sum_{k=1}^{\infty} \beta^k S_k x, \ \forall x \in C,$$

where $\beta^k = \lim_{n\to\infty} \beta_n^k$ *for all* $k \in \mathbb{N}$, *then* $Tx = \lim_{n\to\infty} T_n x$ *and* $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

Proof. (1) and (2) can be deduced directly from Lemma 2.11 in [20]. And the argument of (3) and (4) is similar to the section 4 (Applications) in [17] and so it is omitted. □

In order to prove our main results, we need the following lemmas.

Lemma 2.2 ([16]). *Let* C *be a closed convex subset of a strictly convex Banach space* E. *Let* T_1 *and* T_2 *be two nonexpansive mappings from* C *into itself with* $F(T_1) \bigcap F(T_2) \neq \emptyset$. *Define a mapping* S *by*

$$Sx = \lambda T_1 x + (1-\lambda) T_2 x, \ \forall x \in C,$$

where λ *is a constant in (0, 1). Then* S *is nonexpansive and* $F(S) = F(T_1) \bigcap F(T_2)$.

Lemma 2.3 ([19]). *Let* $\{\alpha_n\}$ *be a sequence of nonnegative numbers satisfying the property:*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + b_n + \gamma_n c_n, \ n \geq 0,$$

where $\{\gamma_n\}, \{b_n\}, \{c_n\}$ *satisfy the restrictions:*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$,
(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 ([18]). *Let* $q > 1$. *Then the following inequality holds:*

$$ab \leq \frac{1}{q} a^q + \frac{q-1}{q} b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.5 ([19]). *Let* E *be a real* q*-uniformly smooth Banach space, then there exists a constant* $C_q > 0$ *such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q \|y\|^q, \ \forall x, y \in E.$$

In particular, if E is a real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2, \ \forall x, y \in E.$$

Lemma 2.6 ([17, 23])**.** *Suppose that $\{T_n\}$ satisfy the AKTT-condition such that*

(i) *$\{T_n x\}$ converges strongly to some point in C for each $x \in C$.*
(ii) *Furthermore, if the mapping $T : C \to C$ is defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$.*

Then $\lim_{n\to\infty} \sup_{\omega\in B} \|T\omega - T_n\omega\| = 0$ for each bounded subset B of C.

Lemma 2.7 ([22])**.** *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E and $T : C \to C$ be a λ-strict pseudo-contraction. For $\alpha \in (0,1)$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then, as $\alpha \in (0,\mu]$, $\mu = \min\{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.8 ([23])**.** *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E which admits weakly sequentially continuous generalized duality mapping j_q from E into E^* (i.e., if for all $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, implies that $j_q(x_n) \overset{*}{\rightharpoonup} j_q(x)$). Let $T : C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.9 ([23])**.** *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E. Let $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then for each $t \in (0, \min\{1, \frac{1}{\tau}\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu V)$ is a contraction with a constant $1 - t\tau$.*

Lemma 2.10 ([23])**.** *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to E$ be a L-Lipschitzian mapping with constant $L \ge 0$ and $T : C \to C$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Let $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $0 \le \gamma L < \tau$, where $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then the sequence $\{x_t\}$ defined by*

$$x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t] \tag{2.1}$$

has following properties:

(i) *$\{x_t\}$ is bounded for each $t \in (0, \min\{1, \frac{1}{\tau}\})$.*
(ii) *$\lim_{t\to 0} \|x_t - Tx_t\| = 0$.*
(iii) *$\{x_t\}$ defines a continuous curve from $(0, \min\{1, \frac{1}{\tau}\})$ into C.*

Lemma 2.11 ([1])**.** *Let C be a closed convex subset of a smooth Banach space E. Let $\tilde{C}$ be a nonempty subset of C. Let $Q : C \to \tilde{C}$ be a retraction and let*

j, j_q be the normalized duality mapping and generalized duality mapping on E, respectively. Then the following are equivalent:

(a) *Q is sunny and nonexpansive.*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle,\ \forall x, y \in C.$
(c) $\langle x - Qx, j(y - Qx)\rangle \leq 0,\ \forall x \in C, y \in \tilde{C}.$
(d) $\langle x - Qx, j_q(y - Qx)\rangle \leq 0,\ \forall x \in C, y \in \tilde{C}.$

Lemma 2.12. *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping j_q from E into E^*. Let Q_C be a sunny nonexpansive retraction from E onto C, $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0, f : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose that $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Let $\{x_t\}$ be defined by (2.1) for each $t \in (0, \min\{1, \frac{1}{\tau}\})$. Then $\{x_t\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of the following variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, j_q(p - x^*)\rangle \leq 0,\ \forall p \in F(T). \tag{2.2}$$

