KYUNGPOOK Math. J. 55(2015), 953-967
http://dx.doi.org/10.5666/KMJ.2015.55.4.953
pISSN 1225-6951 eISSN 0454-8124

On a Class of Spirallike Functions associated with a Fractional Calculus Operator

Kuppathai Appasamy Selvakumaran* and Geetha Balachandar
Department of Mathematics, R. M. K College of Engineering and Technology, Puduvoyal - 601206, Tamil Nadu, India
e-mail : selvaa1826@gmail.com *and* gbalachandar1989@gmail.com

Pugazhenthi Rajaguru
Department of Mathematics, Arignar Anna Govt. Arts College, Attur - 636121, Tamil Nadu, India
e-mail : rajaguru.ap@gmail.com

Abstract. In this article, by making use of a linear multiplier fractional differential operator $D_\lambda^{\delta,m}$, we introduce a new subclass of spiral-like functions. The main object is to provide some subordination results for functions in this class. We also find sufficient conditions for a function to be in the class and derive Fekete-Szegö inequalities.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further let $\mathcal{S}$ denote the family of functions of the form (1.1) which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class of α-spirallike functions of order β in $\mathbb{U}$, denoted by $\mathcal{S}(\alpha, \beta)$, if

$$Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \beta\cos\alpha, \quad z \in \mathbb{U},$$

for $0 \le \beta < 1$ and for some real α with $|\alpha| < \pi/2$. The class $\mathcal{S}(\alpha, \beta)$ was studied by

* Corresponding Author.
Received August 16, 2014; revised February 28, 2015; accepted November 3, 2015.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Analytic functions, α-spirallike functions of order β, Fractional differential operator, Fekete-Szegö problem, Subordination.

Libera [8] and Keogh and Merkes [6]. We observe that $\mathcal{S}(\alpha, 0) = \mathcal{S}(\alpha)$ is the class of spirallike functions introduced by Špaček [20], $\mathcal{S}(0, \beta) = \mathcal{S}^*(\beta)$ is the class of starlike functions of order β and $\mathcal{S}(0, 0) = \mathcal{S}^*$ is the familiar class of starlike functions.

For the constants α, β with $|\alpha| < \pi/2$ and $0 \le \beta < 1$, we define

$$P_{\alpha,\beta}(z) = \frac{1 + e^{-i\alpha}(e^{-i\alpha} - 2\beta\cos\alpha)z}{1 - z}, \quad z \in \mathbb{U}. \tag{1.2}$$

The function $P_{\alpha,\beta}(z)$ maps the open unit disk onto the half plane $H_{\alpha,\beta} = \{z \in \mathbb{C} : Re(e^{i\alpha} z) > \beta\cos\alpha\}$. If $P_{\alpha,\beta}(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$, then it is easy to verify that

$$p_n = 2e^{-i\alpha}(1 - \beta)\cos\alpha, \quad (n = 1, 2, \ldots). \tag{1.3}$$

For functions f given by (1.1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

We will make use of the following definition of fractional derivatives by Owa [13].

The fractional derivative of order δ is defined, for a function f, by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\delta}\, d\xi, \quad (0 \le \delta < 1) \tag{1.4}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z - \xi)^{-\delta}$ is removed by requiring $\log(z - \xi)$ to be real when $(z - \xi) > 0$. It follows from (1.4) that

$$D_z^\delta\, z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-\delta}, \quad (0 \le \delta < 1,\ n \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^\delta f$, Owa and Srivastava [14] introduced the operator $\Omega^\delta : \mathcal{A} \longrightarrow \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral as follows:

$$\begin{aligned} \Omega^\delta f(z) &= \Gamma(2-\delta)\, z^\delta\, D_z^\delta\, f(z), \quad -\infty < \delta < 2 \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\, a_n z^n. \end{aligned} \tag{1.5}$$

Here we note that $\Omega^0 f(z) = f(z)$ and $\Omega^1 f(z) = zf'(z)$.
In [2]Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_\lambda^{\delta,\, m}$ as follows:

$$\begin{aligned}
&D_\lambda^{\delta,0} f(z) = f(z),\\
&D_\lambda^{\delta,\,1} f(z) = (1-\lambda)\Omega^\delta f(z) + \lambda\, z(\Omega^\delta f(z))', \quad (0 \le \delta < 1,\ \lambda \ge 0),\\
&D_\lambda^{\delta,\,2} f(z) = D_\lambda^{\delta,\,1}(D_\lambda^{\delta,\,1} f(z)),\\
&\qquad\vdots
\end{aligned}$$

$$D_\lambda^{\delta,\,m} f(z) = D_\lambda^{\delta,\,1}(D_\lambda^{\delta,\,m-1} f(z)), \qquad m \in \mathbb{N}. \tag{1.6}$$

