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General Theorem for Explicit Evaluations and Reciprocity Theorems for Ramanujan-Göllnitz-Gordon Continued Fraction

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Abstract. In the paper *A new parameter for Ramanujan's theta-functions and explicit values, Arab J. Math. Sc.,* **18** *(2012), 105-119*, Saikia studied the parameter $A_{k,n}$ involving Ramanujan's theta-functions $\phi(q)$ and $\psi(q)$ for any positive real numbers k and n and applied it to find explicit values of $\psi(q)$. As more application to the parameter $A_{k,n}$, in this paper we prove a new general theorem for explicit evaluation of Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ in terms of the parameter $A_{k,n}$ and give examples. We also find some new explicit values of the parameter $A_{k,n}$ and offer reciprocity theorems for the continued fraction $K(q)$.

1. Introduction

For $q := e^{2\pi iz}$, $\mathrm{Im}(z) > 0$, define Ramanujan's theta-functions $\phi(q)$, $\psi(q)$, and $f(-q)$ as

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} = \vartheta_3(0,2z)$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = 2^{-1}q^{-1/8}\vartheta_2(0,z),$$

where ϑ_2 and ϑ_3 [11, p.464] are classical theta-functions and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

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For any positive real numbers k and n, Saikia [7, p.107, (1.7)] defined the parameter $A_{k,n}$ as

$$A_{k,n} = \frac{\phi(-q)}{2\ k^{1/4}q^{k/4}\psi(q^{2k})}, \qquad q = e^{-\pi\sqrt{n/k}} \tag{1.1}$$

and studied its several properties. Saikia [7] also evaluated many explicit values of $A_{k,n}$ and some of the explicit values of $A_{k,n}$ are used to find some particular values of the Ramanujan's theta-function $\psi(q)$.

As more application of the parameter $A_{k,n}$, in this paper we use the particular case $A_{2,n}$ of the parameter $A_{k,n}$ to prove a general theorem for the explicit evaluations of the Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ [6, p.299] defined by

$$K(q) := q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{q^{1/2}}{1+q}\,{}_{+}\,\frac{q^2}{1+q^3}\,{}_{+}\,\frac{q^4}{1+q^5}\,{}_{+\cdots}, \quad |q|<1 \tag{1.2}$$

We also evaluate some new explicit values of the parameter $A_{2,n}$ by proving two new theta-function identities. Previously, Baruah and Saikia [1] established some general theorems for explicit evaluations of $K(q)$ and evaluated some values. Chan and Huang [4] also proved general formulas for explicit evaluation of the continued fraction $K(q)$ in terms of Ramanujan's class invariants. For more results on the continued fraction $K(q)$ see [8] and [9].

The famous Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1}\,{}_{+}\,\frac{q}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q^3}{1}\,{}_{+\cdots}, \qquad |q|<1. \tag{1.3}$$

On page 204 of his second notebook [6], Ramanujan stated that if α and β are both positive and $\alpha\beta = 1$, then

$$\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\alpha})\right)\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\beta})\right) = \frac{5+\sqrt{5}}{2} \tag{1.4}$$

and

$$\left(\frac{\sqrt{5}-1}{2} - R(e^{-2\pi\alpha})\right)\left(\frac{\sqrt{5}-1}{2} - R(e^{-2\pi\beta})\right) = \frac{5-\sqrt{5}}{2}. \tag{1.5}$$

The reciprocity theorems (1.4) and (1.5) are proved by Watson [10]. Ramanthan [5] also proved some reciprocity theorems for $R(q)$ which are analogous to (1.4) and (1.5). Chan [3] proved some reciprocity theorems for the Ramanujan's cubic continued fraction. In this paper, we also prove two reciprocity theorems for the Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ akin to (1.4) and (1.5).

In Section 2, we record some preliminary results for ready references in this paper. In section 3, we prove two new theta-function identities. In section 4, we prove general theorem for explicit evaluation of $K(q)$ and find some new explicit values of the parameter $A_{2,n}$. Finally, in section 5, we prove a reciprocity theorem for the continued fraction $K(q)$.

