J. Appl. Math. & Informatics Vol. **33***(2015), No. 1 - 2, pp. 1 - 12*
http://dx.doi.org/10.14317/jami.2015.001

EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR THE SYSTEMS OF HIGHER ORDER BOUNDARY VALUE PROBLEMS ON TIME SCALES

A. KAMESWARA RAO

Abstract. This paper is concerned with boundary value problems for systems of n-th order dynamic equations on time scales. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using abstract fixed-point theorems.

AMS Mathematics Subject Classification : 34B15, 39B10, 34B18.
Key words and phrases : Boundary value problems, dynamic equations, green's function, positive solutions, multiplicity, cones.

1. Introduction

Let $\mathbb{T}$ be a time scale with $a, \sigma^n(b) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}} = J \cap \mathbb{T}.$$

In this paper we are concerned with the existence and multiplicity of positive solutions for dynamic equation on time scales

$$\begin{aligned} u^{\Delta^{(n)}}(t) + f(t,v) = 0,\ t \in [a,b]_{\mathbb{T}}, \\ v^{\Delta^{(n)}}(t) + g(t,u) = 0,\ t \in [a,b]_{\mathbb{T}}, \end{aligned} \tag{1}$$

satisfying the boundary conditions

$$\begin{aligned} u^{\Delta^{(i)}}(a) = 0,\ 0 \le i \le n-2,\ u(\sigma^n(b)) = 0, \\ v^{\Delta^{(i)}}(a) = 0,\ 0 \le i \le n-2,\ v(\sigma^n(b)) = 0, \end{aligned} \tag{2}$$

where $f, g \in C([a, \sigma^n(b)]_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}^+)$, $f(t,0) \equiv 0,\ g(t,0) \equiv 0$.

Received September 10, 2014. Revised November 3, 2014. Accepted November 7, 2014.

Recently, existence and multiplicity of solutions for boundary value problems of dynamic equations have been of great interest in mathematics and its applications to engineering sciences. To our knowledge, most existing results on this topic are concerned with the single equation and simple boundary conditions.

It should be pointed out that Eloe and Henderson [6] discussed the boundary value problem as follows

$$u^{(n)} + a(t)f(u) = 0,\ 0 < t < 1,$$
$$u^{(i)}(0) = u^{(n-2)}(1) = 0,\ 0 \leq i \leq n-2.$$

By using a Krasnosel'skii fixed point theorem, the existence of solutions are obtained in the case when, either f is superlinear, or f is sublinear. Yang and Sun [15] considered the boundary value problem of the system of differential equations

$$-u'' = f(t,u),\ -v'' = g(t,u),\ u(0) = u(1) = 0,\ v(0) = v(1) = 0.$$

By appealing to the degree theory, the existence of solutions are established. Note, that, there is only one differential equation in [4] and BVP in [15] contains the system of second order differential equations.

The arguments for establishing the existence of solutions of the BVP (1)-(2) involve properties of Green's function that play a key role in defining some cones. A fixed point theorem due to Krasnosel'skii [10] is applied to yield the existence of positive solutions of the BVP (1)-(2). Another fixed point theorem about multiplicity is applied to obtain the multiplicity of positive solutions of BVP (1)-(2).

The rest of this paper is organized as follows. In Section 2, we shall provide some properties of certain Green's functions and preliminaries which are needed later. For the sake of convenience, we also state Krasnosel'skii fixed point theorem in a cone. In Section 3, we establish the existence and multiplicity of positive solutions of the BVP (1)-(2). In Section 4, some examples are given to illustrate our main results.

2. Preliminaries

In this section, we will give some lemmas which are useful in proving our main results.