For a related work also see [3]. If $f(z)$ is given by (1.1), then by (1.6), we have

$$D_\lambda^{\delta,\,m} f(z) = z + \sum_{n=2}^{\infty} \Phi_n(\delta,\lambda,m) a_n z^n, \quad m \in \mathbb{N}_0, \tag{1.7}$$

where

$$\Phi_n(\delta,\lambda,m) = \left(\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} \big[1+(n-1)\lambda\big] \right)^m, \quad (n = 2, 3, \ldots). \tag{1.8}$$

It can be seen that, by specializing the parameters the operator $D_\lambda^{\delta,\,m}$ reduces to many known and new integral and differential operators. In particular, when $\delta = 0$ the operator $D_\lambda^{\delta,\,m}$ reduces to the operator introduced by AL-Oboudi [1] and for $\delta = 0$, $\lambda = 1$ it reduces to the operator introduced by Sălăgean [16]. Further we remark that, when $m = 1$, $\lambda = 0$ the operator $D_\lambda^{\delta,\,m}$ reduces to Owa-Srivastava fractional differential operator [14]. We note here in passing that very recently, Srivastava et al. [23] studied subclasses of analytic functions defined by using a more general fractional derivative operator than $D_\lambda^{\delta,m}$.

In view of (1.7), $D_\lambda^{\delta,\,m} f(z)$ can be expressed in terms of convolution as follows:

$$D_\lambda^{\delta,\,m} f(z) = (g_{\delta,\,\lambda} * f)(z), \quad (m \in \mathbb{N}_0,\ z \in \mathbb{U}),$$

where

$$g_{\delta,\,\lambda}(z) = z + \sum_{n=2}^{\infty} \Phi_n(\delta,\lambda,m) z^n, \quad (z \in \mathbb{U}).$$

Define the function $g_{\delta,\,\lambda}^{(-1)}$ such that

$$g_{\delta,\,\lambda}^{(-1)}(z) * g_{\delta,\,\lambda}(z) = \frac{z}{1-z}, \quad (z \in \mathbb{U}). \tag{1.9}$$

It is easy to note that

$$f(z) = g_{\delta,\,\lambda}^{(-1)}(z) * D_\lambda^{\delta,\,m} f(z). \tag{1.10}$$

We now introduce the following subclass of $\mathcal{A}$:

Definition 1.1. Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\gamma,\,\lambda}^{\delta,\,m}(\alpha, \beta)$ if and only if

$$Re\left(e^{i\alpha}\frac{z(D_\lambda^{\delta,\,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,\,m}f(z)+\gamma z(D_\lambda^{\delta,\,m}f(z))'}\right) > \beta\cos\alpha, \quad (z \in \mathbb{U}). \tag{1.11}$$

It is interesting to note that, by specializing the parameters, the class $\mathcal{S}_{\gamma,\,\lambda}^{\delta,\,m}(\alpha, \beta)$ reduces to many new and known subclasses of analytic functions. We list some of the special cases of the function class $\mathcal{S}_{\gamma,\,\lambda}^{\delta,\,m}(\alpha, \beta)$ in the following remarks.

Remark 1.2. Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$ and $m = 0$. Then $\mathcal{S}_{\gamma,\,\lambda}^{\delta,\,0}(\alpha, \beta) = \mathcal{S}_\gamma(\alpha, \beta) := \left\{f \in \mathcal{A} : Re\left(e^{i\alpha}\frac{zf'(z)}{(1-\gamma)f(z)+\gamma zf'(z)}\right) > \beta\cos\alpha, \quad z \in \mathbb{U}\right\}$. Also, $\mathcal{S}_{0,\,\lambda}^{\delta,\,0}(\alpha, \beta) = \mathcal{S}(\alpha, \beta)$ is the class of α-spirallike functions of order β studied by Libera [8] and $\mathcal{S}_{0,\,\lambda}^{\delta,\,0}(\alpha, 0) = \mathcal{S}(\alpha)$ is the class of spirallike functions introduced by Špaček [20]. Here we remark that some analogous classes of $\mathcal{S}_\gamma(\alpha, \beta)$ have been recently studied by Murugusundaramoorthy in [11] and Orhan et al. in [12].

Remark 1.3. Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$ and $\delta = 0$. Then $\mathcal{S}_{\gamma,\,\lambda}^{0,\,m}(\alpha, \beta) = \mathcal{S}_{\gamma,\,\lambda}^{m}(\alpha, \beta)$ $:= \left\{f \in \mathcal{A} : Re\left(e^{i\alpha}\frac{z(D_\lambda^m f(z))'}{(1-\gamma)D_\lambda^m f(z)+\gamma z(D_\lambda^m f(z))'}\right) > \beta\cos\alpha,\ z \in \mathbb{U}\right\}$, where D_λ^m is the Al-Oboudi operator [1] defined by $D_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^m a_n z^n$.