Since modular equations are key in proving theta-function identities in section 2, we end this introduction by defining Ramanujan's modular equation from Berndt's book [2]. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.6). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. By denoting $z_r = \phi^2(q^r)$, where $q = exp(-\pi K'/K)$, $|q| < 1$, the multiplier m connecting α and β is defined by $m = z_1/z_n$.

2. Preliminary Results

Lemma 2.1.([7, p.111, Theorem 4.1]) *For all positive real numbers k and n, we have*

$$(i) A_{k,1} = 1 \quad and \quad (ii) A_{k,1/n} = 1/A_{k,n}.$$

Lemma 2.2.([6, p.299]) *We have*

$$\frac{1}{K(q)} - K(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{2.1}$$

and

$$\frac{1}{K(q)} + K(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)}. \tag{2.2}$$

Proofs of (2.1) and (2.2) can be found in Berndt's book [2, p.221].

Lemma 2.3.([2, p.43, Entry 27(ii)]) *If α and β are such that the modulus of each exponential argument is less than 1 and $\alpha\beta = \pi$, then*

$$2\sqrt{\alpha}\ \psi(e^{-2\alpha^2}) = \sqrt{\beta}\ e^{\alpha^2/4}\phi(-e^{-\beta^2}) \tag{2.3}$$

and

$$\sqrt{\alpha}\phi(e^{-\alpha^2}) = \sqrt{\beta}\phi(e^{-\beta^2}). \tag{2.4}$$

Lemma 2.4. *We have*

$$\phi(-q) = \sqrt{z_1}(1-\alpha)^{1/4}, \tag{2.5}$$

$$\psi(q^4) = \frac{\sqrt{z_1}\{1-\sqrt{1-\alpha}\ \}^{1/2}}{2\sqrt{2}\ q^{1/2}}, \tag{2.6}$$

For (2.5) see [2, p.122,Entry 10(ii)] and for (2.6) see [2, p.123, ntry 11(iv)].

We also note that if we replace q by q^n in the Lemma 2.4 then z_1 and α will be replaced by z_n and β, respectively, where β has degree n over α.

Lemma 2.5.([2, p.40, Entry 25(vi)]) *We have*

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2).$$

Lemma 2.6.([2, p.280, Entry 13(i)]) *If β has degree 5 over α, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1.$$

Lemma 2.7.([2, p.314, Entry 19(i)]) *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1.$$

3. Theta-Function Identities

The section is devoted to prove two new theta-function identities which will be used in section 4 to find new explicit values of the parameter $A_{2,n}$.

Theorem 2.8. *If* $P = \dfrac{\phi(-q)}{q^{1/2}\psi(q^4)}$ *and* $Q = \dfrac{\phi(-q^5)}{q^{5/4}\psi(q^{20})}$

then $$\left(\frac{P}{Q}\right)^3 - \left(\frac{Q}{P}\right)^3 + \left(\frac{32}{PQ}\right)^2 + (PQ)^2 + 320\left(\frac{1}{Q^2}+\frac{1}{P^2}\right) + 20\left(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right)$$

$$+5\left(\frac{P}{Q}-\frac{Q}{P}\right) + 10(P^2+Q^2) + 120 = 0. \tag{3.1}$$

Proof. Transcribing P and Q by using (2.5) and (2.6) and simplifying, we obtain

$$\sqrt{1-\alpha} = \frac{P^2}{8+P^2} \quad \text{and} \quad \sqrt{1-\beta} = \frac{Q^2}{8+Q^2}, \tag{3.2}$$

where β has degree 5 over α. Equivalently,

$$\alpha = 1 - \left(\frac{P^2}{8+P^2}\right)^2 \quad \text{and} \quad \beta = 1 - \left(\frac{Q^2}{8+Q^2}\right)^2. \tag{3.3}$$

From Lemma 2.6, we note that

$$2(16xy^2)^{1/6} = (1-y) - \sqrt{x}, \tag{3.4}$$

where

$$x := \alpha\beta \quad \text{and} \quad y := \{(1-\alpha)(1-\beta)\}^{1/2}. \tag{3.5}$$