Proof. We firstly show the uniqueness of a solution of the variational inequality (2.2). Suppose that both $\tilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions of (2.2). It follows that

$$\langle \gamma f x^* - \mu V x^*, j_q(\tilde{x} - x^*)\rangle \leq 0, \tag{2.3}$$

$$\langle \gamma f \tilde{x} - \mu V \tilde{x}, j_q(x^* - \tilde{x})\rangle \leq 0. \tag{2.4}$$

Adding up (2.3) and (2.4), we have

$$\langle (\gamma f - \mu V)\tilde{x} - (\gamma f - \mu V)x^*, j_q(x^* - \tilde{x})\rangle \leq 0.$$

Notice that

$$\langle (\gamma f - \mu V)\tilde{x} - (\gamma f - \mu V)x^*, j_q(x^* - \tilde{x})\rangle \geq (\tau - \gamma L)\, \|x^* - \tilde{x}\|^q > 0.$$

Therefore $x^* = \tilde{x}$ and the uniqueness is proved. We use x^* to denote the unique solution of (2.2).

Next, we prove that $x_t \to x^*$ as $t \to 0$.
Since E is reflexive and $\{x_t\}$ is bounded due to Lemma 2.10 (i), there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ and some point $\tilde{x} \in C$ such that $x_{t_n} \rightharpoonup \tilde{x}$. By Lemma 2.10 (ii), we have $\lim_{t\to 0} \|x_{t_n} - Tx_{t_n}\| = 0$. Together with Lemma 2.8, we can get that $\tilde{x} \in F(T)$. Setting $y_t = t\gamma f x_t + (I - t\mu V)Tx_t$, where $t \in (0, \min\{1, \frac{1}{\tau}\})$. Then, we can rewrite (2.1) as $x_t = Q_C y_t$. We claim that $\|x_{t_n} - \tilde{x}\| \to 0$.
Thanks to Lemma 2.11, we have that

$$\langle y_t - Q_C y_t, j_q(\tilde{x} - Q_C y_t)\rangle \leq 0. \tag{2.5}$$

It follows from (2.5) and Lemma 2.9 that

$$\|x_{t_n} - \tilde{x}\|^q = \langle Q_C y_{t_n} - y_{t_n}, j_q(x_{t_n} - \tilde{x})\rangle + \langle y_{t_n} - \tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle$$

$$\leq (1 - t_n\tau)\, \|x_{t_n} - \tilde{x}\|^q + t_n\langle \gamma f x_{t_n} - \mu V\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle.$$

Thus,

$$\begin{aligned}\|x_{t_n} - \tilde{x}\|^q &\leq \frac{1}{\tau}\langle \gamma f x_{t_n} - \mu V\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle \\ &\leq \frac{1}{\tau}[\gamma L\, \|x_{t_n} - \tilde{x}\|^q + \langle \gamma f\tilde{x} - \mu V\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle],\end{aligned}$$

which implies that

$$\|x_{t_n} - \tilde{x}\|^q \leq \frac{\langle \gamma f\tilde{x} - \mu V\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle}{\tau - \gamma L}. \tag{2.6}$$

Using that the duality map j_q is weakly sequentially continuous from E to E^* and noticing (2.6), we get that

$$\lim_{n\to\infty} \|x_{t_n} - \tilde{x}\| = 0. \tag{2.7}$$

Next, we shall prove that $\tilde{x}$ solves the variational inequality (2.2).
Since $x_t = Q_C y_t = Q_C y_t - y_t + t\gamma f x_t + (I - t\mu V)Tx_t$, we derive that

$$(\mu V - \gamma f)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I - T)x_t + \mu(Vx_t - VTx_t).$$

Note that for $\forall z \in F(T)$,

$$\langle (I - T)x_t - (I - T)z, j_q(x_t - z)\rangle \geq \|x_t - z\|^q - \|x_t - z\|^q = 0. \tag{2.8}$$

It thus follows from Lemma 2.11 and (2.8) that

$$\begin{aligned}&\langle (\mu V - \gamma f)x_t, j_q(x_t - z)\rangle \\ &= \frac{1}{t}\langle Q_C y_t - y_t, j_q(x_t - z)\rangle - \frac{1}{t}\langle (I - T)x_t, j_q(x_t - z)\rangle \\ &\quad + \langle \mu(Vx_t - VTx_t), j_q(x_t - z)\rangle \\ &\leq \|x_t - Tx_t\|\, M,\end{aligned} \tag{2.9}$$

where $M = \sup_{n\geq 0}\{\mu k\, \|x_t - z\|^{q-1}\} < \infty$. Now replacing t in (2.9) with t_n and letting $n \to \infty$, noticing (2.7) and Lemma 2.10 (ii), we obtain $\langle (\mu V - \gamma f)\tilde{x}, j_q(\tilde{x} - z)\rangle \leq 0$. That is, $\tilde{x} \in F(T)$ is a solution of (2.2); Hence $\tilde{x} = x^*$ by uniqueness. Therefore $x_{t_n} \to x^*$ as $n \to \infty$. And consequently, $x_t \to x^*$ as $t \to 0$. □

