KYUNGPOOK Math. J. 55(2015), 1097-1104
http://dx.doi.org/10.5666/KMJ.2015.55.4.1097
pISSN 1225-6951 eISSN 0454-8124

Indefinite Generalized Sasakian Space Form Admitting a Lightlike Hypersurface

Dae Ho Jin
Department of Mathematics, Dongguk University, Gyeongju 780-714, Korea
e-mail : jindh@dongguk.ac.kr

Abstract. In this paper, we study the geometry of indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a lightlike hypersurface M subject such that the almost contact structure vector field ζ of $\bar{M}(f_1, f_2, f_3)$ is tangent to M. We prove a classification theorem of such an indefinite generalized Sasakian space form.

1. Introduction

Oubina [12] introduced the notion of a trans-Sasakian manifold $\bar{M}$ of type (α, β), where $\bar{M}$ is a Riemannian manifold. Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of trans-Sasakian manifold such that

$$\alpha = 1,\ \beta = 0; \qquad \alpha = 0,\ \beta = 1; \qquad \alpha = \beta = 0,$$

respectively. Now we say that a trans-Sasakian manifold $\bar{M}$ of type (α, β) is an indefinite trans-Sasakian manifold if $\bar{M}$ is a semi-Riemannian manifold.

Alegre *et. al.* [1] introduced generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, where $\bar{M}(f_1, f_2, f_3)$ is a Riemannian manifold. We say that a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ is an indefinite generalized Sasakian space form if $\bar{M}(f_1, f_2, f_3)$ is a semi-Riemannian manifold. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c denotes constant J-sectional curvatures of each of them.

The theory of lightlike submanifolds is an important topic of research in modern differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [3]

Received September 17, 2013; accepted October 1, 2015.
2010 Mathematics Subject Classification: 53C25, 53C40, 53C50.
Key words and phrases: indefinite generalized Sasakian space form, indefinite trans-Sasakian manifold, lightlike hypersurface.

and later studied by many authors (see two books [4, 5]). The object of this paper is to study the geometry of indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a lightlike hypersurface M subject such that the structure vector field of $\bar{M}(f_1, f_2, f_3)$ is tangent to M. The main result is a classification theorem of such an indefinite generalized Sasakian space form.

2. Lightlike Hypersurfaces

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM of M, of rank 1. Therefore there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM, which is called a *screen distribution* [3], such that

$$TM = TM^{\perp} \oplus_{orth} S(TM), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also we denote by $(2.9)_i$ the i-th equation of the two equations in (2.9). We use same notations for any others. It is well known [3] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ of rank 1 in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.2}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribtion $S(TM)$ respectively.

In the sequel, let X, Y, Z and W be tangent vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.5}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are liner connections on M and $S(TM)$ respectively, B and C are local second fundamental forms on M and $S(TM)$ respectively, A_N and A^*_{ξ} are shape operators on M and $S(TM)$ respectively and τ is a 1-form on TM.

Since the connection $\bar{\nabla}$ is torsion-free, the induced connection ∇ is also torsion-free and B is symmetric. From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$, we show that

B is independent of the choice of screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{2.7}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{2.8}$$

where η is a 1-form such that

$$\eta(X) = \bar{g}(X, N).$$

But the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms B and C are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.9}$$

$$C(X, PY) = g(A_{_N} X, PY), \quad \bar{g}(A_{_N} X, N) = 0. \tag{2.10}$$

From (2.9), the operator A^*_ξ is $S(TM)$-valued self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.11}$$

Denote by $\bar{R}$ and R the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ and the induced connection ∇ of M respectively. We need the following three Gauss-Codazzi equations (for a full set of these equations see [3, 4]).

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z,\, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ + B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \end{aligned} \tag{2.12}$$

$$\bar{g}(\bar{R}(X, Y)Z,\, N) = g(R(X, Y)Z,\, N), \tag{2.13}$$

$$\begin{aligned} g(R(X, Y)PZ,\, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \end{aligned} \tag{2.14}$$

3. Indefinite Generalized Sasakian Space Forms

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exist a tensor field J of type (1, 1), a vector field ζ, which is called the *structure vector field*, and a 1-form θ such that

$$\begin{aligned} &J^2 X = -X + \theta(X)\zeta, \qquad J\zeta = 0, \qquad \theta \circ J = 0, \\ &\theta(\zeta) = 1, \qquad \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y), \end{aligned} \tag{3.1}$$

for any vector fields X and Y in $\bar{M}$, where ϵ is the sign of ζ, *i.e.*, $\epsilon = \bar{g}(\zeta, \zeta) = \pm 1$. The set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

Definition 1. An indefinite almost contact metric manifold $\bar{M}$ is called *indefinite trans-Sasakian manifold* [1, 12] if there exist two functions α and β such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X,Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \epsilon\theta(Y)JX\}, \tag{3.2}$$

for any vector fields X and Y on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$ with respect to the semi-Riemannian metric $\bar{g}$. We say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β).