Remark 1.4. Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, $\delta = 0$ and $\lambda = 1$. Then $\mathcal{S}_{\gamma,\,1}^{0,\,m}(\alpha, \beta) = \mathcal{S}_{\gamma}^{m}(\alpha, \beta)$ $:= \left\{f \in \mathcal{A} : Re\left(e^{i\alpha}\frac{z(D^m f(z))'}{(1-\gamma)D^m f(z)+\gamma z(D^m f(z))'}\right) > \beta\cos\alpha, \quad z \in \mathbb{U}\right\}$, where D^m is the Sălăgean operator [16] defined by $D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n$.

Remark 1.5. Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$, $0 \leq \gamma < 1, \lambda = 0$ and $m = 1$. Then $\mathcal{S}_{\gamma,\,0}^{\delta,\,1}(\alpha, \beta) = \mathcal{S}_{\gamma}^{\delta}(\alpha, \beta) := \left\{f \in \mathcal{A} : Re\left(e^{i\alpha}\frac{z(\Omega^\delta f(z))'}{(1-\gamma)\Omega^\delta f(z)+\gamma z(\Omega^\delta f(z))'}\right) > \beta\cos\alpha,\ z \in \mathbb{U}\right\}$, where Ω^δ is the Owa-Srivastava fractional differential operator [14] defined by (1.5).

2. Membership Characterizations

In this section, we obtain some sufficient conditions for $f(z) \in \mathcal{S}_{\gamma,\,\lambda}^{\delta,\,m}(\alpha, \beta)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ and let η be a real number with $0 \leq \eta < 1$. If*

$$\left|\frac{z(D_\lambda^{\delta,\,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,\,m}f(z)+\gamma z(D_\lambda^{\delta,\,m}f(z))'} - 1\right| < 1-\eta, \quad (z \in \mathbb{U}), \tag{2.1}$$

then $f \in \mathcal{S}_{\gamma,\lambda}^{\delta,\,m}(\alpha, \beta)$ provided, $|\alpha| \leq \cos^{-1}(\frac{1-\delta}{1-\beta})$.

Proof. It follows from (2.1) that $\left[\frac{z(D_\lambda^{\delta,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,m}f(z)+\gamma z(D_\lambda^{\delta,m}f(z))'}\right] = 1+(1-\eta)w(z)$, where $|w(z)|<1$ for $z\in\mathbb{U}$. Thus

$$\begin{aligned} Re\left[e^{i\alpha}\frac{z(D_\lambda^{\delta,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,m}f(z)+\gamma z(D_\lambda^{\delta,m}f(z))'}\right] &= Re[e^{i\alpha}(1+(1-\eta)w(z))]\\ &= \cos\alpha+(1-\eta)\,Re[e^{i\alpha}w(z)]\\ &\geq \cos\alpha-(1-\eta)\left|e^{i\alpha}w(z)\right|\\ &> \cos\alpha-(1-\eta)\\ &\geq \beta\cos\alpha \end{aligned}$$

for $|\alpha|\leq\cos^{-1}(\frac{1-\delta}{1-\beta})$, and the proof is complete. □

Taking $\eta = 1-(1-\beta)\cos\alpha$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $f\in\mathcal{A}$. If*

$$\left|\frac{z(D_\lambda^{\delta,\,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,\,m}f(z)+\gamma z(D_\lambda^{\delta,\,m}f(z))'}-1\right| < (1-\beta)\cos\alpha,$$

then $f\in\mathcal{S}_{\gamma,\lambda}^{\delta,\,m}(\alpha,\beta)$.

Theorem 2.3. *Let $f\in\mathcal{A}$ be of the form* (1.1)*. If*

$$\sum_{n=2}^{\infty}[(1-\gamma)(n-1)\sec\alpha+(1-\beta)(1+(1-n)\gamma)]\Phi_n(\delta,\lambda,m)\,|a_n|\leq 1-\beta, \tag{2.2}$$

where $\frac{-\pi}{2}<\alpha<\frac{\pi}{2}$, $0\leq\beta<1$, $0\leq\gamma<1$ and $\Phi_n(\delta,\lambda,m)$ is defined by (1.8)*, then $f\in\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$.*

Proof. In view of Corollary 2.2, it suffices to show that

$$\left|\frac{z(D_\lambda^{\delta,\,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,\,m}f(z)+\gamma z(D_\lambda^{\delta,\,m}f(z))'}-1\right| < (1-\beta)\cos\alpha. \tag{2.3}$$

Now, we have

$$\begin{aligned} &\left|\frac{z(D_\lambda^{\delta,\,m}f(z))'}{(1-\gamma)D_\lambda^{\delta,\,m}f(z)+\gamma z(D_\lambda^{\delta,\,m}f(z))'}-1\right|\\ &\qquad = (1-\gamma)\left|\frac{\sum_{n=2}^{\infty}(n-1)\Phi_n(\delta,\lambda,m)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}[1+(n-1)\gamma]\Phi_n(\delta,\lambda,m)a_nz^{n-1}}\right|\\ &\qquad < (1-\gamma)\frac{\sum_{n=2}^{\infty}(n-1)\Phi_n(\delta,\lambda,m)\,|a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\gamma]\Phi_n(\delta,\lambda,m)\,|a_n|}. \end{aligned}$$