To obtain solutions of the BVP (1)-(2), we let $G(t,s)$ be the Green's function for the boundary value problem

$$-y^{\Delta^{(n)}} = 0, \tag{3}$$

$$y^{\Delta^{(i)}}(a) = 0,\ 0 \leq i \leq n-2,\ y(\sigma^n(b)) = 0. \tag{4}$$

Using the Cauchy function concept $G(t,s)$ is given by

$$G(t,s)=\frac{1}{(n-1)!}\begin{cases}\displaystyle\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}, & t\le s,\\ \displaystyle\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}-\prod_{i=1}^{n-1}(t-\sigma^i(s)), & \sigma(s)\le t.\end{cases}\tag{5}$$

Lemma 1. *For* $(t,s)\in[a,\sigma^n(b)]_{\mathbb{T}}\times[a,\sigma(b)]_{\mathbb{T}}$, *we have*

$$G(t,s)\le G(\sigma^n(b),s).\tag{6}$$

Proof. For $t\le s$, we have

$$\begin{aligned}G(t,s)&=\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}\\ &\le\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}\\ &=G(\sigma^n(b),s).\end{aligned}$$

Similarly, for $\sigma(s)\le t$ and $\sigma^i(s)\le t,\ i=1,\cdots,n-1$, we have

$$\begin{aligned}G(t,s)&=\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}-\prod_{i=1}^{n-1}(t-\sigma^i(s))\\ &\le\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}\\ &\le\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}\\ &=G(\sigma^n(b),s).\end{aligned}$$

Thus, we have

$$G(t,s)\le G(\sigma^n(b),s),\ \text{for all }(t,s)\in[a,\sigma^n(b)]_{\mathbb{T}}\times[a,\sigma(b)]_{\mathbb{T}}.$$

□

Lemma 2. *Let* $I=[\frac{\sigma^n(b)+3a}{4},\frac{3\sigma^n(b)+a}{4}]_{\mathbb{T}}$. *For* $(t,s)\in I\times[a,\sigma(b)]_{\mathbb{T}}$, *we have*

$$G(t,s)\ge\frac{1}{4^{n-1}}G(\sigma^n(b),s).\tag{7}$$

Proof. The Green's function for the BVP (3)-(4) is given in (5) shows that

$$G(t,s)>0,\ \text{on }(a,\sigma^n(b))_{\mathbb{T}}\times(a,\sigma(b))_{\mathbb{T}}.\tag{8}$$

For $t\le s$, $t\in I$ and $\sigma^{i-1}(a)\le\sigma^{n-2}(a),\ i=1,2,\cdots,n-1$, we have

$$\frac{G(t,s)}{G(\sigma^n(b),s)}=\prod_{i=1}^{n-1}\frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))}{(\sigma^n(b)-\sigma^{i-1}(a))} \\
&\geq \left(\frac{t-\sigma^{n-2}(a)}{\sigma^n(b)-\sigma^{n-2}(a)} \right)^{n-1} \\
&\geq \left(\frac{\frac{\sigma^n(b)+3a}{4}-\sigma^{n-2}(a)}{\sigma^n(b)-\sigma^{n-2}(a)} \right)^{n-1} \\
&\geq \frac{1}{4^{n-1}} \left(\frac{\sigma^n(b)+3a-4\sigma^{n-2}(a)}{\sigma^n(b)-\sigma^{n-2}(a)} \right)^{n-1} \\
&\geq \frac{1}{4^{n-1}}.
\end{aligned}
$$

For $\sigma(s) \leq t$, $t \in I$ and $\sigma^{i-1}(a) \leq \sigma^{n-2}(a)$, $i = 1, 2, \cdots, n-1$, we have

