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Inclusion and Subordination Properties of Multivalent Analytic Functions Involving Cho-Kwon-Srivastava Operator

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Abstract. The object of the present paper is to derive some inclusion and subordination results for certain classes of multivalent analytic functions in the open unit disk, which are defined in terms of the Cho-Kwon-Srivastava operator. Some interesting corollaries are derived and the relevant connection of the results obtained in this paper with various known results are also pointed out.

1. Introduction and Preliminaries

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we denote $\mathcal{A}_1 = \mathcal{A}$.

For functions f and g, analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which (by defintion) is analytic in $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ and $f(z) = g(\omega(z)), z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

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relation (cf., e.g., [13]; see also [15]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\Psi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ and h be univalent in $\mathbb{U}$. If φ is analytic in $\mathbb{U}$ and satisfies the (first-order) differential subordination

$$\Psi\left(\varphi(z), z\varphi'(z); z\right) \prec h(z), \tag{1.2}$$

then φ is called a solution of the differential subordination. A univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $\varphi \prec q$ for all φ satisfying (1.2). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2). We note that the best dominant is unique up to a rotation of $\mathbb{U}$.

For functions $f_j(z) = \sum_{k=0}^{\infty} a_{k,j} z^k$ $(j = 1, 2)$ analytic in $\mathbb{U}$, we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 \star f_2)(z) = \sum_{k=0}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 \star f_1)(z) \quad (z \in \mathbb{U}).$$

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of complex order b and type ρ, that is, $f \in \mathcal{S}_p^*(b;\rho)$, if it satisfies the inequality:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \rho \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, 0 \leq \rho < 1; z \in \mathbb{U}). \tag{1.3}$$

Analogously, a function $f \in \mathcal{A}_p$ is said to be p-valently convex of complex order b and type ρ, that is, $f \in \mathcal{C}_p(b;\rho)$, if it satisfies the inequality:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > \rho \quad (b \in \mathbb{C}^*, 0 \leq \rho < 1; z \in \mathbb{U}). \tag{1.4}$$

From (1.3) and (1.4), it follows that

$$f \in \mathcal{C}_p(b;\rho) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(b;\rho).$$

In particular, the classes $\mathcal{S}_1^*(b;\rho)$ and $\mathcal{C}_1(b;\rho)$ reduces to the classes $\mathcal{S}^*(b;\rho)$ and $\mathcal{C}(b;\rho)$ of starlike functions of complex order b and type ρ, and convex functions of complex order b and type $\rho\,(b \in \mathbb{C}^*; 0 \leq \rho < 1)$, respectively, which were introduced by Frasin [5].

We, further observe that $\mathcal{S}_p^*(p;\alpha/p) = \mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(p;\alpha/p) = \mathcal{C}_p(\alpha)$ are, respectively, the classes of p-valently starlike and p-valently convex functions of order $\alpha\,(0 \leq \alpha < p)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}_1^*(\rho) = \mathcal{S}^*(\rho)$ and $\mathcal{C}_1(\rho) = \mathcal{C}(\rho)$ are the usual classes of starlike and convex functions of order $\rho\,(0 \leq \rho < 1)$ in $\mathbb{U}$. In the special cases, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$.

For $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{\ldots, -2, -1, 0\}$, Saitoh [20] introduced a linear operator $\mathcal{L}_p(a,c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ defined by

$$\mathcal{L}_p(a,c)f(z) = \varphi_p(a,c;z) \star f(z) \quad (f \in \mathcal{A}_p; z \in \mathbb{U}),$$

where φ_p is the incomplete beta function defined by

$$\varphi_p(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (z \in \mathbb{U}), \tag{1.5}$$

and the symbol $(x)_k$ denotes the Pochhammer symbol (or shifted factorial) given by

$$(x)_k = \begin{cases} 1, & (k = 0, x \in \mathbb{C}^*) \\ x(x+1)\cdots(x+k-1), & (k \in \mathbb{N}, x \in \mathbb{C}). \end{cases}$$

The operator $\mathcal{L}_p(a,c)$ is an extension of the Carlson-Shaffer operator [2]. In [3], Cho et al. introduced the family of linear operators $\mathcal{I}_p^{\lambda}(a,c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ analogous to $\mathcal{L}_p(a,c)$ as follows:

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = \varphi_p^{(\dagger)}(a,c;z) \star f(z) \quad (a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U}), \tag{1.6}$$

where $\varphi_p^{(\dagger)}(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following relation

$$\varphi_p(a,c;z) \star \varphi_p^{(\dagger)}(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p; z \in \mathbb{U}), \tag{1.7}$$

where $\varphi_p(a,c;z)$ is given by (1.5). It follows from (1.5) and (1.7) that

$$\varphi_p^{(\dagger)}(a,c;z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c)_k}{(1)_k (a)_k} z^{p+k} \quad (z \in \mathbb{U})$$

so that if $f \in \mathcal{A}_p$ is given by (1.1), then it is easily seen from the above expression and (1.6) that

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{(1)_k (a)_k} a_{p+k} z^{p+k} \quad (z \in \mathbb{U}), \tag{1.8}$$

which readily yields the following identities:

$$z\left(\mathcal{I}_p^{\lambda}(a+1,c)f(z)\right)' = a\,\mathcal{I}_p^{\lambda}(a,c)f(z) - (a-p)\,\mathcal{I}_p^{\lambda}(a+1,c)f(z) \quad (z \in \mathbb{U}) \tag{1.9}$$

and

$$z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)' = (\lambda+p)\,\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\,\mathcal{I}_p^{\lambda}(a,c)f(z) \quad (z \in \mathbb{U}).$$

We observe that

(i) $\mathcal{I}_p^1(p+1,1)f(z) = \mathcal{I}_p^\lambda(\lambda+p,1)f(z) = f(z)$,

(ii) $\mathcal{I}_p^1(p,1)f(z) = \dfrac{zf'(z)}{p}$,

(iii) $\mathcal{I}_p^n(a,a)f(z) = \mathcal{D}^{n+p-1}f(z)$ $(n > -p)$ (see Goel and Sohi [6]),

(iv) $\mathcal{I}_p^{1-\mu}(p-\mu,p+1)f(z) = \Omega_z^{(\mu,p)}f(z)$ $(-\infty < \mu < p+1)$ (see Patel and Mishra [17]),

and

(v) $\mathcal{I}_p^\delta(\delta+p+1,1)f(z) = \mathcal{F}_{\delta,p}(f)(z)$ $(\delta > -p; z \in \mathbb{U})$, the familiar Bernardi-Libera-Livingston integral operator (see, for example [4]).