The last expression is bounded above by $(1-\beta)\cos\alpha$, if

$$\sum_{n=2}^{\infty}(1-\gamma)(n-1)\Phi_n(\delta,\lambda,m)\sec\alpha\,|a_n| \leq (1-\beta)\left\{1-\sum_{n=2}^{\infty}[1+(n-1)\gamma]\Phi_n(\delta,\lambda,m)\,|a_n|\right\}$$

which is equivalent to our condition (2.2) of the theorem. Thus, we conclude from (2.3) that $f(z) \in \mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$. □

In view of the Remarks 1.2–1.5, we state the following corollaries.

Corollary 2.4. *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty}\left[(1-\gamma)(n-1)\sec\alpha+(1-\beta)(1+(n-1)\gamma)\right]|a_n| \leq 1-\beta,$$

where $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{S}_{\gamma}(\alpha,\beta)$.

Corollary 2.5. *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty}\left[(1-\gamma)(n-1)\sec\alpha+(1-\beta)(1+(n-1)\gamma)\right][1+(n-1)\lambda]^m|a_n| \leq 1-\beta,$$

where $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{S}_{\gamma,\lambda}^{m}(\alpha,\beta)$.

Corollary 2.6. *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty}n^m\left[(1-\gamma)(n-1)\sec\alpha+(1-\beta)(1+(n-1)\gamma)\right]|a_n| \leq 1-\beta,$$

where $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{S}_{\gamma}^{m}(\alpha,\beta)$.

Corollary 2.7. *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\left[(1-\gamma)(n-1)\sec\alpha+(1-\beta)(1+(n-1)\gamma)\right]|a_n| \leq 1-\beta,$$

where $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{S}_{\gamma}^{\delta}(\alpha,\beta)$.

We also observe that Corollary 2.4 yields the results of Kwon and Owa [7] and Silverman [18] for particular values of γ and β. Substituting $\gamma = 0$ in Corollary 2.4, we have

Corollary 2.8.([7]) *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty} [(n-1)\sec\alpha + 1 - \beta]\,|a_n| \leq 1 - \beta,$$

where $-\pi/2 < \alpha < \pi/2$ and $0 \leq \beta < 1$, then $f \in \mathcal{S}(\alpha, \beta)$.

Taking $\gamma = 0$ and $\beta = 0$ in Corollary 2.4, we have

Corollary 2.9.([18]) *Let $f \in \mathcal{A}$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty} [1 + (n-1)\sec\alpha]\,|a_n| \leq 1,$$

where $-\pi/2 < \alpha < \pi/2$, then $f \in \mathcal{S}(\alpha)$.

A necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha, \beta)$ is given, in terms of integral representation, in the following theorem.

Theorem 2.10. *A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha, \beta)$ if and only if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1 \quad (z \in \mathbb{U})$, such that*

$$f(z) = g_{\delta,\lambda}^{(-1)}(z) * z \exp\left(\int_0^z \left[\frac{P_{\alpha,\beta}(w(\zeta)) - 1}{1 - \gamma P_{\alpha,\beta}(w(\zeta))}\right] \frac{d\zeta}{\zeta}\right), \quad (z \in \mathbb{U}),$$

where $P_{\alpha,\beta}(z)$ and $g_{\delta,\lambda}^{(-1)}(z)$ are defined by (1.2) *and* (1.9) *respectively.*

Proof. In view of (1.11), $f \in \mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha, \beta)$ if and only if there exists $w(z)$ analytic in $\mathbb{U}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathbb{U}$, such that

$$\frac{z(D_\lambda^{\delta,m} f(z))'}{(1-\gamma) D_\lambda^{\delta,m} f(z) + \gamma z (D_\lambda^{\delta,m} f(z))'} = P_{\alpha,\beta}(w(z)).$$

From the above equation, we get

$$D_\lambda^{\delta,m} f(z) = z \exp\left(\int_0^z \left[\frac{P_{\alpha,\beta}(w(\zeta)) - 1}{1 - \gamma P_{\alpha,\beta}(w(\zeta))}\right] \frac{d\zeta}{\zeta}\right). \tag{2.4}$$

Using (2.4) in (1.10), we obtain

$$f(z) = g_{\delta,\lambda}^{(-1)}(z) * z \exp\left(\int_0^z \left[\frac{P_{\alpha,\beta}(w(\zeta)) - 1}{1 - \gamma P_{\alpha,\beta}(w(\zeta))}\right] \frac{d\zeta}{\zeta}\right), \quad (z \in \mathbb{U}),$$

which completes the proof. □

For $0 \leq \theta \leq 2\pi$ and $0 \leq \tau \leq 1$, we define the function

$$\Psi(z,\theta,\tau) = g^{(-1)}_{\delta,\,\lambda}(z) * z \exp\left(\int_0^z \left[\frac{P_{\alpha,\beta}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right)-1}{1-\gamma P_{\alpha,\beta}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right)}\right]\frac{d\zeta}{\zeta}\right), \tag{2.5}$$

where $P_{\alpha,\beta}(z)$ and $g^{(-1)}_{\delta,\,\lambda}(z)$ are defined by (1.2) and (1.9) respectively. By virtue of Theorem 2.10, the function $\Psi(z,\theta,\tau)$ belongs to the class $\mathcal{S}^{\delta,m}_{\gamma,\lambda}(\alpha,\beta)$.