$$
\begin{aligned}
&\frac{G(t,s)}{G(\sigma^n(b),s)} \\
&= \frac{\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s)) - \prod_{i=1}^{n-1}(t-\sigma^i(s))(\sigma^n(b)-\sigma^{i-1}(a))}{\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\
&\geq \frac{\prod_{i=1}^{n-2}(t-\sigma^{i-1}(a))}{\prod_{i=1}^{n-2}(\sigma^n(b)-\sigma^{i-1}(a))} \\
&\quad \times \frac{[(t-\sigma^{n-2}(a))(\sigma^n(b)-\sigma^{n-1}(s)) - (t-\sigma^{n-1}(s))(\sigma^n(b)-\sigma^{n-2}(a))]}{[(\sigma^n(b)-\sigma^{n-2}(a))(\sigma^n(b)-\sigma^{n-1}(s)) - (\sigma^n(b)-\sigma^{n-1}(s))(\sigma^n(b)-\sigma^{n-2}(a))]} \\
&\geq \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))}{(\sigma^n(b)-\sigma^{i-1}(a))} \\
&\geq \left(\frac{t-\sigma^{n-2}(a)}{\sigma^n(b)-\sigma^{n-2}(a)} \right)^{n-1} \\
&\geq \frac{1}{4^{n-1}}.
\end{aligned}
$$

Therefore

$$
\frac{1}{4^{n-1}} G(\sigma^n(b), s) \leq G(t,s).
$$

□

We note that a pair $(u(t), v(t))$ is a solution of the BVP (1)-(2) if and only if

$$
\begin{aligned}
u(t) &= \int_a^{\sigma(b)} G(t,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r \right) \Delta s, \ a \leq t \leq \sigma^n(b), \\
v(t) &= \int_a^{\sigma(b)} G(t,s)) g(s, u(s)) \Delta s, \ a \leq t \leq \sigma^n(b).
\end{aligned}
$$

Assume throughout that $[a, \sigma^n(b)]_{\mathbb{T}}$ is such that

$$\xi = \min\left\{t \in \mathbb{T} \mid t \geq \frac{3a+\sigma^n(b)}{4}\right\}, \quad \omega = \max\left\{t \in \mathbb{T} \mid t \leq \frac{a+3\sigma^n(b)}{4}\right\}$$

both exist and satisfy

$$\frac{3a+\sigma^n(b)}{4} \leq \xi < \omega \leq \frac{a+3\sigma^n(b)}{4}.$$

Next, let $\tau \in [\xi, \omega]_{\mathbb{T}}$ be defined by

$$\int_{\xi}^{\omega} G(\tau, s)\Delta s = \max_{t\in[\xi,\omega]_{\mathbb{T}}} \int_{\xi}^{\omega} G(t, s)\Delta s.$$

Finally, we define

$$l = \min_{s\in[a,\sigma(b)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma^n(b), s)},$$

and let

$$\gamma = \min\left\{\frac{1}{4^{n-1}}, l\right\}. \tag{9}$$

For our construction, let $E = \{u : [a, \sigma^n(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm

$$\|u\| = \sup_{t\in[a,\sigma^n(b)]_{\mathbb{T}}} |u(t)|.$$

Then $(E, \|.\|)$ is a Banach space. Define

$$P = \{u \in E \mid u(t) \geq 0, \text{ on } [a, \sigma^n(b)]_{\mathbb{T}}, \text{ and } \min_{t\in[\xi,\omega]_{\mathbb{T}}} u(t) \geq \gamma\|u\|\}.$$

It is obivious that P is a positive cone in E. Define an integral operator $T : P \to E$ by

$$Tu(t) = \int_a^{\sigma(b)} G(t, s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r)g(r, u(r))\Delta r\right)\Delta s, \ u \in P. \tag{10}$$

Lemma 3. *If the operator T is defined as* (10)*, then $T : P \to P$ is completely continuous.*

Proof. From the continuity of f and g, and (8) that, for $u \in P$, $Tu(t) \geq 0$ on $[a, \sigma^n(b)]_{\mathbb{T}}$. Also, for $u \in P$, we have from (6) that

$$\begin{aligned} Tu(t) &= \int_a^{\sigma(b)} G(t, s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r)g(r, u(r))\Delta r\right)\Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma^n(b), s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r)g(r, u(r))\Delta r\right)\Delta s \end{aligned}$$

so that

$$\|Tu\| \leq \int_a^{\sigma(b)} G(\sigma^n(b), s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r)g(r, u(r))\Delta r\right)\Delta s.$$