We note that for the function f, given by (1.1)

$$\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta+p}{z^\delta}\int_0^z t^{\delta+p-1}f(t)dt = z^p + \sum_{k=1}^{\infty}\frac{\delta+p}{\delta+k+p}a_{k+p}z^{k+p} \ (\delta > -p; z \in \mathbb{U}). \tag{1.10}$$

Cho et al. [3] established some inclusion relationships and argument properties for certain subclasses of $\mathcal{A}_p$, which were defined in terms of their operator $\mathcal{I}_p^\lambda(a,c)$(see also [18]). For the choices $\lambda = c = 1$ and $a = n+p$, the Cho-Kwon-Srivastava operator $\mathcal{I}_p^\lambda(a,c)$ reduces to the operator $\mathcal{I}_p^1(n+p,1) = \mathcal{I}_{n,p}$ $(n > -p)$, where $\mathcal{I}_{n,p}$ is the integral operator studied by Liu and Noor [9](for details, see [10] and [11]). The Choi-Saigo-Srivastava operator $\mathcal{I}_1^\lambda(\mu+2,1)$ $(\lambda > -1; \mu > -2)$ was studied in [4]. The operator $\Omega_z^{(\mu,p)}$ for $0 \le \mu < 1$ was investigated by Srivastava and Aouf [22] and studied by Srivastava and Mishra [24]. Patel and Mishra [17] also studied certain classes of multivalent analytic functions involving the extended differintegral operator $\Omega_z^{(\mu,p)}$ when $-\infty < \mu < p+1$. We further observe that $\Omega_z^{(\mu,1)} = \Omega_z^\mu$ is the operator introduced and studied by Owa and Srivastava [16].

Using the Cho-Kwon-Srivastava operator $\mathcal{I}_p^\lambda(a,c)$ and the principle of subordination between analytic functions, we now define a subclass of $\mathcal{A}_p$ as follows:

Definition 1.1 For fixed parameters A, B $(-1 \le B < A \le 1), 0 \le \beta \le 1$ and $0 \le \alpha < p$, we say that a function $f \in \mathcal{A}_p$ is in the class $\mathcal{V}_{p,\beta}^\lambda(a,c,\alpha,A,B)$, if it satisfies the following subordination condition:

$$\frac{1}{p-\alpha}\left\{\frac{z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)' + \beta z^2\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)''}{(1-\beta)\mathcal{I}_p^\lambda(a,c)f(z) + \beta z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)'} - \alpha\right\} \prec \frac{1+Az}{1+Bz} \quad (a, c \in \mathbb{R}\setminus\mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U}). \tag{1.11}$$

A subclass of $\mathcal{A}_p$, more general than the class $\mathcal{V}_{p,\beta}^\lambda(a,c,\alpha,A,B)$ has been recently studied by Noor et al. [12]. For convenience, we denote

$$\begin{aligned}
&\mathcal{V}^1_{p,\beta}(p+1,1,\alpha,A,B)=\mathcal{V}^{\lambda}_{p,\beta}(\lambda+p,1,\alpha,A,B)=\mathcal{V}_{p,\beta}(\alpha;A,B)\\
&=\left\{f\in\mathcal{A}_p:\frac{zf'(z)+\beta z^2f''(z)}{(1-\beta)f(z)+\beta zf'(z)}\prec\frac{p+\{pA+(B-A)\alpha\}z}{1+Bz},z\in\mathbb{U}\right\},
\end{aligned}$$

where $0\leq\alpha<p,\lambda>-p;-1\leq B<A\leq 1$ and

$$\begin{aligned}
&\mathcal{V}_{p,\beta}(\alpha,1,-1)=\mathcal{V}_{p,\beta}(\alpha)\ (0\leq\alpha<p)\\
&\qquad=\left\{f\in\mathcal{A}_p:\operatorname{Re}\left(\frac{zf'(z)+\beta z^2f''(z)}{(1-\beta)f(z)+\beta zf'(z)}\right)>\alpha;z\in\mathbb{U}\right\}.
\end{aligned}$$

We observe that $\mathcal{V}_{p,0}(\alpha)=\mathcal{S}^*_p(\alpha),\mathcal{V}_{p,1}(\alpha)=\mathcal{C}_p(\alpha)$ $(0\leq\alpha<p)$ and

$$\mathcal{T}_{\beta}(p;\alpha)=\mathcal{V}^1_{p,\beta}(p+1,1,\alpha,1,-1)\ (0\leq\beta\leq 1;0\leq\alpha<p)$$

is the class studied in [8].

Remark 1.1. If, we write

$$\mathfrak{g}(z)=\frac{(1-\beta)f(z)+\beta zf'(z)}{(1-\beta+p\beta)}\quad(0\leq\beta\leq 1;z\in\mathbb{U}),$$

then it follows that

$$\mathcal{I}^{\lambda}_p(a,c)\mathfrak{g}(z)=\frac{(1-\beta)\mathcal{I}^{\lambda}_p(a,c)f(z)+\beta z\left(\mathcal{I}^{\lambda}_p(a,c)f(z)\right)'}{(1-\beta+p\beta)}\quad(z\in\mathbb{U})$$

so that

$$\begin{aligned}
&\frac{1}{p-\alpha}\left\{\frac{z(\mathcal{I}^{\lambda}_p(a,c)\mathfrak{g})'(z)}{\mathcal{I}^{\lambda}_p(a,c)\mathfrak{g}(z)}-\alpha\right\}\\
&\qquad=\frac{1}{p-\alpha}\left\{\frac{z\left(\mathcal{I}^{\lambda}_p(a,c)f(z)\right)'+\beta z^2\left(\mathcal{I}^{\lambda}_p(a,c)f(z)\right)''}{(1-\beta)\mathcal{I}^{\lambda}_p(a,c)f(z)+\beta z\left(\mathcal{I}^{\lambda}_p(a,c)f(z)\right)'}-\alpha\right\}\quad(z\in\mathbb{U}).
\end{aligned}$$

Thus, with the aid of (1.11), if the function $f\in\mathcal{V}^{\lambda}_{p,\beta}(a,c,\alpha,A,B)$, then $\mathfrak{g}\in\mathcal{V}^{\lambda}_{p,0}(a,c,\alpha,A,B)$.

In the present paper, by using the techniques of differential subordination, we establish some inclusion relationships involving the class $\mathcal{V}^{\lambda}_{p,\beta}(a,c,\alpha,A,B)$ and also obtain certain subordination results for certain classes of functions in $\mathcal{A}_p$ involving the operator $\mathcal{I}^{\lambda}_p(a,c)$. Some interesting corollaries are derived and the relevance of our work with the earlier known results are pointed out.

To establish our results, we shall need the following lemmas.

Lemma 1.1.(Miller and Mocanu [13],[15]) *If* $-1 \leq B < A \leq 1, \beta^* > 0$, *and the complex number* γ^* *is constrained by* $Re(\gamma^*) \geq -\{\beta^*(1-A)\}/(1-B)$, *then the following differential equation:*

$$q(z) + \frac{zq'(z)}{\beta^* q(z) + \gamma^*} = \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

has a univalent solution in $\mathbb{U}$, *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta^*+\gamma^*}(1+Bz)^{\beta^*(A-B)/B}}{\beta^* \displaystyle\int_0^z t^{\beta^*+\gamma^*-1}(1+Bt)^{\beta^*(A-B)/B}\,dt} - \dfrac{\gamma^*}{\beta^*}, & B \neq 0 \\ \dfrac{z^{\beta^*+\gamma^*}\exp(\beta^* Az)}{\beta^* \displaystyle\int_0^z t^{\beta^*+\gamma^*-1}\exp(\beta^* At)\,dt} - \dfrac{\gamma^*}{\beta^*}, & B = 0. \end{cases} \tag{1.12}$$

If the function ϕ, *given by*

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{1.13}$$

satisfies the following subordination relation:

$$\phi(z) + \frac{z\phi'(z)}{\beta^*\phi(z) + \gamma^*} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{1.14}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

and the function q *is the best dominant of* (1.14).