3. Fekete-Szegö Problem

A classical theorem of Fekete-Szegö [5] states that, if $f(z) \in \mathcal{S}$ is given by (1.1), then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu & \text{if} \quad \mu \leq 0, \\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if} \quad 0 \leq \mu \leq 1, \\ 4\mu-3 & \text{if} \quad \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each μ there exists a function in $\mathcal{S}$ such that the equality holds. Later, Pfluger [15] has considered the same problem but for complex values of μ. The problem of finding sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ for different subclasses of $\mathcal{A}$ is known as the Fekete-Szegö problem. In the recent years, this problem has been extensively studied by many authors including [4, 10, 17, 21]. Further, in [22], Srivastava, Mishra and Das investigated the Fekete-Szego coefficient problem rather systematically and extensively for the class

$$\mathcal{C}_1 := \bigcup_{\alpha} \mathcal{C}_1(\alpha), \qquad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right),$$

where $\mathcal{C}_1(\alpha)$ is the class of normalized analytic functions f in $\mathcal{U}$, which are given by (1.1) and satisfying the inequality

$$Re\left(e^{i\alpha}\frac{zf'(z)}{\phi(z)}\right) > 0, \qquad \left(z \in \mathbb{U};\ \phi \in \mathcal{S}^*;\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

In order to obtain sharp upper bound for Fekete-Szegö functional for the class $\mathcal{S}^{\delta,m}_{\gamma,\lambda}(\alpha,\beta)$, the following lemma is required.

Lemma 3.1.([6]) *Let $w(z)$ be analytic in $\mathbb{U}$ and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathbb{U}$. If $w(z) = \sum\limits_{r=1}^{\infty} w_n z^n$, $z \in \mathbb{U}$, then*

$$|w_1| \leq 1, \quad |w_2| \leq 1 - |w_1|^2, \tag{3.1}$$

and

$$|w_2 - s w_1^2| \leq \max\{1, |s|\}, \tag{3.2}$$

for any complex number s.

Theorem 3.2. *Let $f \in \mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ be given by (1.1) and let μ be a real number. Then*

(3.3)
$$|a_3-\mu a_2^2| \le \begin{cases} \frac{(1-\beta)\cos\alpha}{(1-\gamma)^2\Phi_3(\delta,\lambda,m)}\left[\gamma+3-2\beta(1+\gamma)-\frac{4\mu(1-\beta)\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}\right], & \text{if } \mu\le\sigma_1,\\ \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}, & \text{if } \sigma_1\le\mu\le\sigma_2,\\ \frac{(1-\beta)\cos\alpha}{(1-\gamma)^2\Phi_3(\delta,\lambda,m)}\left[\frac{4\mu(1-\beta)\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}+2\beta(1+\gamma)-\gamma-3\right], & \text{if } \mu\ge\sigma_2,\end{cases}$$

where

$$\sigma_1=\frac{(1+\gamma)\Phi_2^2(\delta,\lambda,m)}{2\Phi_3(\delta,\lambda,m)},\ \sigma_2=\frac{2-\beta(1+\gamma)\Phi_2^2(\delta,\lambda,m)}{2(1-\beta)\Phi_3(\delta,\lambda,m)} \tag{3.4}$$

and $\Phi_2(\delta,\lambda,m)$, $\Phi_3(\delta,\lambda,m)$ are defined by (1.8) with $n=2$ and $n=3$ respectively. All the estimates in (3.3) are sharp.

Proof. Suppose that $f \in \mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ is given by (1.1). Then there exists $w(z)=\sum_{r=1}^{\infty} w_r z^r$, analytic in $\mathbb{U}$ with $w(0)=0,\ |w(z)|<1,\ (z\in\mathbb{U})$ such that

$$\frac{z[D_\lambda^{\delta,m}f(z)]'}{(1-\gamma)D_\lambda^{\delta,m}f(z)+\gamma z(D_\lambda^{\delta,m}f(z))'}=P_{\alpha,\beta}(w(z)),\quad z\in\mathbb{U}. \tag{3.5}$$

