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On some Bounds for the Parameter λ in Steffensen's Inequality

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Abstract. The object is to obtain weaker conditions for the parameter λ in Steffensen's inequality and its generalizations and refinements additionally assuming nonnegativity of the function f. Furthermore, we contribute to the investigation of the Bellman-type inequalites establishing better bounds for the parameter λ.

1. Introduction

The well-known Steffensen inequality reads (see [11]):

Theorem 1.1. *Suppose that f is nonincreasing and g is integrable on $[a, b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt. \tag{1.1}$$

The inequalities are reversed for f nondecreasing.

Inequality (1.1) has triggered a huge amount of interest over the years (for a comprehensive survey see [10]). Futher, some newer works on this subject include extension of Steffensen's inequality and its generalizations to the class of convex functions (see [7, 8, 9]). We recall the following generalization obtained by Pečarić in [5].

Theorem 1.2. *Let h be a positive integrable function on $[a, b]$ and f be an integrable function such that f/h is nondecreasing on $[a, b]$. Let g be an integrable function on*

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$[a,b]$ with $0 \leq g \leq 1$. Then

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt \tag{1.2}$$

holds, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.3}$$

If f/h is a nonincreasing function, then the reverse inequality in (1.2) *holds.*

By substitutions $g \to 1-g$ and $\lambda \to b-a-\lambda$ previous theorem becomes:

Theorem 1.3. *Let h be a positive integrable function on $[a,b]$ and f be an integrable function such that f/h is nondecreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt \tag{1.4}$$

holds, where λ is the solution of the equation

$$\int_{b-\lambda}^b h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.5}$$

If f/h is a nonincreasing function, then the reverse inequality in (1.4) *holds.*

In 1959 Bellman gave an L^p generalization of Steffensen's inequality (see [1]). As noted by many mathematicians Bellman's result is incorrect as stated. A comprehensive survey of corrected versions and generalizations of Bellman's result can be found in [10]. In the following theorem we recall generalization of Bellman's result obtained by Pachpatte in [4].

Theorem 1.4. *Let f, g, h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \geq 0$, $h(t) \geq 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \leq g \leq A$, where A is a real positive constant. If $p \geq 1$, then*

$$\left(\int_0^1 g(t)f(t)dt\right)^p \leq A^p \int_0^\lambda f^p(t)dt, \tag{1.6}$$

where λ is the solution of the equation

$$\int_0^\lambda h^p(t)dt = \frac{1}{A^p}\left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

In the main part of this paper we obtain weaker conditions for the parameter λ in the aforementioned generalizations of Steffensen's inequality additionally assuming that the function f is nonnegative. As a consequence we obtain weaker condition for the parameter λ in Steffensen's inequality. Moreover, we obtain better estimation for the parameter λ in Pachpatte's result. We conclude the paper with better estimations for the parameter λ in generalizations of Steffensen's inequality obtained by Mercer, Wu and Srivastava.

2. Main Results

In the following theorems we obtain weaker condition for the parameter λ in generalizations of Steffensen's inequality given in Theorems 1.2 and 1.3.

Theorem 2.1. *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt \tag{2.1}$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \ge \int_a^b h(t)g(t)dt. \tag{2.2}$$

If f/h is nondecreasing, then the reverse inequality in (2.1) *holds, where λ is given by* (2.2) *with the reverse inequality.*

Proof. Since f/h is nonincreasing transformation of the difference between the right-hand side and the left-hand side of inequality (2.1) gives

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = \int_a^{a+\lambda} (1-g(t))f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\ge \frac{f(a+\lambda)}{h(a+\lambda)} \int_a^{a+\lambda} h(t)(1-g(t))dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\ge \frac{f(a+\lambda)}{h(a+\lambda)} \left(\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)g(t)dt \right) - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b g(t)h(t) \left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)} \right) dt \ge 0,
\end{aligned}$$

where we use (2.2) and nonnegativity of function f. □

Theorem 2.2. *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an* *integrable function on* $[a, b]$ *with* $0 \leq g \leq 1$. *Then*

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt \tag{2.3}$$

holds, where λ *is given by*

$$\int_{b-\lambda}^b h(t)dt \leq \int_a^b h(t)g(t)dt. \tag{2.4}$$

If f/h *is nondecreasing, then the reverse inequality in* (2.3) *holds, where* λ *is given by* (2.4) *with the reverse inequality.*

Proof. Similar to the proof of Theorem 2.1. □

Taking $h \equiv 1$ in Theorems 2.1 and 2.2 we obtain the following weaker conditions for the parameter λ in Steffensen's inequality.