Squaring (3.4) and simplifying, we obtain

$$42(16xy^2)^{1/3} = s - 2r\sqrt{x}, \tag{3.6}$$

where

$$r = 1 - y \quad \text{and} \quad s = (1-y)^2 + x. \tag{3.7}$$

Cubing (3.6) and simplifying, we obtain

$$1024xy^2 - s^3 - 12sr^2x = -(6s^2r\sqrt{x} + 8r^3x^{3/2}). \tag{3.8}$$

Squaring (3.8) and simplifying, we obtain

$$(1024xy^2 - s^3 - 12r^2sx)^2 = 36s^4r^2x + 64r^6x^3 + 96s^2r^4x^2. \tag{3.9}$$

Combining (3.5), (3.7), and (3.9), employing (3.2) and (3.3), and then factorizing with the help of *Mathematika*, we find that

$$f(P,Q)\ g(P,Q)\ j(P,Q) = 0, \tag{3.10}$$

where

$$\begin{aligned} f(P,Q) = P^6 &- 1024PQ - 320P^3Q - 20P^5Q + 5P^4Q^2 - 320PQ^3 - 120P^3Q^3 - 10P^5Q^3 \\ &- 5P^2Q^4 - 20PQ^5 - 10P^3Q^5 - P^5Q^5 - Q^6, \end{aligned}$$

$$\begin{aligned} g(P,Q) = P^6 &+ 1024PQ + 320P^3Q + 20P^5Q + 5P^4Q^2 + 320PQ^3 + 120P^3Q^3 + 10P^5Q^3 \\ &- 5P^2Q^4 + 20PQ^5 + 10P^3Q^5 + P^5Q^5 - Q^6, \end{aligned}$$

and

$$\begin{aligned} j(P,Q) = P^{12} &+ 1048576P^2Q^2 + 655360P^4Q^2 + 143360P^6Q^2 + 12800P^8Q^2 + 378P^{10}Q^2 \\ &+ 655360P^2Q^4 + 319488P^4Q^4 + 44544P^6Q^4 + 975P^8Q^4 - 112P^{10}Q^4 \\ &+ 143360P^2Q^6 + 44544P^4Q^6 - 2708P^6Q^6 - 1680P^8Q^6 - 116P^{10}Q^6 \\ &+ 12800P^2Q^8 + 975P^4Q^8 - 1680P^6Q^8 - 360P^8Q^8 - 20P^{10}Q^8 \\ &+ 378P^2Q^{10} - 112P^4Q^{10} - 116P^6Q^{10} - 20P^8Q^{10} - P^{10}Q^{10} + Q^{12}. \end{aligned}$$

By examining the behavior of the first factors $f(P,Q)$ and the last factor $j(P,Q)$ in the left hand side of (3.10) near $q = 0$, it can be seen that there is a neighborhood about the origin, where these factors are not zero. Then the second factor $g(P,Q)$

is zero in this neighborhood. By the identity theorem $g(P,Q)$ is identically zero. Thus, we have

$$g(P,Q) = P^6+1024PQ+320P^3Q+20P^5Q+5P^4Q^2+320PQ^3+120P^3Q^3+10P^5Q^3$$

$$-5P^2Q^4+20PQ^5+10P^3Q^5+P^5Q^5-Q^6=0. \tag{3.11}$$

Dividing (3.11) by P^3Q^3 and rearranging the terms, we arrive at the desired result.□

Theorem 2.9. *If* $P=\dfrac{\phi(-q)}{q^{1/2}\psi(q^4)}$ *and* $Q=\dfrac{\phi(-q^7)}{q^{7/4}\psi(q^{28})}$ *then*

$$P^8-32768PQ-14336P^3Q-1792P^5Q-56P^7Q+7168P^2Q^2+2688P^4Q^2+252P^6Q^2$$

$$-14336PQ^3-7168P^3Q^3-1064P^5Q^3-56P^7Q^3+2688P^2Q^4+1078P^4Q^4+84P^6Q^4$$

$$-1792PQ^5-1064P^3Q^5-224P^5Q^5-14P^7Q^5+252P^2Q^6+84P^4Q^6+7P^6Q^6$$

$$-56PQ^7-56P^3Q^7-14P^5Q^7-P^7Q^7+Q^8=0. \tag{3.12}$$

Proof. Transcribing P and Q by using (2.5) and (2.6) and simplifying, we obtain

$$\sqrt{1-\alpha}=\frac{P^2}{8+P^2} \quad \text{and} \quad \sqrt{1-\beta}=\frac{Q^2}{8+Q^2}, \tag{3.13}$$

where β has degree 7 over α. Equivalently,

$$\alpha=1-\left(\frac{P^2}{8+P^2}\right)^2 \quad \text{and} \quad \beta=1-\left(\frac{Q^2}{8+Q^2}\right)^2. \tag{3.14}$$