Let $P_{\alpha,\beta}(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$. Equating the coefficients of z and z^2 on both sides of (3.5), we obtain

$$a_2=\frac{p_1w_1}{(1-\gamma)\Phi_2(\delta,\lambda,m)},$$
$$a_3=\frac{1}{2(1-\gamma)\Phi_3(\delta,\lambda,m)}\left[\left(\frac{1+\gamma}{1-\gamma}p_1^2+p_2\right)w_1^2+p_1w_2\right].$$

From (1.3), we have $p_1=p_2=2e^{-i\alpha}(1-\beta)\cos\alpha$, and thus we obtain

$$\begin{aligned} a_2&=\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{(1-\gamma)\Phi_2(\delta,\lambda,m)}w_1,\\ a_3&=\frac{e^{-i\alpha}(1-\beta)\cos\alpha}{2(1-\gamma)\Phi_3(\delta,\lambda,m)}\left[\left(\frac{1+\gamma}{1-\gamma}2e^{-i\alpha}(1-\beta)\cos\alpha+1\right)w_1^2+w_2\right].\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}|a_3-\mu a_2^2| \le{}&\frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}\\ &\times\left\{\left|\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{1-\gamma}\left(1+\gamma-\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}\right)+1\right||w_1|^2+|w_2|\right\}.\end{aligned}$$

Making use of Lemma 3.1 we have

$$\begin{aligned}|a_3-\mu a_2^2| \leq & \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)} \\ & \times\left\{1+\left[\left|\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{1-\gamma}\left(1+\gamma-\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}\right)+1\right|-1\right]|w_1|^2\right\}\end{aligned} \tag{3.7}$$

or

$$|a_3-\mu a_2^2| \leq \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}\left[1+\left(\sqrt{1+M(2+M)\cos^2\alpha}-1\right)|w_1|^2\right], \tag{3.8}$$

where

$$M=\frac{2(1-\beta)}{1-\gamma}\left(1+\gamma-\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}\right). \tag{3.9}$$

Let $G(x,y) = 1+\left(\sqrt{1+M(2+M)x^2}-1\right)y^2$ where $x=\cos\alpha$, $y=|w_1|$ and $(x,y)\in[0,1]\times[0,1]$. We can easily verify that, the function $G(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. So the maximum must be attained at a boundary point. As $G(x,0)=1$, $G(0,y)=1$ and $G(1,1)=|1+M|$, the maximal value of $G(x,y)$ may be $G(0,0)=1$ or $G(1,1)=|1+M|$. Therefore, (3.8) gives

$$|a_3-\mu a_2^2| \leq \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}\max\{1,|1+M|\}, \tag{3.10}$$

where M is given by (3.9).

Now let us consider $|1+M|\geq 1$. If $\mu\leq\sigma_1$, where σ_1 is given by (3.4), then $M\geq 0$ and from (3.10), we obtain

$$|a_3-\mu a_2^2| \leq \frac{(1-\beta)\cos\alpha}{(1-\gamma)^2\phi_3(\delta,\lambda,m)}\left[\gamma+3-2\beta(1+\gamma)-\frac{4\mu(1-\beta)\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}\right]$$

which is the first part of the inequality (3.3). If $\sigma_1\leq\mu\leq\sigma_2$, then $|1+M|\leq 1$ and from (3.10) we get

$$|a_3-\mu a_2^2| \leq \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}$$

which is the second part of the inequality (3.3). Finally, if $\sigma_2\geq\mu$ where σ_2 is given by (3.4) then $M\leq -2$ and it follows from (3.10) that

$$|a_3-\mu a_2^2| \leq \frac{(1-\beta)\cos\alpha}{(1-\gamma)^2\Phi_3(\delta,\lambda,m)}\left[\frac{4\mu(1-\beta)\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}+2\beta(1+\gamma)-\gamma-3\right]$$

and this completes the third part of (3.3). In view of Lemma 3.1, the results are sharp for $w(z)=z$ and $w(z)=z^2$ or one of their rotations. From (3.5), we obtain that the extremal functions are $\Psi(z,\theta,1)$ and $\Psi(z,\theta,0)$, defined by (2.5) with $\tau=1$ and $\tau=0$. $\square$

In the following, we give the Fekete-Szegö problem for the class $\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ with complex parameter.

Theorem 3.3. *Let* $f\in\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ *be given by* (1.1) *and let* μ *be a complex number. Then*

(3.11)

$$|a_3-\mu a_2^2| \le \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)}\max\left\{1,\left|\frac{2(1-\beta)\cos\alpha}{(1-\gamma)}\left(\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}-1-\gamma\right)-e^{i\alpha}\right|\right\}.$$

The result is sharp.