Corollary 2.1. *Let* f *be a nonnegative nonincreasing function on* $[a, b]$ *and* g *be an integrable function on* $[a, b]$ *with* $0 \leq g \leq 1$. *Then*

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt \tag{2.5}$$

holds, where

$$\lambda \geq \int_a^b g(t)dt. \tag{2.6}$$

If f *is nondecreasing, then the reverse inequality in* (2.5) *holds, where* λ *is given by* (2.6) *with the reverse inequality.*

Corollary 2.2. *Let* f *be a nonnegative nonincreasing function on* $[a, b]$ *and* g *be an integrable function on* $[a, b]$ *with* $0 \leq g \leq 1$. *Then*

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt \tag{2.7}$$

holds, where

$$\lambda \leq \int_a^b g(t)dt. \tag{2.8}$$

If f *is nondecreasing, then the reverse inequality in* (2.7) *holds, where* λ *is given by* (2.8) *with the reverse inequality.*

In order to obtain Bellman-type inequality we need the following generalization of result given in Theorem 2.1.

Theorem 2.3. *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\int_a^b f^p(t)g(t)dt \le \int_a^{a+\lambda} f^p(t)dt \tag{2.9}$$

holds, where λ is given by

$$\int_a^{a+\lambda} h^p(t)dt \ge \int_a^b h^p(t)g(t)dt. \tag{2.10}$$

If f/h is nondecreasing, then the reverse inequality in (2.9) *holds, where λ is given by* (2.10) *with the reverse inequality.*

Proof. Since f/h is nonincreasing, we have that f^p/h^p is nonincreasing. Hence, we can apply Theorem 2.1 to the function f^p/h^p. □

Similarly, we have the following generalization of result given in Theorem 2.2.

Theorem 2.4. *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\int_a^b f^p(t)g(t)dt \ge \int_{b-\lambda}^b f^p(t)dt \tag{2.11}$$

holds, where λ is given by

$$\int_{b-\lambda}^b h^p(t)dt \ge \int_a^b h^p(t)g(t)dt. \tag{2.12}$$

If f/h is nondecreasing, then the reverse inequality in (2.11) *holds, where λ is given by* (2.12) *with the reverse inequality.*

Proof. Applying Theorem 2.2 to the function f^p/h^p. □

We continue with the following Bellman-type inequality which allows us to obtain better estimation for the parameter λ in Pachpatte's result.

Theorem 2.5. *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\frac{1}{(b-a)^{p-1}} \left(\int_a^b f(t)g(t)dt \right)^p \le \int_a^{a+\lambda} f^p(t)dt \tag{2.13}$$

holds, where λ is given by (2.10).

Proof. Using the Jensen inequality for convex function $\Phi(x) = x^p$ $(p \geq 1)$, we have

$$\left(\int_a^b f(t)g(t)dt\right)^p \leq \left(\int_a^b g(t)dt\right)^{p-1} \int_a^b f^p(t)g(t)dt. \tag{2.14}$$

Since $0 \leq g \leq 1$ we have

$$\left(\int_a^b g(t)dt\right)^{p-1} \int_a^b f^p(t)g(t)dt \leq (b-a)^{p-1} \int_a^b f^p(t)g(t)dt. \tag{2.15}$$

Combining (2.14) and (2.15), and using (2.9) we obtain

$$\frac{1}{(b-a)^{p-1}} \left(\int_a^b f(t)g(t)dt\right)^p \leq \int_a^b f^p(t)g(t)dt \leq \int_a^{a+\lambda} f^p(t)dt.$$

□

Taking $[a,b] = [0,1]$ in Theorem 2.5 we obtain the following corollary.

Corollary 2.3. *Let h be a positive integrable function on $[0,1]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[0,1]$. Let g be an integrable function on $[0,1]$ with $0 \leq g \leq 1$. If $p \geq 1$ then*

$$\left(\int_0^1 f(t)g(t)dt\right)^p \leq \int_0^\lambda f^p(t)dt \tag{2.16}$$

holds, where λ is given by

$$\int_0^\lambda h^p(t)dt \geq \int_0^1 h^p(t)g(t)dt. \tag{2.17}$$

Taking $A = 1$ in Theorem 1.4 we obtain the following corollary.