From Lemma 2.7, we note that

$$y^{1/4}=1-x^{1/8}, \tag{3.15}$$

where

$$x:=\alpha\beta \quad \text{and} \quad y:=\{(1-\alpha)(1-\beta)\}^{1/2}. \tag{3.16}$$

Squaring (3.15) and simplifying, we obtain

$$z-x^{1/4}=-2x^{1/8}, \tag{3.17}$$

where

$$z=\sqrt{y}-1. \tag{3.18}$$

Squaring (3.17) and simplifying, we obtain

$$z^2+\sqrt{x}=(4+2z)x^{1/4}. \tag{3.19}$$

Squaring (3.19) and simplifying, we deduce that

$$z^4+x=(16+2z^2+16z)\sqrt{x}. \tag{3.20}$$

From (3.18), we deduce that

$$z^2=y-1-2(\sqrt{y}-1) \quad \text{and} \quad z^4=(y+1)^2+4y-4(y+1)\sqrt{y}. \tag{3.21}$$

Employing (3.19) and (3.21) in (3.20) and simplifying, we obtain

$$k-4(y+1)\sqrt{y}=(s+12\sqrt{y})\sqrt{x}, \tag{3.22}$$

where

$$k=y^2+6y+x+1 \quad \text{and} \quad s=2+2y. \tag{3.23}$$

Squaring (3.22) and rearranging the terms, we arrive at

$$k^2+16(y+1)^2y-(s^2+144y)x=(24sx+8k(y+1))\sqrt{y}. \tag{3.24}$$

Squaring (3.24), we obtain

$$(k^2+16(y+1)^2y-(s^2+144y)x)^2=(24sx+8k(y+1))^2y. \tag{3.25}$$

Combining (3.23) and (3.25), employing (3.13) and (3.14), and then factorizing with the help of *Mathematika*, we find that

$$f(P,Q)\ g(P,Q)=0, \tag{3.26}$$

where

$$f(P,Q)=P^8+32768PQ+14336P^3Q+1792P^5Q+56P^7Q+7168P^2Q^2+2688P^4Q^2$$
$$+252P^6Q^2+14336PQ^3+7168P^3Q^3+1064P^5Q^3+56P^7Q^3+2688P^2Q^4+1078P^4Q^4$$
$$+84P^6Q^4+1792PQ^5+1064P^3Q^5+224P^5Q^5+14\ P^7Q^5+252P^2Q^6+84P^4Q^6+7P^6Q^6$$
$$+56PQ^7+56P^3Q^7+14P^5Q^7+P^7Q^7+Q^8$$

and

$$g(P,Q)=P^8-32768PQ-14336P^3Q-1792P^5Q-56P^7Q+7168P^2Q^2+2688P^4Q^2$$
$$+252P^6Q^2-14336PQ^3-7168P^3Q^3-1064P^5Q^3-56P^7Q^3+2688P^2Q^4+1078P^4Q^4$$
$$+84P^6Q^4-1792PQ^5-1064P^3Q^5-224P^5Q^5-14P^7Q^5+252P^2Q^6+84P^4Q^6+7P^6Q^6$$

$$-56PQ^7 - 56P^3Q^7 - 14P^5Q^7 - P^7Q^7 + Q^8.$$

By examining the behavior of the first factor $f(P,Q)$ of the left hand side of (3.26) near $q = 0$, it can be seen that there is a neighborhood about the origin, where $f(P,Q)$ factors are not zero. Then the second factor $g(P,Q)$ is zero in this neighborhood. By the identity theorem $g(P,Q)$ is identically zero. Thus, we have $g(P,Q) = 0$. This completes the proof. □

3. Explicit Evaluations of $K(q)$

In this section we prove a general theorem for the explicit evaluation of Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ and give example.