Lemma 2.13 ([20]). *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let $A : C \to E$ be a α-inverse-strongly accretive operator. Then the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(q\alpha - C_q\lambda^{q-1})\, \|Ax - Ay\|^q.$$

In particular, if $0 < \lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.14. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let $M_i : D(M_i) \to 2^E$ be m-accretive with $\overline{D(M_i)} = C$ for i=1,2 and ρ_1, ρ_2 be two arbitrary positive constants. Let $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G : C \to C$ be a mapping defined by*

$$G(x) = J_{M_1,\rho_1}[J_{M_2,\rho_2}(x - \rho_2 Bx) - \rho_1 A J_{M_2,\rho_2}(x - \rho_2 Bx)],\ \forall x \in C.$$

If $0 < \rho_1 \le (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$ and $0 < \rho_2 \le (\frac{q\beta}{C_q})^{\frac{1}{q-1}}$, then $G : C \to C$ is nonexpansive.

Proof. We have by Lemma 2.13 that for all $x, y \in C$,

$$\begin{aligned}\|Gx - Gy\| &\le \|(I - \rho_1 A)J_{M_2,\rho_2}(x - \rho_2 Bx) - (I - \rho_1 A)J_{M_2,\rho_2}(y - \rho_2 By)\| \\ &\le \|x - y\|,\end{aligned}$$

which implies that $G : C \to C$ is nonexpansive. This completes the proof. □

Lemma 2.15. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let $M_i : D(M_i) \to 2^E$ be m-accretive with $\overline{D(M_i)} = C$ for i=1,2 and ρ_1, ρ_2 be two arbitrary positive constants. Then, $(x^*, y^*) \in C \times C$ is a solution of general system (1.7) if and only if $x^* = Gx^*$, where G is defined by Lemma 2.14.*

Proof. Note that

$$\begin{cases} \theta \in x^* - y^* + \rho_1(Ay^* + M_1 x^*), \\ \theta \in y^* - x^* + \rho_2(Bx^* + M_2 y*), \end{cases} \Longleftrightarrow \begin{cases} x^* = J_{M_1,\rho_1}(y^* - \rho_1 Ay^*), \\ y^* = J_{M_2,\rho_2}(x^* - \rho_2 Bx^*), \end{cases}$$

and the above system is equivalent to

$$G(x^*) = J_{M_1,\rho_1}[J_{M_2,\rho_2}(x^* - \rho_2 Bx^*) - \rho_1 A J_{M_2,\rho_2}(x^* - \rho_2 Bx^*)] = x^*.$$

This completes the proof. □

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a strictly convex and real q-uniformly smooth Banach space E, which admits a weakly sequentially continuous generalized duality mapping $j_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C. Assume $A, B : C \to E$ are α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $M_i : D(M_i) \to 2^E$ be m-accretive with $\overline{D(M_i)} = C$ for i=1,2. Suppose that $V : C \to E$ is k-Lipschitz and η-strongly accretive with constants $k, \eta > 0, f : C \to E$ is L-Lipschitz with constant $L \ge 0$. Let $\{S_n : C \to C\}_{n=0}^{\infty}$ be an infinite family of λ_n-strict pseudo-contractions with $\{\lambda_n\} \subset (0,1)$ and $\inf\{\lambda_n : n \ge 0\} = \lambda > 0$, such that $F = \bigcap_{n=1}^{\infty} F(S_n) \bigcap F(G) \ne \emptyset$. Let $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$, $0 < \rho_1 < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, $0 < \rho_2 < (\frac{q\beta}{C_q})^{\frac{1}{q-1}}$, $0 \le \gamma L < \tau$, $0 < \sigma \le d$, where $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$ and $d = \min\ \{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\}$. Define a mapping $T_n x := (1 - \sigma)x + \sigma S_n x$ for all $x \in C$*

and $n \geq 0$. *For arbitrarily given* $x_0 \in C$ *and* $\delta \in (0,1)$, *let* $\{x_n\}$ *be the sequence generated iteratively by*

$$\begin{cases} z_n = J_{M_2,\rho_2}(x_n - \rho_2 B x_n), \\ k_n = J_{M_1,\rho_1}(z_n - \rho_1 A z_n), \\ y_n = \delta T_n x_n + (1-\delta) k_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu V) y_n], \ n \geq 0. \end{cases} \tag{3.1}$$

Assume that $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *are two sequences in (0,1) satisfying the following conditions:*

$$(i) \sum_{n=0}^{\infty} \alpha_n = \infty, \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(ii) \limsup_{n\to\infty} \gamma_n < 1, \ \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

Suppose in addition that $\{S_n\}_{n=1}^{\infty}$ *satisfies the* $AKTT$*-condition. Let* $S : C \to C$ *be the mapping defined by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$ *and suppose that* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in F$, *which is the unique solution of the following variational inequality*

$$\langle \gamma f x^* - \mu V x^*, j_q(p - x^*) \rangle \leq 0, \ \forall p \in F. \tag{3.2}$$