Next, if $u \in P$, we have from (7), (9), and (10) that

$$\begin{aligned}\min_{t\in[\xi,\omega]_{\mathbb{T}}} Tu(t) &= \min_{t\in[\xi,\omega]_{\mathbb{T}}} \int_a^{\sigma(b)} G(t,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r\right)\Delta s \\ &\geq \gamma \int_a^{\sigma(b)} G(\sigma^n(b),s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r\right)\Delta s \\ &\geq \gamma \|Tu\|.\end{aligned}$$

Therefore $T : P \to P$. Since $G(t,s)$, $f(t,u)$ and $g(t,u)$ are continuous, it is easily known that $T : P \to P$ is completely continuous. The proof is complete. $\square$

From above arguments, we know that the existence of positive solutions of (1)-(2) can be transferred to the existence of positive fixed points of the operator T.

Lemma 4 ([3, 4, 10])**.** *Let $(E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open bounded subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If*

$$T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

is a completely continuous operator such that either

$$(i)\ \|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_1,\ \text{and}\ \|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_2,\ \text{or}$$
$$(ii)\ \|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_1,\ \text{and}\ \|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_2,$$

then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 5 ([3, 4, 10])**.** *Let $(E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume that Ω_1, Ω_2 and Ω_3 are open bounded subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $\overline{\Omega}_2 \subset \Omega_3$. If*

$$T : P \cap (\overline{\Omega}_3 \backslash \Omega_1) \to P$$

is a completely continuous operator such that:

$$\begin{aligned}&\|Tu\| \geq \|u\|,\ \forall u \in P \cap \partial\Omega_1;\\ &\|Tu\| \leq \|u\|,\ Tu \neq u, \forall u \in P \cap \partial\Omega_2;\\ &\|Tu\| \geq \|u\|,\ \forall u \in P \cap \partial\Omega_3,\end{aligned}$$

then T has at least two fixed points x^*, x^{**} in $P \cap (\overline{\Omega}_3 \backslash \Omega_1)$, and furthermore $x^* \in P \cap (\Omega_2 \backslash \Omega_1)$, $x^{**} \in P \cap (\overline{\Omega}_3 \backslash \overline{\Omega}_2)$.

3. Main Results

First we give the following assumptions:

(A1) $\lim_{u\to 0^+} \sup_{t\in[a,\sigma^n(b)]} \frac{f(t,u)}{u} = 0$, $\lim_{u\to 0^+} \sup_{t\in[a,\sigma^n(b)]} \frac{g(t,u)}{u} = 0$;

(A2) $\lim_{u\to \infty} \inf_{t\in[a,\sigma^n(b)]} \frac{f(t,u)}{u} = \infty$, $\lim_{u\to \infty} \inf_{t\in[a,\sigma^n(b)]} \frac{g(t,u)}{u} = \infty$;

(A3) $\lim_{u\to 0^+} \inf_{t\in[a,\sigma^n(b)]} \frac{f(t,u)}{u} = \infty$, $\lim_{u\to 0^+} \inf_{t\in[a,\sigma^n(b)]} \frac{g(t,u)}{u} = \infty$;

(A4) $\lim_{u\to\infty}\sup_{t\in[a,\sigma^n(b)]}\frac{f(t,u)}{u}=0$, $\lim_{u\to\infty}\sup_{t\in[a,\sigma^n(b)]}\frac{g(t,u)}{u}=0$;

(A5) $f(t,u)$, $g(t,u)$ are increasing functions with respect to u and, there is a number $N>0$, such that

$$f\left(t,\int_a^{\sigma(b)}N'g(s,N)\Delta s\right)<\frac{N}{N'(\sigma^n(b)-a)}\forall\ t\in[a,\sigma^n(b)]_{\mathbb{T}},\ s\in[a,\sigma(b)]_{\mathbb{T}},$$

where $N'=\prod_{i=1}^{n-1}\frac{(\sigma^n(b)-\sigma^i(a))}{(n-1)!}$.