Theorem 3.1. *We have*

$$K^2(q) - 6 + \frac{1}{K^2(q)} = \left(\frac{\phi(-q)}{q^{1/2}\psi(q^4)}\right)^2.$$

Proof. Combining (2.1) and (2.2), we deduce that

$$2\left(\frac{1}{K(q)} - K(q)\right)^2 - \left(\frac{1}{K(q)} + K(q)\right)^2 = 2\left(\frac{\phi(q^2)}{q^{1/2}\psi(q^4)}\right)^2 - \left(\frac{\phi(q)}{q^{1/2}\psi(q^4)}\right)^2. \tag{3.1}$$

Simplify (3.1), we obtain

$$K^2(q) - 6 + \frac{1}{K^2(q)} = \frac{2\phi^2(q^2) - \phi^2(q)}{q\psi^2(q^4)} \tag{3.2}$$

From Lemma 2.5, we note that

$$2\phi^2(q^2) - \phi^2(q) = \phi^2(-q). \tag{3.3}$$

Employing (3.3) in (3.2) and simplifying, we complete the proof. □

Theorem 3.2. *For $q = e^{-\pi\sqrt{n/2}}$, let*

$$A_{2,n} = \frac{\phi(-q)}{2^{5/4}q^{1/2}\psi(q^4)}.$$

Then

$$\frac{1}{K^2(e^{-\pi\sqrt{n/2}})} + K^2(e^{-\pi\sqrt{n/2}}) = 4\sqrt{2}\ A_{2,n}^2 + 6.$$

Proof. We set $q = e^{-\pi\sqrt{n/2}}$ and use the definition of $A_{2,n}$ in Theorem 3.1 to complete the proof. □

From Theorem 3.2 it is clear that explicit values of $K^2(e^{-\pi\sqrt{n/2}})$ can easily be evaluated if we know the corresponding values of $A_{2,n}$. Saikia [7] evaluated explicit values of $A_{2,n}$ for $n = 1$, 2, 1/2, 3, 1/3, 4, 1/4, 9, and 1/9. Noting $A_{2,1} = 1$ from Lemma 2.1(i), employing in Theorem 3.2 and solving the resulting equation, we evaluate

$$K^2(e^{-\pi/\sqrt{2}}) = 3 + 2\sqrt{2} - 2\sqrt{(2+\sqrt{2})(1+\sqrt{2})}.$$

Next, we evaluate some new explicit values of the parameter $A_{2,n}$ which can be used to evaluate explicit values of $K^2(e^{-\pi/\sqrt{n/2}})$ by appealing to Theorem 3.2. First we state the following remark [7, p. 111, Remarks 4.2]:

Remark 3.3. By using the definitions of $\phi(q)$, $\psi(q)$ and $A_{k,n}$, it can be seen that $A_{k,n}$ has positive real value and that the values of $A_{k,n}$ increases as n increases when $k > 1$. Thus, by Lemma 2.1(i)(Theorem 2.1(i) in [7]), $A_{k,n} > 1$ for all $n > 1$ if $k > 1$.

Theorem 3.4. *We have*

$$\text{(i)}\quad A_{2,5} = \left(6 + 4\sqrt{2} + \sqrt{85 + 60\sqrt{2}}\right)^{1/2},$$

$$\text{(ii)}\quad A_{2,25} = \frac{1}{6}\left(52 + 40\sqrt{2} + 2c + \sqrt{36 + 4\left(26 + 20\sqrt{2} + c\right)^2}\right),$$

where

$$c = \left(80120 + 56650\sqrt{2} - 60\sqrt{33729 + 23850\sqrt{2}}\right)^{1/3} + \left(80120 + 56650\sqrt{2} + 60\sqrt{33729 + 23850\sqrt{2}}\right)^{1/3}.$$

Proof. Setting $q := e^{-\pi\sqrt{n/2}}$ in Theorem 2.8 and employing the definition of $A_{2,n}$, we find that

$$P = 2^{5/4}A_{2,n} \quad \text{and} \quad Q = 2^{5/4}A_{2,25n}. \tag{3.4}$$

Setting $n = 1/5$ in (3.4), employing in (3.1)and then simplifying by noting $A_{2,1/5} = 1/A_{2,5}$ from Lemma 2.1, we obtain

$$A_{2,5}^{12} - 12A_{2,5}^{10} + (5 - 80\sqrt{2})A_{2,5}^{8} - 184A_{2,5}^{6} - 5(1 + 16\sqrt{2})A_{2,5}^{4} - 20A_{2,5}^{2} - 1 = 0. \tag{3.5}$$

Solving (3.5) for $A_{2,5}$ with the help of *Mathematika* and noting the facts in Remark 3.3, we arrive at (i).