Corollary 2.4. *Let f, g, h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \geq 0$, $h(t) \geq 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \leq g(t) \leq 1$, $t \in [0,1]$. If $p \geq 1$, then (2.16) holds, where λ is the solution of the equation*

$$\int_0^\lambda h^p(t)dt = \left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

Remark 2.1. Since $0 \leq g \leq 1$ from (2.17) we have the following

$$\int_0^\lambda h^p(t)dt \geq \int_0^1 h^p(t)g(t)dt \geq \left(\int_0^1 g(t)dt\right)^{p-1} \int_0^1 h^p(t)g(t)dt. \tag{2.18}$$

Hence, the estimation for λ in Corollary 2.3 is better then the one in Pachpatte's result for the case $A = 1$ given in Corollary 2.4.

3. Weaker Conditions for Mercer's and Wu-Srivastava Generalizations

In [3] Mercer proved the following generalization of Steffensen's inequality.

Theorem 3.1. *Let f, g and h be integrable functions on (a, b) with f nonincreasing and $0 \le g \le h$. Then*

$$\int_{b-\lambda}^{b} f(t)h(t)dt \le \int_{a}^{b} f(t)g(t)dt \le \int_{a}^{a+\lambda} f(t)h(t)dt, \tag{3.1}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt. \tag{3.2}$$

Wu and Srivastava in [12] and Liu in [2] noted that the generalization due to Mercer is incorrect as stated. They have proved that it is true if we add the condition:

$$\int_{b-\lambda}^{b} h(t)dt = \int_{a}^{b} g(t)dt. \tag{3.3}$$

Pečarić, Perušić and Smoljak in [6] proved that the corrected version of Mercer's results follows from Theorems 1.2 and 1.3. In the following theorem we obtain weaker conditions for the parameter λ in the corrected version of Mercer's results which follows from Theorems 2.1 and 2.2.

Theorem 3.2. *Let h be a positive integrable function on $[a, b]$ and f be a nonnegative nonincreasing function on $[a, b]$. Let g be an integrable function on $[a, b]$ with $0 \le g \le h$.*

a) *Then*

$$\int_{a}^{b} f(t)g(t)dt \le \int_{a}^{a+\lambda} f(t)h(t)dt$$

holds, where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt \ge \int_{a}^{b} g(t)dt.$$

b) *Then*

$$\int_{b-\lambda}^{b} f(t)h(t)dt \le \int_{a}^{b} f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^{b} h(t)dt \le \int_{a}^{b} g(t)dt.$$

Proof. Using substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 2.1 and 2.2 we obtain the statements of this theorem. □

In [3] Mercer also gave the following theorem. As showed by Pečarić, Perušić and Smoljak in [6] this theorem is equivalent to Theorem 1.2.

Theorem 3.3. *Let f, g, h and k be integrable functions on (a, b) with $k > 0$, f/k nonincreasing and $0 \le g \le h$. Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ is given by

$$\int_a^{a+\lambda} h(t)k(t)dt = \int_a^b g(t)k(t)dt. \tag{3.4}$$

In the following theorem we relax condition (3.4) for the above Mercer's result and the corresponding condition for result equivalent to Theorem 1.3.

Theorem 3.4. *Let k be a positive integrable function on $[a, b]$ and f be a nonnegative integrable function such that f/k is nonincreasing on $[a, b]$. Let g, h be integrable functions on $[a, b]$ with $0 \le g \le h$.*

a) *Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)k(t)dt \ge \int_a^b g(t)k(t)dt.$$

b) *Then*

$$\int_{b-\lambda}^b f(t)h(t)dt \le \int_a^b f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^b h(t)k(t)dt \le \int_a^b g(t)k(t)dt.$$

Proof. Using substitutions $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 2.1 and 2.2 we obtain the statements of this theorem. □

Next, we recall the corrected and refined version of Mercer's result given by Wu and Srivastava in [12].

Theorem 3.5. *Let f, g and h be integrable functions on $[a, b]$ with f nonincreasing and let $0 \leq g \leq h$. Then the following integral inequalities hold true*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leq \int_{b-\lambda}^{b} \left(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{b} f(t)g(t)dt \\ &\leq \int_{a}^{a+\lambda} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{a+\lambda} f(t)h(t)dt,\end{aligned}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt = \int_{b-\lambda}^{b} h(t)dt. \tag{3.5}$$

In the following theorem we relax condition (3.5) by separating the above result into two parts and assuming nonnegativity of the function f.