Lemma 1.2.(Wilken and Feng [25]) *Let* ν *be a positive measure on* $[0,1]$. *Let* $h(z,t)$ *be a complex-valued function defined on* $\mathbb{U} \times [0,1]$ *such that* $h(\cdot,t)$ *is analytic in* $\mathbb{U}$ *for each* $t \in [0,1]$ *and* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* $z \in \mathbb{U}$. *In addition, suppose that* $Re\{h(z,t)\} > 0, h(-r,t)$ *is real and*

$$Re\left(\frac{1}{h(z,t)}\right) \geq \frac{1}{h(-r,t)} \quad for \quad |z| \leq r < 1 \quad and \quad t \in [0,1].$$

If the function $\mathcal{H}$ is defined by $\mathcal{H}(z) = \int_0^1 h(z,t)d\nu(t)$ $(z \in \mathbb{U})$, then

$$\mathrm{Re}\left(\frac{1}{\mathcal{H}(z)}\right) \geq \frac{1}{\mathcal{H}(-r)}.$$

For real or complex numbers a_1, a_2, b_1 $(b_1 \notin \mathbb{Z}_0^-)$, the *Gauss Hypergeometric* function ${}_2F_1$ is defined by

$${}_2F_1(a_1, a_2; b_1; z) = 1 + \frac{a_1 a_2}{b_1}\frac{z}{1!} + \frac{a_1(a_1+1)a_2(a_2+1)}{b_1(b_1+1)}\frac{z^2}{2!} + \cdots .$$

We note that the function ${}_2F_1$ represented by the above series converges absolutely for $z \in \mathbb{U}$ and hence represents an analytic function in $\mathbb{U}$ (see, for details, [26, Chapter 14]).

The following identities (asserted by Lemma 1.3 are well-known (cf., e.g., [26, Chapter 14]).

Lemma 1.3. *For real or complex numbers a_1, a_2, b_1 ($b_1 \notin \mathbb{Z}_0^-$), we have*

$$\int_0^1 t^{a_2-1}(1-t)^{b_1-a_2-1}(1-tz)^{-a_1}\,dt = \frac{\Gamma(a_2)\Gamma(b_1-a_2)}{\Gamma(b_1)}\,{}_2F_1(a_1,a_2;b_1;z); \quad (Re(b_1) > Re(a_2) > 0) \tag{1.15}$$

$$ {}_2F_1(a_1,a_2;b_1;z) = (1-z)^{-a_1}\,{}_2F_1\left(a_1, b_1-a_2; b_1; \frac{z}{z-1}\right); \tag{1.16}$$

$$ {}_2F_1(a_1,a_2;b_1;z) = {}_2F_1(a_2,a_1;b_1;z) \tag{1.17}$$

$$ (a_2+1)\,{}_2F_1(1,a_2;a_2+1;z) = (a_2+1) + a_2 z\,{}_2F_1(1,a_2+1;a_2+2;z); \tag{1.18}$$

Lemma 1.4.([21]) *Let $\widetilde{\beta} \in \mathbb{C}$ and $\widetilde{\gamma} \in \mathbb{C}^*$. Let q be a convex univalent function in $\mathbb{U}$ such that*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\widetilde{\beta}}{\widetilde{\gamma}}\right)\right\}.$$

If ϕ is analytic in $\mathbb{U}$ with $\phi(0) = q(0)$ and

$$\widetilde{\beta}\phi(z) + \widetilde{\gamma}z\phi'(z) \prec \widetilde{\beta}q(z) + \widetilde{\gamma}zq'(z) \quad (z \in \mathbb{U}),$$

then

$$\phi(z) \prec q(z) \quad (z \in \mathbb{U}) \tag{1.19}$$

and the function q is the best dominant of (1.19).

Lemma 1.5.([14]) *Let q be univalent in $\mathbb{U}$, and let θ and ϕ be analytic in a domain Ω containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ for $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) Q is univalent starlike in $\mathbb{U}$,

(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U}).$

If g is analytic in $\mathbb{U}$ with $g(0) = q(0), g(\mathbb{U}) \subseteq \Omega$ and

$$\theta(g(z)) + zg'(z)\phi(g(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathbb{U}),$$

then

$$g(z) \prec q(z) \quad (z \in \mathbb{U}) \tag{1.20}$$

and the function q is the best dominant of (1.20).

2. Inclusion Relationships for the Function Class $\mathcal{V}_{p,\beta}^{\lambda}(a,c,\alpha,A,B)$

Unless otherwise mentioned, we assume throughout the sequel that

$$a>0, c>0, 0\le\alpha<p, \lambda>-p, 0\le\beta\le 1, -1\le B<A\le 1,$$

and the powers appearing in some expressions are principal ones.

Theorem 2.1. *If $f\in\mathcal{V}_{p,\beta}^{\lambda}(a,c,\alpha,A,B)$ and*

$$a(1-B)-(p-\alpha)(A-B)\ge 0, \tag{2.1}$$

then

$$\frac{1}{p-\alpha}\left\{\frac{z\left(\mathcal{J}_p^{\lambda}(a+1,c)f(z)\right)'+\beta z^2\left(\mathcal{J}_p^{\lambda}(a+1,c)f(z)\right)''}{(1-\beta)\mathcal{J}_p^{\lambda}(a+1,c)f(z)+\beta z\left(\mathcal{J}_p^{\lambda}(a+1,c)f(z)\right)'}-\alpha\right\} \tag{2.2}$$
$$\prec\frac{1}{p-\alpha}\left\{\frac{1}{\mathcal{Q}(z)}-(a+\alpha-p)\right\}=q(z)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}),$$

where

$$\mathcal{Q}(z)=\begin{cases}\displaystyle\int_0^z t^{a-1}\left(\frac{1+Bzt}{1+Bz}\right)^{(p-\alpha)(A-B)/B}dt, & B\neq 0\\ \displaystyle\int_0^z t^{a-1}\exp\left((p-\alpha)Az(t-1)\right)\,dt, & B=0,\end{cases} \tag{2.3}$$

and the function q is the best dominant of (2.2). *If, in addition to* (2.1),

$$A\le-\frac{(a+\alpha+1-p)B}{p-\alpha}\quad\textit{with}\quad -1\le B<0,$$

then

$$\mathcal{V}_{p,\beta}^{\lambda}(a,c,\alpha,A,B)\subset\mathcal{V}_{p,\beta}^{\lambda}(a+1,c,\alpha,1-2\kappa,-1), \tag{2.4}$$

where

$$\kappa=\frac{1}{p-\alpha}\left[a\left\{{}_2F_1\left(1,\frac{(p-\alpha)(B-A)}{B};a+1;\frac{B}{B-1}\right)\right\}^{-1}-(a+\alpha-p)\right].$$