Setting $n = 1$ in (3.4), employing in (3.1)and then simplifying by noting $A_{2,1} = 1$ from Lemma 2.1, we obtain

$$A_{2,25}^{6} - 4(13 + 10\sqrt{2})A_{2,25}^{5} + 5A_{2,25}^{4} - 40(3 + 2\sqrt{2})A_{2,25}^{3} \tag{3.6}$$

$$-5A_{2,25}^2-4(13+10\sqrt{2})A_{2,25}-1=0.$$

Dividing (3.6) by $A_{2,25}^3$ and rearranging the terms, we obtain

$$\left(A_{2,25}^3-\frac{1}{A_{2,25}^3}\right)-4(13+10\sqrt{2})\left(A_{2,25}^2+\frac{1}{A_{2,25}^2}\right)$$
$$+5\left(A_{2,25}-\frac{1}{A_{2,25}}\right)-40(3+2\sqrt{2})=0. \tag{3.7}$$

Set

$$L=A_{2,25}-\frac{1}{A_{2,25}}, \tag{3.8}$$

so that

$$A_{2,25}^2+\frac{1}{A_{2,25}^2}=L^2-2 \quad \text{and} \quad A_{2,25}^3-\frac{1}{A_{2,25}^3}=L^3+3L. \tag{3.9}$$

Employing (3.8) and (3.9) in (3.7) and simplifying, we obtain

$$L^3-4(13+10\sqrt{2})L^2+8L-32(7+5\sqrt{2})=0. \tag{3.10}$$

Solving (3.10) for L with the help of *Mathematika*, we obtain

$$L=\frac{2}{3}\left(26+20\sqrt{2}+c\right), \tag{3.11}$$

where

$$c=\left(80120+56650\sqrt{2}-60\sqrt{33729+23850\sqrt{2}}\right)^{1/3}$$
$$+\left(80120+56650\sqrt{2}+60\sqrt{33729+23850\sqrt{2}}\right)^{1/3}.$$

Employing (3.11) in (3.8), solving the resulting equation, and noting the facts in Remark 3.3, we complete the proof of (ii). □

Remark 3.5. Explicit values of $A_{2,1/5}$ and $A_{2,1/25}$ can also be evaluated by employing the values of $A_{2,5}$ and $A_{2,25}$, respectively in the result $A_{2,1/n}=1/A_{2,n}$ of Lemma 2.1.

Theorem 3.6. *We have*

(i) $A_{2,7}$

$$=\frac{1}{\sqrt{2}}\left(16+10\sqrt{2}+2\sqrt{130+92\sqrt{2}}+\sqrt{-4+4\left(8+5\sqrt{2}+\sqrt{130+92\sqrt{2}}\right)}\right)^{1/2},$$

(ii) $A_{2,49}$

$$=\frac{1}{2}\left(26+88\sqrt{2}+3c+\frac{d}{\sqrt{b}}+\frac{1}{\sqrt{c}}\left(\left(-4b+\left(126+88\sqrt{2}+3c\right)\sqrt{b}+d\right)^2\right)^{1/2}\right),$$

where

$$b=\sqrt{1743+1232\sqrt{2}},\qquad c=\sqrt{3486+2462\sqrt{2}},$$

and

$$d=\sqrt{14\left(263488+186314\sqrt{2}+4463\sqrt{b+3156c}\right)}\ .$$

Proof. Setting $q=e^{-\pi\sqrt{n/2}}$ in Theorem 2.9 and employing the definition of $A_{2,n}$, we find that

$$P=2^{5/4}A_{2,n}\quad\text{and}\quad Q=2^{5/4}A_{2,49n}. \tag{3.12}$$

Setting $n=1/7$ in (3.12), employing in (3.12), and then simplifying by noting $A_{2,1/7}=1/A_{2,7}$ from Lemma 2.1, we obtain

$$A_{2,7}^{16}-56A_{2,7}^{14}+(252-448\sqrt{2})A_{2,7}^{12}+56(-35+12\sqrt{2})A_{2,7}^{10}+(1526-2048\sqrt{2})A_{2,7}^{8}$$

$$+56(-35+12\sqrt{2})A_{2,7}^{6}+(252-448\sqrt{2})A_{2,7}^{4}-56A_{2,7}^{2}+1=0. \tag{3.13}$$