Proof. Assume that $f\in\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$. Making use of (3.6), we obtain

$$|a_3-\mu a_2^2| \le \frac{(1-\beta)\cos\alpha}{(1-\gamma)\Phi_3(\delta,\lambda,m)} \times\left|w_2-\left[\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{1-\gamma}\left(\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}-1-\gamma\right)-1\right]w_1^2\right|.$$

The inequality (3.11) follows by applying Lemma 3.1 with

$$s=\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{1-\gamma}\left(\frac{2\mu\Phi_3(\delta,\lambda,m)}{\Phi_2^2(\delta,\lambda,m)}-1-\gamma\right)-1.$$

The functions $\Psi(z,\theta,1)$ and $\Psi(z,\theta,0)$ defined by (2.5) with $\tau=1$ and $\tau=0$ show that the inequality (3.11) is sharp. $\square$

Theorems 3.2 and 3.3 include several other results for particular values of the parameters m, α, β, γ. In particular, Theorem 3.2 yields the results obtained by Keogh and Merkes in [6] and Srivastava et al., in [22] for special values of the parameters.

4. Subordination Results

In view of Theorem 2.3, we now introduce the subclass $\widetilde{\mathcal{S}}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)\subset\mathcal{S}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ which consist of functions $f\in\mathcal{A}$ whose coefficients satisfy the condition (2.2). In this section, we prove a subordination theorem for the class $\widetilde{\mathcal{S}}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$. To state and prove our result we need the following definitions and lemma.

Definition 4.1.(cf. [9]) Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z)\prec g(z)$ if there exists an analytic function $w(z)$ with $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(w(z))$ $(z\in\mathbb{U})$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z)\prec g(z)\quad(z\in\mathbb{U})\Longleftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

Definition 4.2.([24]) (Subordinating Factor Sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n\, b_n\, z^n \prec f(z) \qquad (z \in \mathbb{U};\ a_1 := 1).$$

Lemma 4.3.([24]) *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left(1 + 2\sum_{n=1}^{\infty} b_n\, z^n\right) > 0 \qquad (z \in \mathbb{U}).$$

Theorem 4.4. *Let the function $f(z)$ defined by* (1.1) *be in the class $\widetilde{\mathcal{S}}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$. If $g(z) \in \mathcal{K}$, the class of convex functions, then*

$$\frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta + ((1-\gamma)\sec\alpha + (1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}(f * g)(z) \prec g(z), \tag{4.1}$$

$$\left(z \in \mathbb{U},\ |\alpha| < \frac{\pi}{2},\ 0 \le \beta < 1,\ 0 \le \gamma < 1\right)$$

and

$$Re\{f(z)\} > -\frac{[1-\beta + ((1-\gamma)\sec\alpha + (1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}, \quad (z \in \mathbb{U}), \tag{4.2}$$

where $\Phi_n(\delta,\lambda,m)$ is given by (1.8). *The following constant factor in the subordination result* (4.1)*:*

$$\frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta + ((1-\gamma)\sec\alpha + (1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}$$

cannot be replaced by a larger one.

Proof. Let $f(z) \in \widetilde{\mathcal{S}}_{\gamma,\lambda}^{\delta,m}(\alpha,\beta)$ and suppose that

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}.$$

Then we readily have

$$\begin{aligned}&\frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta + ((1-\gamma)\sec\alpha + (1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}(f * g)(z)\\ &= \frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta + ((1-\gamma)\sec\alpha + (1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}\left(z + \sum_{n=2}^{\infty} a_n\, c_n z^n\right).\end{aligned}$$

Thus, by Definition 4.2, the subordination result (4.1) will hold true if

$$\left\{\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta+((1-\gamma)\sec\alpha+(1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}\,a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 4.3 this is equivalent to the following inequality:
(4.3)

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,a_n\,z^n\right\}>0\quad(z\in\mathbb{U}).$$

Since $[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_n(\delta,\lambda,m),\ (n\geq 2)$ is an increasing function of n, we have

$$\begin{aligned}
&Re\left\{1+\sum_{n=1}^{\infty}\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,a_n\,z^n\right\}\\
&\quad= Re\left\{1+\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,z\right.\\
&\qquad\left.+\sum_{n=2}^{\infty}\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,a_n\,z^n\right\}\\
&\quad\geq 1-\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,r\\
&\qquad-\frac{\sum_{n=2}^{\infty}[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_n(\delta,\lambda,m)\,|a_n|\,r^n}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\\
&\quad> 1-\frac{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,r\\
&\qquad-\frac{1-\beta}{1-\beta+[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\,\Phi_2(\delta,\lambda,m)}\,r\\
&\quad= 1-r>0\qquad(|z|=r<1),
\end{aligned}$$

where we have also made use of the assertion (2.2) of Theorem 2.3. This evidently proves the inequality (4.3), and hence also the subordination result (4.1) asserted by Theorem 4.4. The inequality (4.2) follows from (4.1) upon setting