Theorem 3.6. *Let h be a positive integrable function on $[a, b]$ and f be a nonnegative nonincreasing function on $[a, b]$. Let g be an integrable function on $[a, b]$ with $0 \leq g \leq h$.*

a) *Then*

$$\begin{aligned}\int_{a}^{b} f(t)g(t)dt &\leq \int_{a}^{a+\lambda} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{a+\lambda} f(t)h(t)dt\end{aligned}$$

holds, where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt \geq \int_{a}^{b} g(t)dt.$$

b) *Then*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leq \int_{b-\lambda}^{b} \left(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{b} f(t)g(t)dt\end{aligned}$$

holds, where λ is given by

$$\int_{b-\lambda}^{b} h(t)dt \leq \int_{a}^{b} g(t)dt.$$

Proof. The proof is based on the following inequalities:

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) dt + \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt, \tag{3.6}$$

and

$$\int_a^b f(t)g(t)dt \ge \int_{b-\lambda}^b \left(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\right) dt + \int_a^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt. \tag{3.7}$$

Let us prove the first one. Transformation of the right-hand side of (3.6) gives the following

$$\begin{aligned}
&\int_a^{a+\lambda} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) dt + \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt \\
&= \int_a^{a+\lambda} f(t)g(t)dt + f(a+\lambda)\int_a^{a+\lambda}(h(t) - g(t))dt + \int_{a+\lambda}^b f(t)g(t)dt - f(a+\lambda)\int_{a+\lambda}^b g(t)dt \\
&= \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^{a+\lambda}(h(t) - g(t))dt - \int_{a+\lambda}^b g(t)dt\right] \\
&= \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^{a+\lambda} h(t)dt - \int_a^b g(t)dt\right] \\
&\ge \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^b g(t)dt - \int_a^b g(t)dt\right] \\
&= \int_a^b f(t)g(t)dt
\end{aligned}$$

where in the inequality we use nonnegativity of f and a definition of λ, i.e. $\int_a^{a+\lambda} h(t)dt \ge \int_a^b g(t)dt$.

Inequality (3.7) can be proved in a similar manner.

Since f is nonincreasing on $[a, b]$ we get $f(t) \ge f(a+\lambda)$ for all $t \in [a, a+\lambda]$ and $f(t) \le f(a+\lambda)$ for all $t \in [a+\lambda, b]$. Then

$$\int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt \le 0$$

and

$$\int_a^{a+\lambda} [f(t) - f(a+\lambda)][h(t) - g(t)]dt \ge 0.$$

Using (3.6) and above inequalities we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &\le \int_a^{a+\lambda} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) dt \\ &\le \int_a^{a+\lambda} f(t)h(t)dt.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &\ge \int_{b-\lambda}^{b} \left(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\right) dt \\ &\ge \int_{b-\lambda}^{b} f(t)h(t)dt.\end{aligned}$$

□

Wu and Srivastava also proved a new sharpened and generalized version of inequality (3.1) (see [12]). We additionally assume that f is nonnegative to obtain the following weaker conditions for the parameter λ. Original result can be found in [12].

Theorem 3.7. *Let f, g, h and ψ be integrable functions on $[a, b]$ with f nonnegative nonincreasing and let $0 \le \psi \le g \le h - \psi$.*

a) *Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b |f(t) - f(a+\lambda)|\, \psi(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \ge \int_a^b g(t)dt.$$

b) *Then*

$$\int_{b-\lambda}^b f(t)h(t)dt + \int_a^b |f(t) - f(b-\lambda)|\psi(t)dt \le \int_a^b f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^b h(t)dt \le \int_a^b g(t)dt.$$

Proof. Since f is nonincreasing on $[a, b]$ we get $f(t) \ge f(a+\lambda)$ for all $t \in [a, a+\lambda]$ and $f(t) \le f(a+\lambda)$ for all $t \in [a+\lambda, b]$. Hence, using inequality (3.6) and the fact that $0 \leq \psi \leq g \leq h - \psi$ we get

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)h(t)dt - \int_a^b f(t)g(t)dt \\
&\quad \geq \int_a^{a+\lambda} [f(t) - f(a+\lambda)][h(t) - g(t)]dt - \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt \\
&\quad = \int_a^{a+\lambda} |f(t) - f(a+\lambda)|[h(t) - g(t)]dt + \int_{a+\lambda}^b |f(t) - f(a+\lambda)|g(t)dt \\
&\quad \geq \int_a^{a+\lambda} |f(t) - f(a+\lambda)|\psi(t)dt + \int_{a+\lambda}^b |f(a+\lambda) - f(t)|\psi(t)dt \\
&\quad = \int_a^b |f(t) - f(a+\lambda)|\psi(t)dt
\end{aligned}$$

and the proof is established. □

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