Dividing (3.13) by $A_{2,7}^8$ and rearranging the terms, we obtain

$$\left(A_{2,7}^{8}+\frac{1}{A_{2,7}^{8}}\right)-56\left(A_{2,7}^{6}+\frac{1}{A_{2,7}^{6}}\right)+(252-448\sqrt{2})\left(A_{2,7}^{4}+\frac{1}{A_{2,7}^{4}}\right)$$

$$+56(-35+12\sqrt{2})\left(A_{2,7}^{2}+\frac{1}{A_{2,7}^{2}}\right)+(1526-2048\sqrt{2})=0. \tag{3.14}$$

Next, set

$$T=A_{2,7}^{2}+\frac{1}{A_{2,7}^{2}}, \tag{3.15}$$

so that

(3.16)

$$A_{2,7}^{4}+\frac{1}{A_{2,7}^{4}}=T^2-2,\quad A_{2,7}^{6}+\frac{1}{A_{2,7}^{6}}=T^3-3T\quad\text{and}\quad A_{2,7}^{8}+\frac{1}{A_{2,7}^{8}}=(T^2-2)^2-2.$$

Employing (3.15) and (3.16) in (3.14) and simplifying, we obtain

$$T^4-56T^3+(248-448\sqrt{2})T^2+224(-8+3\sqrt{2})T+(1024-1152\sqrt{2})=0. \tag{3.17}$$

Solving (3.17) for T, we obtain

$$T=2\left(8+5\sqrt{2}+\sqrt{130+92\sqrt{2}}\right). \tag{3.18}$$

Employing (3.18) in (3.15), solving for $A_{2,7}$, and noting the facts in Remarks 3.3, we arrive at (i).

Again setting $n=1$ in (3.12), employing in (3.12) and simplifying, we obtain

$$A_{2,49}^8-8(63+44\sqrt{2})A_{2,49}^7+28(17+12\sqrt{2})A_{2,49}^6-56(27+20\sqrt{2})A_{2,49}^5$$
$$+14(77+48\sqrt{2})A_{2,49}^4-56(27+20\sqrt{2})A_{2,49}^3$$
$$+28(17+12\sqrt{2})A_{2,49}^2-8(63+44\sqrt{2})A_{2,49}+1=0. \tag{3.19}$$

Dividing (3.19) by $A_{2,49}^4$ and rearranging the terms, we obtain

$$\left(A_{2,49}^4+\frac{1}{A_{2,49}^4}\right)-8(63+44\sqrt{2})\left(A_{2,49}^3+\frac{1}{A_{2,49}^3}\right)+28(17+12\sqrt{2})\left(A_{2,49}^2+\frac{1}{A_{2,49}^2}\right)$$
$$-56(27+20\sqrt{2})\left(A_{2,49}+\frac{1}{A_{2,49}}\right)+14(77+48\sqrt{2})=0. \tag{3.20}$$

Next, set

$$E=A_{2,49}+\frac{1}{A_{2,49}} \tag{3.21}$$

so that

$$A_{2,49}^2+\frac{1}{A_{2,49}^2}=E^2-2,\quad A_{2,49}^3+\frac{1}{A_{2,49}^3}=E^3-3E,\quad \text{and}\quad A_{2,49}^4+\frac{1}{A_{2,49}^4}=(E^2-2)^2-2. \tag{3.22}$$