From (3.1) and (3.2), we show that

$$\bar{\nabla}_X \zeta = -\epsilon\alpha JX + \epsilon\beta(X - \theta(X)\zeta), \qquad d\theta(X,Y) = \alpha\bar{g}(X, JY). \tag{3.3}$$

Definition 2. An indefinite almost contact metric manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form* [1] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(X,Y)Z \;=\; & f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + f_2\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & \qquad + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}, \end{aligned} \tag{3.4}$$

for any vector fields X, Y and Z on $\bar{M}$, where $\bar{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}(f_1, f_2, f_3)$.

In the entire discussion of this article, let M be a lightlike hypersurface of a indefinite trans-Sasakian manifold $\bar{M}$ and we shall assume that the vector field ζ is tangent to M, such M is called a *tangential lightlike hypersurface* of $\bar{M}$. Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which many authors assumed in their works [5, 11]. We also assume this result. In this case,

$$\theta(\xi) = \epsilon g(\zeta, \xi) = 0, \quad \theta(N) = \epsilon g(\zeta, N) = 0. \tag{3.5}$$

It is well known [3, 6] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$, $J(TM^{\perp})$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1. Therefore, there exist two almost complex distributions D_o and D with respect to the structure tensor J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = \{J(TM^{\perp}) \oplus J(tr(TM))\} \oplus_{orth} D_o,$$
$$D = TM^{\perp} \oplus_{orth} J(TM^{\perp}) \oplus_{orth} D_o.$$

Using these two distributions, the decomposition form (2.1) is reduced to

$$TM = D \oplus J(tr(TM)).$$

Consider two local lightlike vector fields U and V and their 1-forms such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{3.6}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{3.7}$$

where F is a tensor field of type $(1,1)$ globally defined on M by

$$FX = JSX.$$

Applying $\bar{\nabla}_X$ to $(3.6)_{1,2}$ and using (2.3)∼(2.7), (3.2), (3.6) and (3.7), we have

$$v(A^*_\xi X) = B(X,U) = C(X,V) = u(A_N X), \tag{3.8}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.9}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \beta u(X)\zeta. \tag{3.10}$$

Theorem 3.1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, endowed with indefinite trans-Sasakian sutucture of type (α, β), admitting a tangential lightlike hypersurface M satisfies* (1) *α is a constant and* (2) $\alpha\beta = 0$.

(i) *In case $\alpha = 0$: $\epsilon\zeta[\beta] + \beta^2 = f_3 - \epsilon f_1$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu space form.*

(ii) *In case $\alpha \neq 0$: $\alpha^2 = \epsilon f_1 - f_3$, $\beta = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form.*

Proof. Applying $\bar{\nabla}_X$ to $(3.5)_{1,2}$ and using (3.1), (3.3) and (3.6), we have

$$B(X,\zeta) = -\epsilon\alpha u(X), \quad C(X,\zeta) = \epsilon\beta\eta(X) - \epsilon\alpha v(X). \tag{3.11}$$

Substituting (3.7) into (3.3) and using (2.3), we have

$$\nabla_X \zeta = -\epsilon\alpha FX + \epsilon\beta(X - \theta(X)\zeta). \tag{3.12}$$

Applying $\bar{\nabla}_X$ to $u(Y) = g(Y,V)$ and using (3.7) and (3.10), we get

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - \epsilon\beta\theta(Y)u(X) - B(X,FY). \tag{3.13}$$

Substituting (3.4) into (2.12), we have

$$\begin{aligned} & f_2\{u(Y)\bar{g}(X,JZ) - u(X)\bar{g}(Y,JZ) + 2u(Z)\bar{g}(X,JY)\} \\ &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y). \end{aligned}$$