Proof. **Step 1.** We show that sequences $\{x_n\}$ is bounded. By condition (ii) there is a positive number b such that $\limsup_{n\to\infty} \gamma_n < b < 1$. Applying condition (i) and (ii), we may assume, without loss of generality, that $\{\gamma_n\} \subset (0, b]$ and $\{\alpha_n\} \subset (0, (1-b)\min\{1, \frac{1}{\tau}\})$. From Lemma 2.9, we deduce that $\|((1-\gamma_n)I - \alpha_n \mu V)x - ((1-\gamma_n)I - \alpha_n \mu V)y\| \leq ((1-\gamma_n) - \alpha_n \tau) \|x - y\|$ for $\forall\ x, y \in C$. For $x^* \in F$, it follows from Lemma 2.15 that

$$x^* = J_{M_1,\rho_1}[J_{M_2,\rho_2}(x^* - \rho_2 B x^*) - \rho_1 A J_{M_2,\rho_2}(x^* - \rho_2 B x^*)].$$

Putting $y^* = J_{M_2,\rho_2}(x^* - \rho_2 B x^*)$, then we can get that $x^* = J_{M_1,\rho_1}(y^* - \rho_1 A y^*)$. By Lemma 2.13, we obtain

$$\|k_n - x^*\| \leq \|(I - \rho_1 A) z_n - (I - \rho_1 A) y^*\| \leq \|x_n - x^*\|. \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned} \|y_n - x^*\| &\leq \delta \|T_n x_n - x^*\| + (1-\delta) \|k_n - x^*\| \\ &\leq \delta \|[(1-\sigma)x_n + \sigma S_n x_n] - [(1-\sigma)x^* + \sigma S_n x^*]\| \\ &\quad + (1-\delta) \|x_n - x^*\|. \end{aligned} \tag{3.4}$$

Combining Lemma 2.7 and the condition of $0 < \sigma \leq d = \min\left\{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\right\}$, we can deduce that

$$\|[(1-\sigma)x_n + \sigma S_n x_n] - [(1-\sigma)x^* + \sigma S_n x^*]\| \leq \|x_n - x^*\|. \tag{3.5}$$

Substituting (3.5) into (3.4) and simplifying, we have that

$$\|y_n - x^*\| \leq \|x_n - x^*\|.$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|Q_C[\alpha_n\gamma f x_n+\gamma_n x_n+((1-\gamma_n)I-\alpha_n\mu V)y_n]-x^*\|\\ &\le \|\alpha_n\gamma f x_n+\gamma_n x_n+[(1-\gamma_n)I-\alpha_n\mu V]y_n-x^*\|\\ &\le [1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|+\alpha_n\|\gamma f x^*-\mu V x^*\|\\ &\le \max\{\|x_0-x^*\|,\frac{\|\gamma f x^*-\mu V x^*\|}{\tau-\gamma L}\}.\end{aligned}$$

Hence, $\{x_n\}$ is bounded. $\{y_n\},\{k_n\}$ and $\{z_n\}$ are also bounded.

Step 2. We shall claim that $\|x_{n+1}-x_n\|\to 0$, as $n\to\infty$. We observe that

$$\begin{aligned}\|k_{n+1}-k_n\| &= \|J_{M_1,\rho_1}(z_{n+1}-\rho_1 A z_{n+1})-J_{M_1,\rho_1}(z_n-\rho_1 A z_n)\|\\ &\le \|(I-\rho_2 B)x_{n+1}-(I-\rho_2 B)x_n\|\\ &\le \|x_{n+1}-x_n\|.\end{aligned}\tag{3.6}$$

This together with Lemma 2.7 implies that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \delta(\|x_{n+1}-x_n\|+\sigma\|S_{n+1}x_n-S_n x_n\|)+(1-\delta)\|x_{n+1}-x_n\|\\ &\le \|x_{n+1}-x_n\|+\|S_{n+1}x_n-S_n x_n\|.\end{aligned}\tag{3.7}$$

At the same time, we observe that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le \|[\alpha_{n+1}\gamma f x_{n+1}+\gamma_{n+1}x_{n+1}+((1-\gamma_{n+1})I-\alpha_{n+1}\mu V)y_{n+1}]\\ &\quad -[\alpha_n\gamma f x_n+\gamma_n x_n+((1-\gamma_n)I-\alpha_n\mu V)y_n]\|\\ &\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\ &\quad +[(1-\gamma_{n+1})-\alpha_{n+1}\tau]\|y_{n+1}-y_n\|+|\alpha_{n+1}-\alpha_n|\gamma\|f x_n\|\\ &\quad +|\alpha_{n+1}-\alpha_n|\mu\|V y_n\|+|\gamma_{n+1}-\gamma_n|\|y_n-x_n\|.\end{aligned}\tag{3.8}$$