Employing (3.21) and (3.22) in (3.20) and simplifying, we obtain

$$E^4-8(63+44\sqrt{2})E^3+8(59+42\sqrt{2})E^2-64\sqrt{2}E+128=0. \tag{3.23}$$

Solving (3.23) for E using *Mathematica*, we obtain

$$E=\frac{1}{4}(504+352\sqrt{2})+3\sqrt{3486+2462\sqrt{2}}$$
$$+\sqrt{2\left(31241+22092\sqrt{2}+186314\sqrt{\frac{14}{249+176\sqrt{2}}}+\frac{1844416}{\sqrt{1743+1232\sqrt{2}}}\right)}. \tag{3.24}$$

Employing (3.24) in (3.21), solving the resulting equation, and noting the facts in Remark 3.3, we complete the proof of (ii). □

Remark 3.7. Explicit values of $A_{2,1/7}$ and $A_{2,1/49}$ can also be evaluated by employing the values of $A_{2,7}$ and $A_{2,49}$, respectively in the result $A_{2,1/n} = 1/A_{2,n}$ of Lemma 2.1.

4. Reciprocity Theorem of $K(q)$

In this section we prove reciprocity theorems for the continued fraction $K(q)$.

Theorem 4.1. *If r and s are both positive and $2rs = 1$, then*

$$\left(K^2(e^{-\pi r}) - 6 + \frac{1}{K^2(e^{-\pi r})}\right)\left(K^2(e^{-\pi s}) - 6 + \frac{1}{K^2(e^{-\pi s})}\right) = 32.$$

Proof. From Theorem 3.1, we deduce that

$$\left(K^2(e^{-\pi r}) - 6 + \frac{1}{K^2(e^{-\pi r})}\right)\left(K^2(e^{-\pi s}) - 6 + \frac{1}{K^2(e^{-\pi s})}\right) = \left(\frac{\phi(-e^{-\pi r})\phi(-e^{-\pi s})}{e^{-\pi(r+s)/2}\psi(e^{-4\pi r})\psi(e^{-4\pi s})}\right)^2. \tag{4.1}$$

Using (2.3) and noting $2rs = 1$, we find that

$$\left(\frac{\phi(-e^{-\pi r})}{e^{-\pi s/2}\ \psi(e^{-4\pi s})}\right)^2 = 2^{5/2}\sqrt{\frac{s}{r}}. \tag{4.2}$$

Similarly, interchanging the role of r and s, we obtain

$$\left(\frac{\phi(-e^{-\pi s})}{e^{-\pi r/2}\ \psi(e^{-4\pi r})}\right)^2 = 2^{5/2}\sqrt{\frac{r}{s}}. \tag{4.3}$$

Employing (4.2) and (4.3) in (4.1) and simplifying, we arrive at the desired result. □

Theorem 4.2. *If r and s are both positive and $2rs = 1$, then*

$$\left(\frac{1 + K^2(e^{-\pi r})}{1 - K^2(e^{-\pi r})}\right)\left(\frac{1 + K^2(e^{-\pi s})}{1 - K^2(e^{-\pi s})}\right) = \sqrt{2}.$$

Proof. Combining (2.1) and (2.2), we deduce that

$$\frac{1 + K^2(q)}{1 - K^2(q)} = \frac{\phi(q)}{\phi(q^2)}. \tag{4.4}$$

Using (4.4), we find that

$$\left(\frac{1 + K^2(e^{-\pi r})}{1 - K^2(e^{-\pi r})}\right)\left(\frac{1 + K^2(e^{-\pi s})}{1 - K^2(e^{-\pi s})}\right) = \frac{\phi(e^{-\pi r})\phi(e^{-\pi s})}{\phi(e^{-2\pi r})\phi(e^{-2\pi s})}. \tag{4.5}$$

From (2.4), we deduce that

$$\frac{\phi(e^{-\pi r})}{\phi(e^{-2\pi s})} = 2^{1/4} \, (s/r)^{1/4} \tag{4.6}$$

and

$$\frac{\phi(e^{-\pi s})}{\phi(e^{-2\pi r})} = 2^{1/4}(r/s)^{1/4}. \tag{4.7}$$

Employing (4.6) and (4.7) in (4.5) and simplifying, we arrive at the desired result.□

Remark 4.3. If we know $K(e^{-\pi r})$ (or $K(e^{-\pi s})$) then $K(e^{-\pi/2r})$ (or $K(e^{-\pi/2s})$) can easily be evaluated by appealing to Theorem 4.1 or 4.2.

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