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in\mathcal{K}.$$

Next we consider the function

$$q(z):=z-\frac{1-\beta}{[(1-\gamma)\sec\alpha+(1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}\,z^2, \tag{4.4}$$

$$(-\pi/2<\alpha<\pi/2,\ 0\leq\beta<1,\ 0\leq\gamma<1),$$

where $\Phi_n(\delta, \lambda, m)$ is given by (1.8), which is a member of the class $\widetilde{\mathbb{S}}^{\delta,m}_{\gamma,\lambda}(\alpha, \beta)$. Then, by using (4.1), we have

$$\frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta+((1-\gamma)\sec\alpha+(1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}\,q(z) \prec \frac{z}{1-z} \qquad (z \in \mathbb{U}).$$

It is also easily verified for the function $q(z)$ defined by (4.4) that

$$\min\left\{Re\left(\frac{[(1-\gamma)\sec\alpha + (1-\beta)(1+\gamma)]\Phi_2(\delta,\lambda,m)}{2[1-\beta+((1-\gamma)\sec\alpha+(1-\beta)(1+\gamma))\,\Phi_2(\delta,\lambda,m)]}\,q(z)\right)\right\} = -\frac{1}{2},$$

$(z \in \mathbb{U})$, which completes the proof of Theorem 4.4. □

Here, we remark that Theorem 4.4 yields the results obtained by Singh [19] for special values of the parameters m, α, β, γ. Also, in view of the Remarks 1.2–1.5 and by taking suitable values for the parameters in Theorem 4.4, we can give analogous results for the subclasses defined in these remarks and we choose to omit further details in this regard.

Acknowledgements. The authors thank the referees for their valuable comments and helpful suggestions.

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci., **2004(25-28)**, 1429–1436.

[2] F. M. Al-Oboudi and K. A. Al-Amoudi, *On classes of analytic functions related to conic domains*, J. Math. Anal. Appl., **339(1)**(2008), 655–667.

[3] F. M. Al-Oboudi and K. A. Al-Amoudi, *Subordination results for classes of analytic functions related to conic domains defined by a fractional operator*, J. Math. Anal. Appl., **354(2)**(2009), 412–420.

[4] E. Deniz, M. Çağlar and H. Orhan, *The Fekete-Szegö problem for a class of analytic functions defined by Dziok-Srivastava operator*, Kodai Math. J., **35(3)**(2012), 439–462.

[5] M. Fekete and G. Szegö, *Eine Bemerkung Uber Ungerade Schlichte Funktionen*, J. London Math. Soc., **8**(1933), 85–89.

[6] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20**(1969), 8–12.

[7] O. S. Kwon and S. Owa, *The subordination theorem for λ-spirallike functions of order α*, Int. J. Appl. Math., **11(2)**(2002), 113–119.

[8] R. J. Libera, *Univalent α-spiral functions*, Canad. J. Math., **19**(1967), 449–456.

[9] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[10] A. K. Mishra and P. Gochhayat, *Fekete-Szegö problem for a class defined by an integral operator*, Kodai Math. J., **33(2)**(2010), 310–328.

[11] G. Murugusundaramoorthy, *Subordination results for spiral-like functions associated with the Srivastava-Attiya operator* Integral Transforms Spec. Funct., **23(2)**(2012), 97–103.

[12] H. Orhan, D. Răducanu, M. Çağlar and Mustafa Bayram, *Coefficient estimates and other properties for a class of spirallike functions associated with a differential operator*, Abstr. Appl. Anal., **2013**, Art. ID 415319, 7 pp.

[13] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J., **18(1)**(1978), 53–59.

[14] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39(5)**(1987), 1057–1077.

[15] A. Pfluger, *The Fekete-Szegö inequality for complex parameters*, Complex Variables Theory Appl., **7(1-3)**(1986), 149–160.

[16] G. Ş. Sălăgean, *Subclasses of univalent functions*, in Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362–372, Lecture Notes in Math., 1013, Springer, Berlin.

[17] C. Selvaraj and K. A. Selvakumaran, *Fekete-Szegö problem for some subclasses of analytic functions*, Far East J. Math. Sci., (FJMS) **29(3)**(2008), 643–652.

[18] H. Silverman, *Sufficient conditions for spiral-likeness*, Internat. J. Math. Math. Sci., **12(4)**(1989), 641–644.

[19] S. Singh, *A subordination theorem for spirallike functions*, Int. J. Math. Math. Sci., **24(7)**(2000), 433–435.

[20] L. Špaček, *Contribution à la théorie des fonctions univalentes*, Časopis Pro Pěstování Matematiky a Fysiky, **62**(1933), 12–19.

[21] H. M. Srivastava and A. K. Mishra, *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Comput. Math. Appl., **39(3-4)**(2000), 57–69.

[22] H. M. Srivastava, A. K. Mishra and M. K. Das, *The Fekete-Szegő problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl., **44(2)**(2001), 145–163.

[23] H. M. Srivastava, A. K. Mishra and S. N. Kund, *Certain classes of analytic functions associated with iterations of the Owa-Srivastava fractional derivative operator*, Southeast Asian Bull. Math., **37(3)**(2013), 413–435.

[24] H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle*, Proc. Amer. Math. Soc., **12**(1961), 689–693.