Substituting (3.7) into (3.8), we have that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\|+\|S_{n+1}x_n-S_n x_n\|\\ &\quad +(|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|)M',\end{aligned}\tag{3.9}$$

where $M'=\sup_{n\ge 0}\{\mu\|V y_n\|+\gamma\|f x_n\|,\|y_n-x_n\|\}<\infty$. Thanks to $\{S_n\}_{n=1}^{\infty}$ satisfying the $AKTT$-condition, we deduce that

$$\sum_{n=1}^{\infty}\|S_{n+1}x_n-S_n x_n\|\le\sum_{n=1}^{\infty}\sup_{\omega\in\{x_n\}}\|S_{n+1}\omega-S_n\omega\|<\infty\tag{3.10}$$

From (i), (ii), (3.9), (3.10) and Lemma 2.3, we deduce that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.11}$$

Notice that

$$\begin{aligned}\|y_n-x_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-y_n\|\\ &\le \|x_{n+1}-x_n\|+\alpha_n\|\gamma f x_n-\mu V y_n\|+\gamma_n\|x_n-y_n\|,\end{aligned}$$

which implies that

$$\|y_n - x_n\| \le \frac{1}{1-\gamma_n}(\|x_{n+1} - x_n\| + \alpha_n \|\gamma f x_n - \mu V y_n\|). \tag{3.12}$$

Combining conditions (i), (ii), (3.11) and (3.12), we deduce that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.13}$$

For any bounded subset B of C, we observe that

$$\begin{aligned}\sup_{\omega\in B} \|T_{n+1}\omega - T_n\omega\| &= \sup_{\omega\in B} \|[(1-\sigma)\omega + \sigma S_{n+1}\omega] - [(1-\sigma)\omega + \sigma S_n\omega]\| \\ &\le \sup_{\omega\in B} \|S_{n+1}\omega - S_n\omega\|.\end{aligned}$$

Since $\{S_n\}$ satisfies the $AKTT$-condition, we have that

$$\sum_{n=1}^{\infty} \sup_{\omega\in B} \|T_{n+1}\omega - T_n\omega\| < \infty.$$

That is, $\{T_n\}$ satisfies the $AKTT$-condition. Define a mapping $T : C \to C$ by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. It follows that

$$Tx = \lim_{n\to\infty} T_n x = \lim_{n\to\infty} [(1-\sigma)x + \sigma S_n x] = (1-\sigma)x + \sigma Sx,\ \forall x \in C. \tag{3.14}$$

Noticing that

$$\begin{aligned}\langle Sx - Sy, j_q(x-y)\rangle &= \lim_{n\to\infty} \langle S_n x - S_n y, j_q(x-y)\rangle \\ &\le \|Sx - Sy\|^q - \lambda \|(I-S)x - (I-S)y\|^q,\end{aligned}$$

we deduce that $S : C \to C$ is a λ-strict pseudo-contraction. In view of (3.14), Lemma 2.6 and the condition of $0 < \sigma \le d$, where $d = \min\,\{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\}$, we have that $T : C \to C$ is a nonexpansive and $F(T) = F(S)$. Hence we have $F(T) = \bigcap_{n=0}^{\infty} F(S_n) = \bigcap_{n=0}^{\infty} F(T_n)$.
Let $W : C \to C$ be the mapping defined by

$$Wx = \delta Tx + (1-\delta) J_{M_1,\rho_1}(I - \rho_1 A) J_{M_2,\rho_2}(I - \rho_2 B)x. \tag{3.15}$$

In view of Lemma 2.2, we see that W is nonexpansive such that

$$F(W) = F(T) \bigcap F(J_{M_1,\rho_1}(I-\rho_1 A) J_{M_2,\rho_2}(I-\rho_2 B)) = \bigcap_{n=0}^{\infty} F(S_n) \bigcap F(G) = F.$$

Noting that

$$\|Wx_n - y_n\| = \delta \|Tx_n - T_n x_n\|,$$

we obtain

$$\|Wx_n - x_n\| \le \|Wx_n - y_n\| + \|y_n - x_n\| \le \delta \|Tx_n - T_n x_n\| + \|y_n - x_n\|. \tag{3.16}$$

From Lemma 2.6, we can get that

$$\limsup_{n\to\infty} \|Tx_n - T_n x_n\| \le \lim_{n\to\infty} \sup_{\omega\in\{x_i : i\ge 0\}} \|T\omega - T_n\omega\| = 0. \tag{3.17}$$

Combing (3.13), (3.16) and (3.17), we deduce that

$$\|Wx_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Define $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Wx_t]$. From Lemma 2.11, we deduce that $\{x_t\}$ converges strongly to $x^* \in F(W) = F$, which is the unique solution of the variational inequality of (3.2).