The inclusion relationship in (2.4) *is the best possible.*

Proof. Let $f\in\mathcal{V}_{p,\beta}^{\lambda}(a,c,\alpha,A,B)$. Setting

$$g(z)=z\left(\frac{(1-\beta)\mathcal{J}_p^{\lambda}(a+1,c)f(z)+\beta z\left(\mathcal{J}_p^{\lambda}(a+1,c)f(z)\right)'}{(1-\beta+p\beta)z^p}\right)^{1/(p-\alpha)} \tag{2.5}$$

and $\widetilde{r} = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$, we see that g is single-valued and analytic in $|z| < \widetilde{r}$. Differentiating (2.5) logarithmically and using the identity (1.9) in the resulting equation, it follows that the function

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{J}_p^\lambda(a+1,c)f(z)\right)' + \beta z^2\left(\mathcal{J}_p^\lambda(a+1,c)f(z)\right)''}{(1-\beta)\mathcal{J}_p^\lambda(a+1,c)f(z) + \beta z\left(\mathcal{J}_p^\lambda(a+1,c)f(z)\right)'} - \alpha\right) \tag{2.6}$$

is of the form (1.13) and analytic in $|z| < \widetilde{r}$. Again, by using the identity (1.9) in (2.6) and carrying out the logarithmic differentiation in the resulting expression, we deduce that

$$\frac{1}{p-\alpha}\left\{\frac{z\left(\mathcal{J}_p^\lambda(a,c)f(z)\right)' + \beta z^2\left(\mathcal{J}_p^\lambda(a,c)f(z)\right)''}{(1-\beta)\mathcal{J}_p^\lambda(a,c)f(z) + \beta z\left(\mathcal{J}_p^\lambda(a,c)f(z)\right)'} - \alpha\right\} = \phi(z) + \frac{z\phi'(z)}{(p-\alpha)\phi(z) + (a+\alpha-p)} \prec \frac{1+Az}{1+Bz} \quad (|z| < \widetilde{r}). \tag{2.7}$$

Thus, an application of Lemma 1.1 yields

$$\phi(z) \prec \frac{1}{p-\alpha}\left\{\frac{1}{\mathcal{Q}(z)} - (a+\alpha-p)\right\} = q(z) \prec \frac{1+Az}{1+Bz} \quad (|z| < \widetilde{r}), \tag{2.8}$$

where q, given by (1.12) with $\beta^* = p-\alpha$ and $\gamma^* = a+\alpha-p$. is the best dominant of (2.2).

For $-1 \leq B < A \leq 1$, we note that $\operatorname{Re}\{(1+Az)/(1+Bz)\} > 0$, so that by (2.8), we get $\operatorname{Re}(\phi(z)) > 0$ in $|z| < \widetilde{r}$. Now, (2.6) shows that g is starlike in $|z| < \widetilde{r}$. Thus, it is not possible that g vanishes on $|z| = \widetilde{r}$, if $\widetilde{r} = 1$. So, we must have $\widetilde{r} = 1$ and the function ϕ becomes analytic in $\mathbb{U}$. Therefore, by (2.8)

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

provided p, α, a, A and B satisfy (2.1). This proves the assertion (2.2).

Next, we show that

$$\inf_{z\in\mathbb{U}}\{\operatorname{Re}(q(z))\} = q(-1). \tag{2.9}$$

Letting

$$a_1 = \frac{(p-\alpha)(B-A)}{B}, \quad a_2 = a, \quad \text{and} \quad b_1 = a+1,$$

we find that $b_1 > a_2 > 0$. From (2.3), by making use of (1.15),(1.16), (1.17) and (1.18), we obtain for $B \neq 0$,

$$\mathcal{Q}(z) = (1+Bz)^{a_1}\int_0^z t^{a_2-1}(1+Bzt)^{-a_1}\,dt = \frac{\Gamma(a_2)}{\Gamma(b_1)}\,{}_2F_1\left(1, a_1; b_1; \frac{Bz}{Bz+1}\right). \tag{2.10}$$

To prove (2.9), it suffices to show that

$$\mathrm{Re}\left(\frac{1}{\mathcal{Q}(z)}\right) \geq \frac{1}{\mathcal{Q}(-1)} \quad (z \in \mathbb{U}). \tag{2.11}$$

Since

$$A < -\frac{(a+\alpha+1-p)B}{p-\alpha} \quad \text{with} \quad -1 \leq B < 0,$$

we find that $b_1 > a_1 > 0$. Thus, by using (1.15), we deduce from (2.10) that

$$\mathcal{Q}(z) = \int_0^1 h(z,t)d\nu(t),$$

where

$$h(z,t) = \frac{1+Bz}{1+(1-t)Bz} \quad \text{and} \quad d\nu(t) = \frac{\Gamma(a_2)}{\Gamma(a_1)\Gamma(b_1)} t^{a_1-1}(1-t)^{b_1-a_1-1}dt \ (0 \leq t \leq 1)$$

which is a positive measure on $[0,1]$. Further, for $-1 \leq B < 0$, it may be noted that $\mathrm{Re}\{h(z,t)\} > 0, h(-r,t)$ is real for $0 < |z| \leq r < 1$ and $0 \leq t \leq 1$. Therefore, by Lemma 1.2, we obtain

$$\mathrm{Re}\left(\frac{1}{\mathcal{Q}(z)}\right) \geq \frac{1}{\mathcal{Q}(-r)} \quad (|z| \leq r < 1),$$

which upon letting $r \to 1^-$ yields (2.11).

For the case $A = -\{(a+\alpha+1-p)B\}/(p-\alpha)$, by taking

$$A \to \left(-\frac{(a+\alpha+1-p)B}{p-\alpha}\right)^+$$

and using (2.2), we get (2.4).

The inclusion relationship in (2.4) is the best possible as the function q is the best dominant of (2.2). This completes the proof of Theorem 2.1. □

Setting $a = p, c = \lambda = A = 1$ and $B = -1$ in Theorem 2.1, we get the following result which gives the corresponding work of Patel et al. [18, Corollary 1] for $\beta = 0$.

Corollary 2.1. *If $(p-1)/2 \leq \alpha < p$ and $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1 + \frac{(1+\beta)zf''(z) + \beta z^2 f'''(z)}{f'(z) + \beta z f''(z)}\right\} > \alpha \quad (z \in \mathbb{U}),$$

then $f \in \mathcal{V}_{p,\beta}(\varkappa)$, where

$$\varkappa = p\left\{{}_2F_1\left(1, 2(p-\alpha); p+1; \frac{1}{2}\right)\right\}^{-1} \tag{2.12}$$

and the result is the best possible.

For the choice $\beta = 1$, Corollary 2.1 yields

Corollary 2.2. *If* $(p-1)/2 \leq \alpha < p$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$Re\left\{1 + \frac{z\,(2f''(z) + zf'''(z))}{f'(z) + zf''(z)}\right\} > \alpha \quad (z \in \mathbb{U}),$$

then $f \in \mathcal{C}_p(\varkappa)$, *where* $\varkappa$ *is given by* (2.12) *and the result is the best possible.*

Theorem 2.1 with $a = \delta + p, c = 1$ and $\lambda = \delta$ yields the following results obtained by Patel et al. [18, Theorem 2(Part (ii))].