Theorem 1. *If* $(A1)$ *and* $(A2)$ *are satisfied, then* (1)-(2) *has at least one positive solution* $(u,v)\in C^n([a,\sigma^n(b)]_{\mathbb{T}},\mathbb{R}^+)\times C^n([a,\sigma^n(b)]_{\mathbb{T}},\mathbb{R}^+)$ *satisfying* $u(t)>0$, $v(t)>0$.

Proof. From $(A1)$ there is a number $H_1\in(0,1)$ such that for each $(t,u)\in[a,\sigma^n(b)]_{\mathbb{T}}\times(0,H_1)$, one has

$$f(t,u)\le\eta u,\ g(t,u)\le\eta u,$$

where $\eta>0$ satisfies

$$\eta\int_a^{\sigma(b)}G(\sigma^n(b),t)\Delta t\le 1.$$

For every $u\in P$ and $\|u\|=H_1/2$, note that

$$\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\le\eta\int_a^{\sigma(b)}G(\sigma(r),r)u(r)\Delta r\le\|u\|=\frac{H_1}{2}<H_1,$$

thus

$$\begin{aligned}Tu(t)&=\int_a^{\sigma(b)}G(t,s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s\\&\le\eta\int_a^{\sigma(b)}G(\sigma^n(b),s)\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\Delta s\\&\le\eta^2\int_a^{\sigma(b)}G(\sigma^n(b),s)\int_a^{\sigma(b)}G(\sigma^n(b),r)u(r)\Delta r\Delta s\\&\le\eta^2\|u\|\int_a^{\sigma(b)}G(\sigma^n(b),s)\int_a^{\sigma(b)}G(\sigma^n(b),r)\Delta r\Delta s\\&\le\|u\|.\end{aligned}$$

So, $\|Tu\|\le\|u\|$. If we set

$$\Omega_1=\{u\in E:\|u\|<H_1/2\},$$

then

$$\|Tu\|\le\|u\|,\ \text{for } u\in P\cap\partial\Omega_1. \tag{11}$$

On the other hand, from $(A2)$ there exist four positive numbers μ, μ', C_1 and C_2 such that

$$f(t,u)\ge\mu u-C_1,\ \forall(t,u)\in[a,\sigma^n(b)]_{\mathbb{T}}\times\mathbb{R}^+,$$

$$g(t,u) \geq \mu' u - C_2,\ \forall (t,u) \in [a, \sigma^n(b)]_{\mathbb{T}} \times \mathbb{R}^+,$$

where μ and μ' satisfy

$$\mu\gamma \int_\xi^\omega G(\tau,s)\Delta s \geq 2,\ \mu'\gamma \int_\xi^\omega G(\sigma^n(b),s)\Delta s \geq 1.$$

For $u \in P$, we have

$$\begin{aligned}
Tu(\tau) &= \int_a^{\sigma(b)} G(\tau,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r\right)\Delta s \\
&\geq \int_a^{\sigma(b)} G(\tau,s)\left[\mu \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r - C_1\right]\Delta s \\
&= \mu \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r \Delta s - C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s \\
&\geq \mu \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(s),r)[\mu' u(r) - C_2]\Delta r \Delta s - C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s \\
&= \mu\mu' \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(s),r) u(r)\Delta r \Delta s - C_3,
\end{aligned}$$

where

$$\begin{aligned}
C_3 &= \mu C_2 \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(s),r)\Delta r \Delta s + C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s \\
&\leq \mu C_2 \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma^n(b),r)\Delta r \Delta s + C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s.
\end{aligned}$$

Therefore

$$Tu(\tau) \geq \mu \int_\xi^\omega G(\tau,s)\Delta s.\ \gamma\mu' \int_\xi^\omega G(\sigma^n(b),r) u(r)\Delta r - C_3 \geq 2\|u\| - C_3,$$

from which it follows that $\|Tu\| \geq Tu(\tau) \geq \|u\|$ as $\|u\| \to \infty$.