Step 3. We show that

$$\limsup_{n\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*)\rangle \le 0,$$

where x^* is the solution of the variational inequality of (3.2). To show this, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*) = \lim_{i\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_{n_i} - x^*)\rangle.$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup z$ for some point $z \in C$ due to reflexivity of the Banach space E and boundness of $\{x_n\}$. It follows from (3.18) and Lemma 2.8 that $z \in F(W)$. Since the Banach space E has a weakly sequentially continuous generalized duality mapping $j_q : E \to E^*$, we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*) &= \lim_{i\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_{n_i} - x^*)\rangle \\ &= \langle \gamma f x^* - \mu V x^*, j_q(z - x^*)\rangle \le 0.\end{aligned}$$

Step 4. Finally we prove that $\lim_{n\to\infty}\|x_n - x^*\|$. Setting $h_n = \alpha_n\gamma f x_n + \gamma_n x_n + [(1-\gamma_n)I - \alpha_n\mu V]y_n, \forall n \ge 0$. Then by (3.1) we can write $x_{n+1} = Q_C h_n$. It follows from Lemmas 2.3 and Lemmas 2.10 that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &= \langle Q_C h_n - h_n, j_q(x_{n+1} - x^*)\rangle + \langle h_n - x^*, j_q(x_{n+1} - x^*)\rangle \\ &\le \langle h_n - x^*, j_q(x_{n+1} - x^*)\rangle \\ &\le [(1-\gamma_n) - \alpha_n\tau]\,\|y_n - x^*\|\,\|x_{n+1} - x^*\|^{q-1} \\ &\quad + \gamma_n\|x_n - x^*\|\,\|x_{n+1} - x^*\|^{q-1} + \alpha_n\langle \gamma f x_n - \gamma f x^*, j_q(x_{n+1} - x^*)\rangle \\ &\quad + \alpha_n\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\ &\le [1 - \alpha_n(\tau - \gamma L)][\frac{1}{q}\|x_n - x^*\|^q + \frac{q-1}{q}\|x_{n+1} - x^*\|^q] \\ &\quad + \alpha_n\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle,\end{aligned}$$

which implies

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \frac{1 - \alpha_n(\tau - \gamma L)}{1 + (q-1)(\tau - \gamma L)\alpha_n}\|x_n - x^*\|^q \\ &\quad + \frac{q\alpha_n}{1 + (q-1)(\tau - \gamma L)\alpha_n}\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\ &\le [1 - \alpha_n(\tau - \gamma L)]\,\|x_n - x^*\|^q \\ &\quad + \frac{q\alpha_n}{1 + (q-1)(\tau - \gamma L)\alpha_n}\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle.\end{aligned} \tag{3.19}$$

Put $\gamma_n = \alpha_n(\tau - \gamma L)$ and $c_n = \frac{q\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle}{[1+(q-1)(\tau-\gamma L)\alpha_n](\tau-\gamma L)}$. Applying Lemma 2.2 to (3.19), we obtain that $x_n \to x^* \in F$ as $n \to \infty$. This completes the proof. □

Remark 3.1. Compared with the known results in the literature, our results are very different from those in the following aspects:

(i) The results in this paper improve and extend corresponding results in [7, 8, 9, 10, 11, 12, 13]. Especially, Our results extend their results from 2-uniformly smooth Banach spaces or Hilbert spaces to more general q-uniformly smooth Banach spaces.

(ii) Our Theorem 3.1 extends one nonexpansive mapping in Theorem 2.1 in [6] or one λ-strict pseudo-contraction in Theorem 3.1 in [8] and an infinitely family of nonexpansive mappings in Theorem 3.1 in [10] to an infinite family of λ_i-strict pseudo-contractions. And our Theorem 3.1 gets a common element of the common fixed point set of an infinite family of λ_i-strict pseudo-contractions and the solution set of general system of variational inclusions for two inverse strongly accretive mappings in a q-uniformly smooth Banach space.

(iii) We by $f(x_n)$ replace the u which is a fixed element in iterative scheme (1.8), where f is a L-Lipschitzian operator. And we also add a Lipschitzian and strong accretive operator V in our scheme (3.1). In particular, whenever $C = E, f = u, V = I, \{T_n\}_{n=0}^{\infty} = \{T\}$, our scheme (3.1) reduces to (1.8).

(iv) It is worth noting that, the Banach space E does not have to be uniformly convex in our Theorem 3.1. However, it is very necessary in Theorem 3.1 of Qin et al. [8] and many other literatures.

Remark 3.2. The variational inequality problem in a q-uniformly smooth Banach space E: finding x^* such that

$$\langle \gamma f x^* - \mu V x^*, j_q(p - x^*)\rangle \le 0, \ \forall p \in \mathcal{M} \tag{3.20}$$

is also very interesting and important. As we can see that:

(i) If $\mathcal{M} := C$, then it follows from Lemma 2.11 that the variational inequality problem (3.20) is equivalent to a fixed point problem: find $x^* \in C$ such that it satisfies the following equation:

$$x^* = Q_C[x^* - \varsigma(\mu V - \gamma f)x^*],$$

where $\varsigma > 0$ is a constant.