Corollary 2.3 *Suppose that* $0 \leq \alpha < p, -1 \leq B < 0$ *and*

$$\delta \geq \max\left\{-\frac{p(1-A) + (A-B)\alpha}{1-B}, -\frac{(p-\alpha)A}{B} - \alpha - 1\right\}.$$

If $f \in \mathcal{V}_{p,\beta}(\alpha; A, B)$, *then* $\mathcal{F}_{\delta,p}(f) \in \mathcal{V}_{p,\beta}(\varrho)$, *where*

$$\varrho = (\delta + p)\left\{{}_2F_1\left(1, \frac{(p-\alpha)(B-A)}{B}; \delta + p + 1; \frac{B}{B-1}\right)\right\}^{-1} - \delta$$

and $\mathcal{F}_{\delta,p}$ *is given by* (1.10). *The result is the best possible.*

Corollary 2.3 with $\beta = 0$ (or $\beta = 1$, respectively) yields the following results obtained by Patel et al. [18, Remark 2].

Corollary 2.4. *For* $0 \leq \alpha < p$ *and* $\delta \geq \max\{-\alpha, p - 2\alpha - 1\}$, *we have*

$$\mathcal{F}_{\delta,p}\left(\mathcal{S}_p^*(\alpha)\right) \subset \mathcal{S}_p^*(\xi) \qquad \textit{and} \qquad \mathcal{F}_{\delta,p}\left(\mathcal{C}_p(\alpha)\right) \subset \mathcal{C}_p(\xi),$$

where

$$\xi = (\delta + p)\left\{{}_2F_1\left(1, 2(p-\alpha); \delta + p + 1; \frac{1}{2}\right)\right\}^{-1} - \delta$$

and $\mathcal{F}_{\delta,p}$ *is given by* (1.10). *The results are the best possible.*

Remark 2.1. (i) Theorem 2.1 improves the result due to Noor et al. [12, Theorem 1] for the function $\chi(z) = (1 + Az)/(1 + Bz),\ z \in \mathbb{U}$.

(ii) Substituting $a = p - \mu, c = p + 1$ and $\lambda = 1 - \mu$ $(-\infty < \mu < p+1)$ in Theorem 2.1, we obtained the corresponding work of Patel and Mishra [17, Theorem 1].

3. Subordination Results

In this section, we derive certain results for functions in $\mathcal{A}_p$ involving the operator $\mathcal{I}_p(a,c)$.

Theorem 3.1. *Let* $\gamma, \mu \in \mathbb{C}^*$ *and* q *be univalent in* $\mathbb{U}$ *such that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -aRe\left(\frac{\gamma}{\mu}\right)\right\}. \tag{3.1}$$

If $f \in \mathcal{A}_p$ satisfies the subordination relation:

$$(3.2) \qquad (1-\mu)\left(\frac{\mathcal{J}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma + \mu\left(\frac{\mathcal{J}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma \frac{\mathcal{J}_p^\lambda(a,c)f(z)}{\mathcal{J}_p^\lambda(a+1,c)f(z)} \prec q(z) + \frac{\mu}{a\gamma}zq'(z) \quad (z \in \mathbb{U}),$$

then

$$(3.3) \qquad \left(\frac{\mathcal{J}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma \prec q(z) \quad (z \in \mathbb{U})$$

and the function q is the best dominant of (3.3).

Proof. Letting

$$(3.4) \qquad h(z) = \left(\frac{\mathcal{J}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma \quad (z \in \mathbb{U}),$$

differentiating (3.4) logarithmically and using the identity (1.9) in the resulting expression, we get

$$h(z) + \frac{zh'(z)}{a\gamma} = \frac{\mathcal{J}_p^\lambda(a,c)f(z)}{\mathcal{J}_p^\lambda(a+1,c)f(z)} \quad (z \in \mathbb{U})$$

so that by using (3.4) again, the above equation yields

$$(3.5) \qquad h(z) + \frac{\mu}{a\gamma}zh'(z) \prec q(z) + \frac{\mu}{a\gamma}zq'(z) \quad (z \in \mathbb{U}).$$

Thus, by applying Lemma 1.4 to the subordination condition (3.5) with $\widetilde{\beta} = 1$ and $\widetilde{\gamma} = \mu/a\gamma$, we get the the desired assertion (3.3). □

Remark 3.1. If, we let $q(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1; z \in \mathbb{U})$ in Theorem 3.1, the condition (3.1) reduces to

$$(3.6) \qquad \mathrm{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -a\mathrm{Re}\left(\frac{\gamma}{\mu}\right)\right\} \quad (z \in \mathbb{U}).$$

It is easy to verify that the function $\psi(z) = (1-Bz)/(1+Bz)$ $(z \in \mathbb{U})$ is convex(univalent) in $\mathbb{U}$. Since $\psi(\overline{z}) = \overline{\psi(z)}$ for all $z \in \mathbb{U}$, the image of $\mathbb{U}$ under the function ψ is a convex domain and symmetrical with respect to the real axis. Thus,

$$\inf\left\{\mathrm{Re}\left(\frac{1-Bz}{1+Bz}\right) : z \in \mathbb{U}\right\} = \frac{1-|B|}{1+|B|} > 0$$

from which, it follows that the inequality (3.6) is equivalent to

$$a\mathrm{Re}\left(\frac{\gamma}{\mu}\right) \geq \frac{|B|-1}{|B|+1}.$$

Letting $q(z) = (1 + Az)/(1 + Bz)$ in Theorem 3.1 and using the Remark 3.1, we arrive at the following result.

Corollary 3.1. *Let* $-1 \leq B < A \leq 1$ *and*

$$\frac{1-|B|}{1+|B|} \geq \max\left\{0, -aRe\left(\frac{\gamma}{\mu}\right)\right\}.$$

If $f \in \mathcal{A}_p$ *satisfies*

$$(1-\mu)\left(\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma + \mu\left(\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma \frac{\mathcal{I}_p^\lambda(a,c)f(z)}{\mathcal{I}_p^\lambda(a+1,c)f(z)}$$
$$\prec \frac{1+Az}{1+Bz} + \frac{\mu}{a\gamma}\frac{(A-B)z}{(1+Bz)^2} \quad (z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^p}\right)^\gamma \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

and the function $(1 + Az)/(1 + Bz)$ *is the best dominant.*

For the choices $a = p, c = \lambda = \gamma = 1, A = 1 - 2\rho$ and $B = -1$, Corollary 3.1 yields the following result.

Corollary 3.2. *Let* $\mu \in \mathbb{C}^*$ *be such that* $Re(1/\mu) \geq 0$. *If* $f \in \mathcal{A}_p$ *satisfies the following subordination relation:*

$$(1-\mu)\frac{f(z)}{z^p} + \frac{\mu}{p}\frac{f'(z)}{z^{p-1}} \prec \frac{1+(1-2\rho)z}{1-z} + \frac{2(1-\rho)\mu\, z}{p(1-z)^2} \quad (0 \leq \rho < 1; z \in \mathbb{U}),$$

then

$$\frac{f(z)}{z^p} \prec \widetilde{q}(z) \quad (z \in \mathbb{U}),$$

where $\widetilde{q}$, *defined by*

$$\widetilde{q}(z) = \frac{1+(1-2\rho)z}{1-z} \quad (0 \leq \rho < 1; z \in \mathbb{U}) \tag{3.7}$$

is the best dominant.