Let $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then for $u \in P$ and $\|u\| = H_2 > 0$ sufficient by large, we have

$$\|Tu\| \geq \|u\|,\ \text{for } u \in P \cap \partial\Omega_2. \tag{12}$$

Thus, from (11), (12) and Lemma 4, we know that the operator T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete. □

Theorem 2. *If* (*A*3) *and* (*A*4) *are satisfied, then* (1)-(2) *has at least one positive solution* $(u,v) \in C^n([a,\sigma^n(b)]_{\mathbb{T}}, \mathbb{R}^+) \times C^n([a,\sigma^n(b)]_{\mathbb{T}}, \mathbb{R}^+)$ *satisfying* $u(t) > 0$, $v(t) > 0$.

Proof. From (*A*3) there is a number $\widehat{H}_3 \in (0,1)$ such that for each $(t,u) \in [a,\sigma^n(b)]_{\mathbb{T}} \times (0, \widehat{H_3})$, one has

$$f(t,u) \geq \lambda u,\ g(t,u) \geq \lambda' u,$$

where $\lambda >$ and λ' satisfy

$$\lambda\gamma \int_{\xi}^{\omega} G(\tau,t)\Delta t \geq 1,\ \lambda'\gamma \int_{\xi}^{\omega} G(\sigma^n(b),s)\Delta s \geq 1.$$

From $g(t,0) \equiv 0$ and the continuity of $g(t,u)$, we know that there exists a number $H_3 \in (0, \widehat{H}_3)$ small enough such that

$$g(t,u) \leq \frac{\widehat{H}_3}{\int_a^{\sigma(b)} G(\sigma(t),t)\Delta t},\ \forall (t,u) \in [a,\sigma^n(b)]_{\mathbb{T}} \times (0,H_3).$$

For every $u \in P$ and $\|u\| = H_3$, note that

$$\int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r \leq \int_a^{\sigma(b)} G(\sigma(s),r)\frac{\widehat{H}_3}{\int_a^{\sigma(b)} G(\sigma(r),r)\Delta r}\Delta r \leq \widehat{H}_3,$$

thus

$$\begin{aligned}
Tu(\tau) &= \int_a^{\sigma(b)} G(\tau,s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s \\
&\geq \int_{\xi}^{\omega} G(\tau,s)\lambda \int_{\xi}^{\omega} G(\sigma(s),r)\lambda' u(r)\Delta r\Delta s \\
&\geq \gamma^2\|u\|\lambda \int_{\xi}^{\omega} G(\tau,s)\lambda' \int_{\xi}^{\omega} G(\sigma^n(b),r)\Delta r\Delta s \\
&\geq \|u\|.
\end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we set

$$\Omega_3 = \{u \in E : \|u\| < H_3\},$$

then

$$\|Tu\| \geq \|u\|,\ \text{for } u \in P \cap \partial\Omega_3. \tag{13}$$

On the other hand, we know from $(A4)$ that there exist three positive numbers η', C_4, and C_5 such that for every $(t,u) \in [a,\sigma^n(b)]_{\mathbb{T}} \times \mathbb{R}^+$,

$$f(t,u) \leq \eta' u + C_4,\ g(t,u) \leq \eta' u + C_5,$$

where

$$\eta' \int_a^{\sigma(b)} G(\sigma^n(b),t)\Delta t \leq \frac{1}{2}.$$

Thus we have

$$\begin{aligned}
Tu(t) &= \int_a^{\sigma(b)} G(t,s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s \\
&\leq \int_a^{\sigma(b)} G(\sigma^n(b),s)\left[\eta' \int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r + C_4\right]\Delta s \\
&\leq \eta' \int_a^{\sigma(b)} G(\sigma^n(b),s)\Delta s \int_a^{\sigma(b)} G(\sigma^n(b),r)[\eta' u(r) + C_5]\Delta r\Delta s
\end{aligned}$$