Replacing Z by ζ to this equation and using (3.11), we have

$$(\nabla_X B)(Y,\zeta) - (\nabla_Y B)(X,\zeta) + \epsilon\alpha\{u(X)\tau(Y) - u(Y)\tau(X)\} = 0. \tag{3.14}$$

Applying ∇_X to $B(Y,\zeta) = -\epsilon\alpha u(Y)$ and using (3.12) and (3.13), we have

$$\begin{aligned} (\nabla_X B)(Y,\zeta) = & -\epsilon(X\alpha)u(Y) - \epsilon\beta B(X,Y) \\ & + \alpha\beta\{\theta(Y)u(X) - \theta(X)u(Y)\} \\ & + \epsilon\alpha\{u(Y)\tau(X) + B(X,FY) + B(Y,FX)\}. \end{aligned}$$

Substituting this equation into (3.14), we have

$$\{\epsilon(X\alpha) + 2\alpha\beta\theta(X)\}u(Y) = \{\epsilon(Y\alpha) + 2\alpha\beta\theta(Y)\}u(X).$$

Replacing Y by U to this equation, we obtain

$$\epsilon(X\alpha) + 2\alpha\beta\theta(X) = \epsilon(U\alpha)u(X). \tag{3.15}$$

Applying $\bar{\nabla}_X$ to $v(Y) = g(Y, U)$ and using (3.6) $\sim$ (3.9), we get

$$(\nabla_X v)(Y) = v(Y)\tau(X) - \epsilon\theta(Y)\{\alpha\eta(X) + \beta v(X)\} - g(A_{_N}X, FY). \tag{3.16}$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y, N)$ and using (2.4), we have

$$(\nabla_X \eta)Y = -g(A_{_N}X, Y) + \tau(X)\eta(Y). \tag{3.17}$$

Substituting (2.14) and (3.4) into (2.13) with $Z = PZ$, we have

$$\begin{aligned} & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ & + f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\} \\ & + f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} \\ & = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & \qquad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \end{aligned} \tag{3.18}$$

Replacing PZ by ζ to the last equation and using $(3.11)_2$, we have

$$\begin{aligned} & (\epsilon f_1 - f_3)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\} \\ & = (\nabla_X C)(Y, \zeta) - (\nabla_Y C)(X, \zeta) + \epsilon\beta\{\eta(X)\tau(Y) - \eta(Y)\tau(X)\} \\ & \qquad + \epsilon\alpha\{v(Y)\tau(X) - v(X)\tau(Y)\}. \end{aligned} \tag{3.19}$$

Applying ∇_Y to $(3.11)_2$ and using (3.12), (3.16) and (3.17), we have

$$\begin{aligned} (\nabla_X C)(Y, \zeta) = {} & \epsilon(X\beta)\eta(Y) - \epsilon(X\alpha)v(Y) \\ & + \epsilon\alpha\{g(A_{_N}X, FY) + g(A_{_N}Y, FX) - v(Y)\tau(X)\} \\ & + \epsilon\beta\{-g(A_{_N}X, Y) - g(A_{_N}Y, X) + \eta(Y)\tau(X)\} \\ & + \alpha\beta\{\theta(Y)v(X) - \theta(X)v(Y)\} \\ & + \alpha^2\theta(Y)\eta(X) + \beta^2\theta(X)\eta(Y). \end{aligned}$$

Substituting this equation into (3.19) and using (3.15), we have

$$\begin{aligned} & \{\epsilon(X\beta) + A\theta(X)\}\eta(Y) - \{\epsilon(Y\beta) + A\theta(Y)\}\eta(X) \\ & = \epsilon(U\alpha)\{u(X)v(Y) - u(Y)v(X)\}. \end{aligned} \tag{3.20}$$

where $A = \epsilon f_1 - f_3 - \alpha^2 + \beta^2$. Taking $X = U$ and $Y = V$ to (3.20), we have $U\alpha = 0$. Therefore (3.15) is reduced to

$$\epsilon(X\alpha) + 2\alpha\beta\theta(X) = 0. \tag{3.21}$$

Replacing Y by ξ to (3.20) and then, replacing X by ζ, we have

$$X\beta = (\xi\beta)\eta(X) - \epsilon A\theta(X), \tag{3.22}$$
$$\epsilon(\zeta\beta) + \{\epsilon f_1 - f_3 - \alpha^2 + \beta^2\} = 0. \tag{3.23}$$