Taking $a = \delta + p, c = \gamma = 1, \lambda = \delta, A = 1 - 2\rho$ and $B = -1$ in Corollary 3.1, we obtain

Corollary 3.3. *Let* $\mu \in \mathbb{C}^*$ *be such that* $Re(1/\mu) \geq 0$. *If* $f \in \mathcal{A}_p$ *satisfies the following subordination relation:*

$$(1-\mu)\frac{\mathcal{F}_{\delta,p}(f)(z)}{z^p} + \mu\frac{f(z)}{z^p}$$
$$\prec \frac{1+(1-2\rho)z}{1-z} + \frac{2(1-\rho)\mu\, z}{(\delta+p)(1-z)^2} \quad (0 \leq \rho < 1, \delta > -p; z \in \mathbb{U})$$
$$= \mathfrak{g}(z) \ (say),$$

then

$$\frac{\mathcal{F}_{\delta,p}(f)(z)}{z^p} \prec \widetilde{q}(z) \quad (z \in \mathbb{U}),$$

where $\mathcal{F}_{\delta,p}$ *is given by* (1.10) *and the function* $\widetilde{q}$, *defined by* (3.7) *is the best dominant.*

Remark 3.2. We now investigate the image of the unit disk $\mathbb{U}$ under the function

$$\mathfrak{g}(z) = \frac{1+(1-2\rho)z}{1-z} + \frac{2(1-\rho)\mu\, z}{(\delta+p)(1-z)^2} \quad (0 \leq \rho < 1, \delta > -p; z \in \mathbb{U}).$$

We note that $g \in \mathcal{A}, \mathfrak{g}(0) = 1$ and $\mathfrak{g}(-1) = \rho - \{\mu(1-\rho)\}/\{2(\delta+p)\}$. The boundary curve of the image $\mathfrak{g}(\mathbb{U})$ is given by

$$\mathfrak{g}(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta}), -\pi < \theta < \pi,$$

where

$$u(e^{i\theta}) = \rho + \frac{\mu(1-\rho)}{(\delta+p)(\cos\theta - 1)} \quad \text{and} \quad v(e^{i\theta}) = -\frac{(1-\rho)\sin\theta}{\cos\theta - 1}.$$

Eliminating θ from the above equations, we get

$$v^2 = -\frac{2(p+\delta)(1-\rho)}{\mu}\left\{u - \left(\rho - \frac{\mu(1-\rho)}{2(\delta+p)}\right)\right\}, \tag{3.8}$$

which represents a parabola opening towards the left with vertex at the point $\left(\rho - \frac{\mu(1-\rho)}{2(\delta+p)}, 0\right)$ and negative real axis as its axis. Thus, $\mathfrak{g}(\mathbb{U})$ is the exterior of the parabola given by (3.8) and it includes the right half plane

$$u > \rho - \frac{\mu(1-\rho)}{2(\delta+p)}.$$

Theorem 3.2. *Let* $\mu \in \mathbb{C}^*$ *and* $\gamma \in \mathbb{C}$. *Let* q *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$, $q(z) \neq 0$ *in* $\mathbb{U}$ *and satisfies*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.9}$$

If $f \in \mathcal{A}_p$ *satisfies*

$$\frac{(1-\beta)\mathcal{I}_p^\lambda(a,c)f(z) + \beta z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)'}{(1-\beta+p\beta)z^p} \neq 0 \quad (z \in \mathbb{U}) \tag{3.10}$$

and

$$\begin{aligned} 1 + \mu\gamma &\left\{\frac{z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)' + \beta z^2\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)''}{(1-\beta)\mathcal{I}_p^\lambda(a,c)f(z) + \beta z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)'} - p\right\} \\ &\prec 1 + \gamma\frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}), \end{aligned} \tag{3.11}$$

then

$$\left\{\frac{(1-\beta)\mathcal{I}_p^{\lambda}(a,c)f(z)+\beta z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'}{(1-\beta+p\beta)z^p}\right\}^{\mu}\prec q(z)\quad(z\in\mathbb{U}) \tag{3.12}$$

and the function q is the best dominant of (3.12).

Proof. Consider the function h defined by

$$h(z)=\left\{\frac{(1-\beta)\mathcal{I}_p^{\lambda}(a,c)f(z)+\beta z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'}{(1-\beta+p\beta)z^p}\right\}^{\mu}\quad(z\in\mathbb{U}). \tag{3.13}$$

In view of (3.10), the function h is analytic in $\mathbb{U}$ and $h(0)=1$. Differentiating both the sides of (3.13) logarithmically followed by the use of the identity (1.9) in the resulting expression, we get

$$\frac{zh'(z)}{h(z)}=\mu\left\{\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'+\beta z^2\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)''}{(1-\beta)\mathcal{I}_p^{\lambda}(a,c)f(z)+\beta z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'}-p\right\}\quad(z\in\mathbb{U}). \tag{3.14}$$

By setting

$$\theta(w)=1\ (w\in\mathbb{C})\quad\text{and}\quad\phi(w)=\frac{\gamma}{w}\ (w\in\mathbb{C}^*),$$

it is easily observed that θ is analytic in $\mathbb{C}$ and $\phi(w)\neq 0$ in $\mathbb{C}^*$. Further, if we let

$$Q(z)=zq'(z)\phi(q(z))=\gamma\frac{zq'(z)}{q(z)}\quad\text{and}\quad g(z)=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)},$$

then by (3.9), the function Q is univalent starlike in $\mathbb{U}$. Also, by (3.9)

$$\operatorname{Re}\left\{\frac{zg'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0\quad(z\in\mathbb{U}).$$

Using (3.14) in (3.11), we get

$$1+\gamma\frac{zh'(z)}{h(z)}\prec 1+\gamma\frac{zq'(z)}{q(z)}\quad(z\in\mathbb{U}),$$

which is equivalent to

$$\theta(h(z))+zh'(z)\phi(h(z))\prec\theta(q(z))+zq'(z)\phi(q(z))\quad(z\in\mathbb{U}).$$

Thus, by making use of Lemma 1.5, we have

$$h(z)\prec q(z)\quad(z\in\mathbb{U})$$

and the function q is the best dominant. This completes the proof of Theorem 3.2. □

Letting $a = p+1, c = \lambda = \gamma = 1$ and $q(z) = (1+Az)/(1+Bz)$ in Theorem 3.2, we obtain the following result.

Corollary 3.4. *Let $\mu \in \mathbb{C}^*$. If $f \in \mathcal{A}_p$ satisfies $\{(1-\beta)f(z)+\beta z f'(z)\}/z^p \neq 0$ in $\mathbb{U}$ and*

$$1+\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-p\right\} \prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)} \quad (z\in\mathbb{U}),$$

then

$$\left\{\frac{(1-\beta)f(z)+\beta z f'(z)}{z^p}\right\}^{\mu} \prec (1-\beta+p\beta)^{\mu}\,\frac{1+Az}{1+Bz} \quad (z\in\mathbb{U})$$

and the function $(1+Az)/(1+Bz)$ is the best dominant.