(ii) When $E := H$ which is a real Hilbert space, $\mathcal{M} := F(T)$ and $\mu = 1$, problem (3.20) reduces to finding $x^* \in C$ such that

$$\langle \gamma f x^* - V x^*, p - x^*\rangle \le 0, \ \forall p \in F(T),$$

which is the optimality condition for the minimization problem:

$$\min_{x \in C} \frac{1}{2}\langle Vx, x\rangle - h(x),$$

where $F(T)$ is the fixed point set of a nonexpansive mapping T and h is a potential function for γf (i.e.,$h'(x) = \gamma f(x)$ for all $x \in H$). Furthermore, if $\gamma = 1, V = I$ and $f(x) = u$ for all $x \in C$, then problem (3.20) reduces to finding $x^* \in F(T)$ such that

$$\langle u - x^*, p - x^* \rangle \leq 0, \ \forall p \in F(T),$$

which is equivalent to finding $x^* \in F(T)$ such that

$$x^* = P_{F(T)}u = \arg \min_{x \in F(T)} \frac{1}{2} \|u - x\|^2 .$$

Corollary 3.2. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping $j : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C. Assume the mappings $A, B : C \to E$ are α-inverse-strongly accretive and β-inverse-strongly accretive, respectively.Let $M_i : D(M_i) \to 2^E$ be m-accretive with $\overline{D(M_i)} = C$ for i=1,2. Suppose $V : C \to E$ is a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to E$ is a L-Lipschitzian with constant $L \geq 0$. Let $0 < \mu < \frac{\eta}{K^2 k^2}$, $0 < \rho_1 < \frac{\alpha}{K^2}$, $0 < \rho_2 < \frac{\beta}{K^2}$ and $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - K^2 \mu k^2)$. Let $T : C \to C$ be a nonexpansive with $F = F(T) \bigcap F(G) \neq \emptyset$. For arbitrarily given $\delta \in (0, 1)$ and $x_0 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$\begin{cases} z_n = J_{M_2,\rho_2}(x_n - \rho_2 B x_n), \\ k_n = J_{M_1,\rho_1}(z_n - \rho_1 A z_n), \\ y_n = \delta T x_n + (1 - \delta) k_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n \mu V) y_n], \ n \geq 0. \end{cases} \tag{3.21}$$

Assume that $\{\alpha_n\}$ and $\{\gamma_n\}$ are two sequences in (0, 1) satisfying the following conditions:

$$(i) \sum_{n=0}^{\infty} \alpha_n = \infty, \ \lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(ii) \limsup_{n \to \infty} \gamma_n < 1, \ \sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

Then $\{x_n\}$ defined by (3.21) converges strongly to $x^* \in F$, which is the unique solution of the following variational inequality:

$$\langle \gamma f x^* - \mu V x^*, j(p - x^*) \rangle \leq 0, \ \forall p \in F.$$

4. Conclusion

In this research, a general iterative algorithm is proposed for finding a common element of the common fixed point set of an infinite family of λ_i-strict pesudo-contractions and the solution set of a general system of variational inclusions for two inverse strongly accretive operators in q-uniformly smooth Banach spaces.

Then we analyzed the strong convergence of the iterative sequence generated by the proposed iterative algorithm under very mild conditions. The methods in the paper are different from those in the early and recent literature. Our results can be viewed as the improvement, supplementation, and extension of the corresponding results in some references.