$$+ C_4 \int_a^{\sigma(b)} G(\sigma^n(b), s)\Delta s$$
$$\leq \frac{1}{4}\|u\| + C_6,$$

where

$$C_6 = C_5\eta' \int_a^{\sigma(b)} G(\sigma^n(b), s)\Delta s \int_a^{\sigma(b)} G(\sigma^n(b), r)\Delta r + C_4 \int_a^{\sigma(b)} G(\sigma^n(b), s)\Delta s,$$

from which it follows that $Tu(t) \leq \|u\|$ as $\|u\| \to \infty$. Let $\Omega_4 = \{u \in E : \| u \| < H_4\}$. For each $u \in P$ and $\|u\| = H_4 > 0$ large enough, we have

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_4. \tag{14}$$

From (13), (14) and Lemma 4, we know that the operator T has a fixed point in $P \cap (\overline{\Omega}_4 \backslash \Omega_3)$. The proof is complete. □

Theorem 3. *If* $(A2)$, $(A3)$ *and* $(A5)$ *are satisfied, then* (1)-(2) *has at least two distinct positive solutions* $(u_1, v_1), (u_2, v_2) \in C^n([a, \sigma^n(b)]_{\mathbb{T}}, \mathbb{R}^+) \times C^n([a, \sigma^n(b)]_{\mathbb{T}}, \mathbb{R}^+)$ *satisfying* $u_i(t) > 0$, $v_i(t) > 0$ $(i = 1, 2)$.

Proof. Note that $G(t, s) \leq \prod_{i=1}^{n-1} \frac{(\sigma^n(b) - \sigma^i(a))}{(n-1)!} = N'$. Let $B_N = \{u \in E : \|u\| < N\}$. Then from $(A5)$, for every $u \in \partial B_N \cap P$, $t \in [a, \sigma^n(b)]_{\mathbb{T}}$, we have

$$Tu(t) = \int_a^{\sigma(b)} G(t, s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r))\Delta r\right)\Delta s$$
$$\leq N' \int_a^{\sigma(b)} f\left(s, \int_a^{\sigma(b)} N' g(r, N)\Delta r\right)\Delta s$$
$$< N' \int_a^{\sigma(b)} \frac{N}{N'(\sigma^n(b) - a)}\Delta s \leq N.$$

Thus

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial B_N. \tag{15}$$

And from $(A2)$ and $(A3)$ we have

$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_2, \tag{16}$$

$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_3. \tag{17}$$

We can choose H_2, H_3 and N such that $H_3 \leq N \leq H_2$ and (15)-(17) are satisfied. From Lemma 5, T has at least two fixed points in $P \cap (\overline{\Omega}_2 \setminus B_N)$ and $P \cap (\overline{B}_N \setminus \Omega_2)$, respectively. The proof is complete. □

4. Examples

Some examples are given to illustrate our main results.

Example 1. Consider the following dynamic equations

$$u^{\Delta^{(3)}}(t) + f(t,v) = 0,\ v^{\Delta^{(3)}}(t) + g(t,u) = 0,\ t \in [0,1]_{\mathbb{T}}, \tag{18}$$

satisfying the boundary conditions

$$u(0) = u^{\Delta}(0) = 0, u(\sigma^3(1)) = 0,\ v(0) = v^{\Delta}(0) = 0, v(\sigma^3(1)) = 0, \tag{19}$$

where $f(t,v) = v^3$, $g(t,u) = u^2$, then conditions of Theorem 1 are satisfied. From Theorem 1, the BVP (18)-(19) has at least one positive solution.