Using the fact that (cf., e.g., [19]) the function

$$q(z) = (1-z)^{-2(1-\rho)\mu\, b}\ (b,\mu\in\mathbb{C}^*, 0\le\rho<1; z\in\mathbb{U})$$

is univalent in $\mathbb{U}$, if and only if either $|2(1-\rho)\mu b-1|\le 1$ or $|2(1-\rho)\mu b+1|\le 1$, taking q defined as above and putting $a = p+1, c=\lambda=1, \gamma = 1/\mu\, b$ in Theorem 3.2, we get the following result which reduces to the corresponding work of Aouf et al. [1, Corollary 6] for $\beta=\rho=0$.

Corollary 3.5. *Let $\mu, b\in\mathbb{C}^*$ be such that either*

$$|2(1-\rho)\mu b-1|\le 1 \quad \textit{or} \quad |2(1-\rho)\mu b+1|\le 1\ (0\le\rho<1).$$

If $f\in\mathcal{A}_p$ satisfies $\{(1-\beta)f(z)+\beta z f'(z)\}/z^p\neq 0$ in $\mathbb{U}$ and

$$1+\frac{1}{b}\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-p\right\} \prec \frac{1+(1-2\rho)z}{1-z} \quad (0\le\rho<1; z\in\mathbb{U}),$$

then

$$\left\{\frac{(1-\beta)f(z)+\beta z f'(z)}{z^p}\right\}^{\mu} \prec \frac{(1-\beta+p\beta)^{\mu}}{(1-z)^{2\mu b(1-\rho)}} \quad (z\in\mathbb{U})$$

and the function $1/(1-z)^{2\mu b(1-\rho)}$ is the best dominant.

Putting $\beta=0$ (or $\beta=1$, respectively) in Corollary 3.5, we get the following result obtained by Srivastava and Lashin [23] (also see [7, Corollary 3.8]) for $\mu = p = 1$ and $\rho=0$.

Corollary 3.6. *Let $\mu, b\in\mathbb{C}^*$ be such that either*

$$|2(1-\rho)\mu b-1|\le 1 \quad \textit{or} \quad |2(1-\rho)\mu b+1|\le 1.$$

Then

$$f\in\mathcal{S}_p^*(b;\rho) \Longrightarrow \left\{\frac{f(z)}{z^p}\right\}^{\mu} \prec \frac{1}{(1-z)^{2\mu b(1-\rho)}} \quad (0\le\rho<1; z\in\mathbb{U}),$$

and

$$f \in \mathcal{C}_p(b;\rho) \Longrightarrow \left\{\frac{f'(z)}{z^{p-1}}\right\}^{\mu} \prec \frac{p^{\mu}}{(1-z)^{2\mu b(1-\rho)}} \quad (0 \leq \rho < 1; z \in \mathbb{U})$$

and the function $1/(1-z)^{2\mu b(1-\rho)}$ *is the best dominant for both the results.*

For $a = p+1, c = \lambda = \gamma = 1$ and $q(z) = (1+Bz)^{\mu(A-B)/B}$ $(\mu \in \mathbb{C}^*, B \neq 0)$ in Theorem 3.2, we obtain the following result which, in turn yields the work of Aouf et al. [1, Corollary 7] for $\beta = 0$.

Corollary 3.7. *Let* $\mu \in \mathbb{C}^*$ *and* $B \neq 0$ *be such that either*

$$\left|\frac{\mu(A-B)}{B} - 1\right| \leq 1 \qquad or \qquad \left|\frac{\mu(A-B)}{B} + 1\right| \leq 1.$$

If $f \in \mathcal{A}_p$ *satisfies* $\{(1-\beta)f(z) + \beta z f'(z)\}/z^p \neq 0$ *in* $\mathbb{U}$ *and*

$$1 + \mu\left\{\frac{zf'(z) + \beta z^2 f''(z)}{(1-\beta)f(z) + \beta z f'(z)} - p\right\} \prec \frac{1 + \{B + \mu(A-B)\}z}{1+Bz} \quad (z \in \mathbb{U}),$$

then

$$\left\{\frac{(1-\beta)f(z) + \beta z f'(z)}{z^p}\right\}^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}} \quad (z \in \mathbb{U})$$

and the function $(1+Bz)^{\mu(A-B)/B}$ *is the best dominant.*

Theorem 3.3. *Let* $\mu \in \mathbb{C}^*$ *and* $\eta \in \mathbb{C}$. *Let* q *be a univalent function in* $\mathbb{U}$ *with* $q(0) = 1$ *and*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -Re(\eta)\} \quad (z \in \mathbb{U}). \tag{3.15}$$

If $f \in \mathcal{A}_p$ *satisfies* (3.10) *and*

$$\Psi(z) \prec \eta q(z) + \gamma\, z q'(z) \quad (z \in \mathbb{U}), \tag{3.16}$$

where

$$\begin{aligned}\Psi(z) = &\left\{\frac{(1-\beta)\mathcal{J}_p^{\lambda}(a,c)f(z) + \beta z(\mathcal{J}_p^{\lambda}(a,c)f(z))'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \times \\ &\left\{\eta + \mu\gamma\left(\frac{z\left(\mathcal{J}_p^{\lambda}(a,c)f(z)\right)' + \beta z^2\left(\mathcal{J}_p^{\lambda}(a,c)f(z)\right)''}{(1-\beta)\mathcal{J}_p^{\lambda}(a,c)f(z) + \beta z\left(\mathcal{J}_p^{\lambda}(a,c)f(z)\right)'} - p\right)\right\} \quad (z \in \mathbb{U}),\end{aligned} \tag{3.17}$$

then

$$\left\{\frac{(1-\beta)\mathcal{J}_p^{\lambda}(a,c)f(z) + \beta z(\mathcal{J}_p^{\lambda}(a,c)f(z))'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \prec q(z) \quad (z \in \mathbb{U}) \tag{3.18}$$

and the function q *is the best dominant of* (3.18).