References

1. S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl., **44** (1973), 57–70.
2. Q. Dong, S. He, F. Su, *Strong convergence of an iterative algorithm for aninfinite family of strict pseudo-contractions in Banach spaces*, Appl. Math. Comput., **216** (2010), 959–969.
3. H. Zhou, *Convergence Theorems for λ-strict Pseudo-contractions in q-uniformly Smooth Banach Spaces,* Acta. Math. Sin., **26** (2010), 743–758.
4. G. Cai, C.S. Hu, *Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces,* Comput. Math. Appl., **59** (2010), 149–160.
5. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff Int, Leyden, The Netherlands, 1976.
6. S.S. Zhang, J. Lee, C.K. Chan, *Algorithms of common solutions to quasi variational inclusion and fixed point problems,* Appl. Math. Mech., **29** (2008), 571–581.
7. J. Peng, Y. Wang, D. Shyu, J.C. Yao, *Common Solutions of an Iterative Scheme for Variational Inclusions, Equilibrium Problems, and Fixed Point Problems,* J. Inequa. Appl., **2008**(2008), (Article ID 720371).
8. X. Qin, S.S. Chang, Y.J. Cho, S.M. Kang, *Approximation of Solutions to a System of Variational Inclusions in Banach Spaces,* J. Inequa. Appl., **2010** (2010), (Article ID 916806).
9. Y. Yao, M. Noor, K. Noor, Y. Liou, *Modified extragradient methods for a system of variational inequalities in Banach spaces,* Acta Appl. Math., **110** (2010), 1211–1224.
10. Y. Hao, *On variational inclusion and common fixed point problems in Hilbert spaces with applications,* Appl. Math. Comput., **217** (2010), 3000–3010.
11. Y. Yao, J. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings,* Appl. Math. Comput., **186** (2007), 155–1558.
12. W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings,* J. Optim. Theor. Appl., **118** (2003), 417–428.
13. X. Qin, Y. Cho, J. Kang, S. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces,* J. Comput. Anal. Appl., **230** (2009), 121–127.
14. G.M. Korpelevich, *The extragradient method for finding saddle points and for other problems,* Ekon. Mat. Metody, **12** (1976), 747–756.
15. Y.L. Song, L.C. Zeng, *Strong convergence of a new general iterative method for variational inequality problems in Hilbert spaces,* Fixed Point Theory and Applications, **46** (2012).
16. R.E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces,* Trans. Am. Math Soc., **179** (1973), 251–262.
17. K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, *Approximation of common fixed point of a countable family of nonexpansive mapping in a Banach space,* Nonlinear Anal. Th Methods Appl., **67** (2007), 2350–2360.
18. D.S. Mitrinović, *Analytic Inequalities,* Springer-Verlag, New York, 1970.
19. H.K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Anal., **16** (1991), 1127–1138.
20. Y.L. Song, L.C. Ceng, *Convergence Theorems for Accretive Operators with Nonlinear Mappings in Banach Spaces*, Abstr. Appl. Anal., **2014** (2014), (Art. ID 928950).

21. S. Matsushita, W. Takahashi, *Strong convergence theorems for nonexpansive nonself-mappings without boundary conditions,* Nolinear Anal., **68** (2008), 412–419.
22. H. Zhang, Y.F. Su, *Strong convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces,* Nonlinear Anal., **70** (2009), 3236–3242.
23. S. Pongsakorn, K. Poom, *Iterative methods for variational inequality problems and fixed point problems of a countable family of strict pseudo-contractions in a q-uniformly smooth Banach space,* Fixed Point Theory and Applications, **65** (2012).
24. F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces,* J. Math. Anal. Appl., **20** (1967), 197–228.
25. Y. Shehu, *An iterative method for fixed point problems, variational inclusions and generalized equilibrium problems,* Math. Comput. Model, **54** (2011), 1394–1404.
26. L. Ceng, C. Wang, JC. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities,* Math. Methods Oper. Res., **67** (2008), 375–390.
27. N. Shioji, W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces,* Proc. Amer. Math. Soc., **12** (1997), 3461–3465.
28. A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems,* J. Nonlinear Convex Anal., **9** (2008), 37–43.
29. S. Takahashia, W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space,* Nonlinear Anal., **69** (2008), 1025–1033.
30. Y. Shehu, *Iterative methods for family of strictly pseudo-contractive mappings and system of generalized mixed equilibrium problems and variational inequality problems,* Fixed Point Theory and Applications, **22** (2011), (Article ID 852789).
31. I.K. Argyros, Y.J. Cho, X. Qin, *On the implicit iterative process for strictly pseudo-contractive mappings in Banach spaces,* J. Comput. Appl. Math., **233**(2009), 208–216.
32. O. Scherzer, *Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems,* J. Math. Anal. Appl., **194** (1995), 911–933.
33. G. Stampacchia, *Forms bilineaires coercivities sur les ensembles convexes,* CR Acad Sci Paris, **258** (1964), 4413–4416.
34. H. Brézis, *Équations et inéquations non linéares dans les espaces vectoriels en dualité,* Ann. Inst. Fourier (Grenoble), **18** (1968), 115–175.
35. J.J. Moreau, *Proximite dualite dans un espaces hilbertien,* Bull. Soc. Math. France, **93** (1965), 273–299.
36. Y. Yao, M. Noor, K. Noor, Y. Liou, H. Yaqoob, *Modified Extragradient Methods for a System of Variational Inequalities in Banach Spaces,* Acta Appl Math., **110** (2010), 1211–1224.

Yonggang Pei received MS degree from Henan Normal University. and Ph.D at Shanghai Normal University. Since 2002 he has been at Henan Normal University as a teacher. His research interests include nonlinear optimization and fixed point theorems.

College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China.
`e-mail: ygpei59@gmail.com`

Fujun Liu received MS degree from zhengzhou University. Since 2007 he has been at Henan Institute Engineering college as a teacher. His research interests include numerical methods of differential equations and fixed point theorems.

Faculty of Science, Henan Institute of Engineering, Zhengzhou, 451191, China.
`e-mail: liufujun1981@126.com`

Qinghui Gao received MS degree from Henan Normal University. and Ph.D from Beihang University. She is currently a teacher at Henan Normal University since 2013. Her research interests are computational biology and iterative method.

College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China.
e-mail: qinghuigaocb@163.com