Example 2. Consider the following dynamic equations

$$u^{\Delta^{(3)}}(t) + f(t,v) = 0,\ v^{\Delta^{(3)}}(t) + g(t,u) = 0,\ t \in [0,1]_{\mathbb{T}}, \tag{20}$$

satisfying the boundary conditions

$$u(0) = u^{\Delta}(0) = 0, u(\sigma^3(1)) = 0,\ v(0) = v^{\Delta}(0) = 0, v(\sigma^3(1)) = 0, \tag{21}$$

where $f(t,v) = v^{2/3}$, $g(t,u) = u^{3/4}$, then conditions of Theorem 2 are satisfied. From Theorem 2, the BVP (20)-(21) has at least one positive solution.

Example 3. Consider the following system of boundary value problems

$$u^{\Delta^{(3)}}(t) + f(t,v) = 0,\ v^{\Delta^{(3)}}(t) + g(t,u) = 0,\ t \in [0,1]_{\mathbb{T}}, \tag{22}$$

$$u(0) = u^{\Delta}(0) = 0, u(\sigma^3(1)) = 0,\ v(0) = v^{\Delta}(0) = 0, v(\sigma^3(1)) = 0, \tag{23}$$

where

$$\mathbb{T} = \left\{ \left(\frac{2}{5}\right)^n : n \in \mathbb{N}_0 \right\} \cup \{0\} \cup [1,2],$$

$f(t,v) = \frac{v^{2/3}+v^4}{8}$, $g(t,u) = u^{2/3} + u^4$. Then $N' = 17.8$. We can choose $N = 100$, then conditions of Theorem 3 are satisfied. From Theorem 3, the BVP (22)-(23) has at least two positive solutions.

References

1. R.P. Agarwal, D. O'Regan, and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, The Netherlands, 1999.
2. M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, Mass, USA, 2001.
3. K. Deimling, *Nonlinear Functional Analysis*, Springer-verlag, Berlin, 1985.
4. L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **120** (1994), 743-748.
5. L.H. Erbr and A. Peterson, *Positive solutions for a nonlinear differential equation on a measure chain*, Math. Comp. Model., **32** (2000), 571-585.
6. P.W. Eloe and J. Henderson, *Positive solutions for higher order differential equations*, Elec. J. Diff. Eqns., **1995** (1995), 1-8.
7. J.R. Graef and B. Yang, *Boundary value problems for second order nonlinear ordinary differential equations*, Comm. Applied Anal., **6** (2002), 273-288.
8. D.J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New Yark, 1988.
9. L. Hu and L. Wang, *Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations*, J. Math. Anal. Appl., **335** (2007), 1052-1060.

10. M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd, Groningen, The Netherlands (1964).
11. W.T. Li and H.R. Sun, *Multiple positive solutions for nonlinear dynamical systems on a measure chain*, J. Comp. Appl. Math., **162** (2004), 421-430.
12. R.Y. Ma, *Multiple nonnegative solutions of second-order systems of boundary value problems*, Nonlinear Anal, (T. M. A), **42** (2000), 1003-1010.
13. H.R. Sun, *Existence of positive solutions to second-order time scale systems*, Comp. Math. Appl., **49** (2005), 131-145.
14. H. Wang, *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl., **281** (2003), 287-306.
15. Z.L. Yang and J.X. Sun, *Positive solutions of boundary value problems for system of nonlinear second order ordinary differential equations*, Acta. Math. Sinica., **47** (2004), 111-118.

A. K. Rao received M. Sc degree from Andhra University, Visakhapatnam. M. Phil degree from M. K. University, Madhurai and Ph. D from Andhra University under the esteemed guidence of Prof. K. R. Prasad. His research interest focuses on Boundary Value Problems.

Department of Mathematics, Gayatri Vidya Parishad College of Engineering for Women, Madhurawada, Visakhapatnam, 530 048, India.
e-mail: kamesh_1724@yahoo.com