Proof. The proof of this theorem being much similar to that of Theorem 3.2, we give the main steps only. We consider the function h, given by (3.13). Then by (3.14), we have

$$zh'(z) = \mu h(z) \left\{ \frac{z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)' + \beta z^2 \left(\mathcal{I}_p^\lambda(a,c)f(z)\right)''}{(1-\beta)\mathcal{I}_p^\lambda(a,c)f(z) + \beta z \left(\mathcal{I}_p^\lambda(a,c)f(z)\right)'} - p \right\} \quad (z \in \mathbb{U}). \tag{3.19}$$

Taking $\theta(w) = \eta w, \phi(w) = \gamma \ (w \in \mathbb{C}), Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z)$ and

$$g(z) = \theta(q(z)) + Q(z) = \eta q(z) + \gamma zq'(z),$$

we find from (3.15) that Q is univalent starlike in $\mathbb{U}$. Also, by the hypothesis (3.15)

$$\mathrm{Re}\left\{\frac{zg'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\eta + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Furthermore, by substituting the expression for h from (3.13) and $zh'(z)$ from (3.19), we have

$$\theta(h(z)) + zh'(z)\phi(h(z)) = \eta h(z) + \gamma zh'(z) = \Psi(z) \quad (z \in \mathbb{U}).$$

Thus, the hypothesis (3.16) reduces to

$$\theta(h(z)) + zh'(z)\phi(h(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathbb{U})$$

and an application of Lemma 1.5 gives the required assertion, This completes the proof of Theorem 3.3. □

Noting that

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} = \mathrm{Re}\left\{\frac{1-Bz}{1+Bz}\right\} > \frac{1-|B|}{1+|B|} \quad (z \in \mathbb{U})$$

and taking $\beta = 0, \gamma = 1, q(z) = (1+Az)/(1+Bz)$ in Theorem 3.3, we have

Corollary 3.8. *Let $\mu \in \mathbb{C}^*$ and $\eta = (|B|-1)/(|B|+1)$. If $f \in \mathcal{A}_p$ satisfies $\mathcal{I}_p^\lambda(a,c)f(z)/z^p \neq 0$ in $\mathbb{U}$ and*

$$\left\{\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right\}^\mu \left\{\eta + \mu\left(\frac{z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)'}{\mathcal{I}_p^\lambda(a,c)f(z)} - p\right)\right\} \prec \frac{\eta(1+Az)}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2} \quad (z \in \mathbb{U}),$$

then

$$\left\{\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right\}^\mu \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

and the function $(1+Az)/(1+Bz)$ *is the best dominant.*

In the special case when $\lambda = c = 1, A = 1-2\rho, B = -1$ (so that $\eta = 0$) and $a = p+1$ (or $a = p$, respectively), Corollary 3.8 gives the following result.

Corollary 3.9. *Let* $\mu \in \mathbb{C}^*, 0 \leq \rho < 1$ *and* $f \in \mathcal{A}_p$. *If*

(i) $\left\{\frac{f(z)}{z^p}\right\}^\mu \left\{\frac{zf'(z)}{f(z)} - p\right\} \prec \frac{2(1-\rho)z}{(1-z)^2} \quad (z \in \mathbb{U})$, *then*

$$\left\{\frac{f(z)}{z^p}\right\}^\mu \prec \frac{1+(1-2\rho)z}{1-z} \quad (z \in \mathbb{U})$$

and

(ii) $\left\{\frac{f'(z)}{pz^{p-1}}\right\}^\mu \left\{\frac{zf''(z)}{f'(z)} - (p-1)\right\} \prec \frac{2(1-\rho)z}{(1-z)^2} \quad (z \in \mathbb{U})$, *then*

$$\left\{\frac{f'(z)}{pz^{p-1}}\right\}^\mu \prec \frac{1+(1-2\rho)z}{1-z} \quad (z \in \mathbb{U}).$$

The function $\{1+(1-2\rho)z\}/(1-z)$ *is the best dominant for both the results.*

Remark 3.3. The results of Section 3 can be extended for the special cases of the operator $\mathfrak{I}_p^\lambda(a,c)$(as discussed in the introduction) by suitably choosing the parameters involved.

References

[1] M. K. Aouf, A. Shamandy, A. O. Mostafa and E. A. Adwan, *Differential sandwich theorems for multivalent analytic functions defined by the Srivastava-Attiya operator*, Bull. Math. Anal. Appl., **3(3)**(2011), 227-238.

[2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **159**(2004), 737-745.

[3] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl., **292**(2004), 470-483.

[4] J. H. Choi, M. Saigo and H. M. Srivastava, *Some inclusion properties of acertain family of integral operators*, J. Math. Anal. Appl., **276**(2002), 432-445.

[5] B. A. Frasin, *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi.(N.S.), **22**(2006), 179-191.

[6] R. M. Goel and N. S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78 (1980), 353-357.

[7] S. P. Goyal, P. Goswami and H. Silverman, *Subordination and superordination results for a class of analytic multivalent functions*, Internat. J. Math. Math. Sci., Vol. **2008**, Art. ID **561638**, DOI: **10:1155/2008/561638**, 1-12.

[8] H. Imrak and R. K. Raina, *The starlikeness and convexity of multivalent functions involving certain inequalities*, Rev. Mat. Complut., **16(2)**(2003), 391-398.

[9] J.-L. Liu and K. I. Noor, *Some properties of Noor integral operator*, J. Natur. Math., **21**(2002), 81-90.

[10] K. I. Noor, *Some classes of p-valent analytic functions defined by certain integral operator*, Appl. Math. Comput., **157**(2004), 835-840.

[11] K. I. Noor and M. A. Noor, *On certain classes of analytic functions defined by Noor integral operator*, J. Math. Anal. Appl., **281**(2003), 244-252.

[12] K. I. Noor, S. Z. H. Bukhari, M. Arif and M. Nazir, *Some properties of p-valent analytic functions involving Cho-Kwon-Srivastava integral operator*, J. Classical Anal., **3(1)**(2013), 35-43.

[13] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[14] S. S. Miller and P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential subordinations*, J. Differential Equations, **58**(1985), 297-309.

[15] S. S. Miller and P. T. Mocanu, *Differential Subordinations:Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. **225**, Marcel Dekker Inc., New York and Basel, 2000.

[16] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39(5)**(1987), 1057-1077.

[17] J. Patel and A. K. Mishra, *On certain subclasses of multivalent functions associated with an extended fractional differintegral operator*, J. Math. Anal. Appl., **332(1)**(2007), 109-122.

[18] J. Patel, N. E. Cho and H. M. Srivastava, *Certain subclasses of multivalent functions associated with a family of linear operators*, Math. Comput. Modelling, **43**(2006), 320-338.

[19] W. C. Royster, *On the univalence of certain integral*, Michigan Math. J., **12**(1965), 385-387.

[20] H. Saitoh, *A linear operator and its application of first order differential subordinations*, Math. Japon., **44**(1996), 31-38.

[21] T. N. Shanmugam, S. Sivasubramanian and H. M. Srivastava, *Differential sandwich theorems for certain subclasses of analytic functions involing multiplier transformations*, Integral Transforms Spec. Funct., **17(12)**(2006), 889-899.

[22] H. M. Srivastava and M. K. Aouf, *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients I*, J. Math. Anal. Appl., **171(1)**(1992), 1-13.

[23] H. M. Srivastava and A. Y. Lashin, *Some applications of Briot-Bouquet differential subordination*, J. Ineq. Pure Appl. Math., **6(2)**(2005), 1-7.

[24] H. M. Srivastava and A. K. Mishra, *A fractional differintegral operator and its applications to a nested class of multivalent functions with negative coefficients*, Adv. Stud. Contemp. Math. (Kyungshang), **7(2)**(2003), 203-214.

[25] D. R. Wilken and J. Feng, *A remark on convex and starlike functions*, J. London Math. Soc., **21(2)**(1980), 287-290.

[26] E. T. Whittaker and G. N. Watson, *A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processess and of Analytic Functions: With an Account of the Principal Transcendental Functions*, Fourth Edn., Cambridge University Press, 1927 (Reprinted).