Applying ∇_Y to (3.21) and using (3.21) and (3.22), we have

$$\epsilon XY\alpha + 2\alpha\beta X(\theta(Y)) - 2\alpha\epsilon(2\beta^2 + A)\theta(X)\theta(Y) + 2\alpha(\xi\beta)\eta(X)\theta(Y) = 0.$$

Using this equation and the fact $[X, Y] = XY - YX$, we obtain

$$2\alpha^2\beta\bar{g}(X, JY) = \alpha(\xi\beta)\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}. \tag{3.24}$$

Taking $X = U$ and $Y = \xi$ to (3.24), we have $\alpha\beta = 0$. As $\alpha\beta = 0$, we show that α is a constant by (3.21). Therefore $\alpha = 0$ or $\beta = 0$.

(i) In case $\alpha = 0$, from (3.23) we have $\epsilon\zeta\beta + \beta^2 = f_3 - \epsilon f_1$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu space form.

(ii) In case $\alpha \neq 0$, we get $\beta = 0$. Therefore $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form. From (3.23) we show that $\alpha^2 = \epsilon f_1 - f_3$. □

Corollary 1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, of type (α, β), admitting a tangential lightlike hypersurface M is either an indefinite β-Kenmotsu space form or an indefinite α-Sasakian space form.*

Definition 3. We say that M is *screen totally umbilical* [3] if there exist a smooth function γ on $\mathcal{U}$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X, PY) = \gamma g(X, Y). \tag{3.25}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that M is *screen totally geodesic.*

Theorem 3.2. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a screen totally umbilical lightlike hypersurface is a semi-Euclidean space, i.e., $\bar{M}(f_1, f_2, f_3)$ is satisfied $f_1 = f_2 = f_3 = 0$.*

Proof. As M is screen totally umbilical, from $(3.11)_2$ and (3.25) we have

$$\gamma\theta(X) = \beta\eta(X) - \alpha v(X).$$

Taking $X = \zeta$, $X = V$ and $X = \xi$ to this equation by turns, we have $\gamma = 0$, $\alpha = 0$ and $\beta = 0$ respectively. Thus M is screen totally geodesic and $\bar{M}$ is an indefinite cosymplectic manifold. As $C = 0$, (3.18) is reduce to

$$\begin{aligned} & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ & + f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\} \\ & + f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} = 0. \end{aligned}$$

Replacing Y by ξ to this equation, we obtain

$$f_1 g(X, PZ) + f_2\{v(X)u(PZ) + 2u(X)v(PZ)\} - f_3\theta(X)\theta(PZ) = 0.$$

Taking $X = V$, $PZ = U$; $X = U$, $PZ = V$ and $X = PZ = \zeta$ by turns, we have

$$f_1 + f_2 = 0, \quad f_1 + 2f_2 = 0, \quad \epsilon f_1 = f_3.$$

From the first two equations of these results, we show that $f_2 = 0$. As $\bar{M}$ is an indefinite cosymplectic manifold, we have $f_1 = f_2 = f_3 = \frac{c}{4}$. Thus we obtain $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is a semi-Euclidean space. □

References

[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian space form*, Israel J. Math., **141**(2004), 157-183.

[2] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi (Romania, 1998).

[3] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[4] K. L. Duggal and D. H. Jin, *Null curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[5] K. L. Duggal and B. Sahin, *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, Birkhäuser, 2010.

[6] D. H. Jin, *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. of Pure and Applied Math., **41(4)**(2010), 569-581.

[7] D. H. Jin, *The curvatures of lightlike hypersurfaces of an indefinite Kenmotsu manifold*, Balkan J. Geo. Its Appl., **17(1)**(2012), 49-57.

[8] D. H. Jin, *Non-existence of screen homothetic half lightlike submanifolds of an indefinite Kenmotsu manifold,* Balkan J. Geo. Its Appl., **18(1)**(2013), 22-30.

[9] D. H. Jin, *Geometry of lightlike hypersurfaces of an indefinite cosymplectic manifold*, Commun. Korean Math. Soc., **27(1)**(2012), 185-195.

[10] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite cosymplectic manifold*, Math. Prob. in Engineering, 2011, Art ID 610986, 1-16.

[11] T. H. Kang, S. D. Jung, B. H. Kim, H. K. Pak and J. S. Pak, *Lightlike hypersurfaces of indefinite Sasakian manifolds*, Indian J. Pure and Apple., Math., **34**(2003), 1369-1380.

[12] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen, **32**(1985), 187-193.