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Convergence of an Iterative Algorithm for Systems of Variational Inequalities and Nonlinear Mappings in Banach Spaces

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Abstract. In this paper, we consider the problem of convergence of an iterative algorithm for a general system of variational inequalities, a nonexpansive mapping and an η-strictly pseudo-contractive mapping. Strong convergence theorems are established in the framework of real Banach spaces.

1. Introduction

In this paper, we are concerned with a general system of variational inequalities in Banach spaces, which involves finding $(x^*, y^*) \in C \times C$ such that

$$(1.1) \qquad \begin{cases} \langle \lambda \sum_{i=1}^N \lambda_i A_i y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu \sum_{i=1}^N \mu_i B_i x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C, \end{cases}$$

where E is a real Banach space, $C \subset E$ is a nonempty closed convex set, $A, B : C \to E$ are two nonlinear mappings, $j \in J$, $J : E \to 2^{E^*}$ is the duality mapping and λ, μ, λ_i, μ_i are positive real numbers for all $i = 1, 2, \cdots, N$ with $\sum_{i=1}^N \lambda_i = \sum_{i=1}^N \mu_i = 1$.

Special cases

(I) If $E = H$ is a real Hilbert space, then (1.1) reduces to

$$(1.2) \qquad \begin{cases} \langle \lambda \sum_{i=1}^N \lambda_i A_i y^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu \sum_{i=1}^N \mu_i B_i x^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C. \end{cases}$$

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In particular, if $A_i = B_i$ for all $i = 1, 2, \cdots, N$, then (1.2) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda \sum_{i=1}^{N} \lambda_i A_i y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu \sum_{i=1}^{N} \mu_i A_i x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{1.3}$$

Further, if $A_i = A$ for all $i = 1, 2, \cdots, N$, $\lambda = \mu = 1$ and $x^* = y^*$, then (1.3) reduces to the following classical variational inequality of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.4}$$

This problem (1.4) is a fundamental problem in Variational Analysis. Many algorithms for solving this problem are projection algorithms. One can see that the variational inequality (1.4) is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality (1.4) if and only if $x^* \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where P_C is the metric projection of H onto C, I is the identity mapping and $\lambda > 0$ is a constant.

(II) If E is still a real Banach space, $A_i = B_i$ for all $i = 1, 2, \cdots, N$, $\lambda = \mu$ and $x^* = y^*$, then (1.1) reduces to

$$\langle \sum_{i=1}^{N} \lambda_i A_i x^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{1.5}$$

which was considered by Kangtunyakarn [6]. In particular, if $A_i = A$ for all $i = 1, 2, \cdots, N$, then (1.5) reduces to

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{1.6}$$

which was considered by Aoyama et al. [1]. Note that this problem is connected with the fixed point problem for nonlinear mapping, the problem of finding a zero point of a nonlinear operator and so on. It is clear that problem (1.6) extends problem (1.4) from Hilbert spaces to Banach spaces.

Aoyama et al [1] introduced an iterative method for finding an element of the solution set of variational inequalities for an α-inverse strongly accretive mapping. To be more precise, they proved the following theorem.

Theorem 1.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let A be an α-inverse strongly accretive mapping of C into E with $S(C, A) = \{u \in C : \langle Au, j(u - v) \rangle \geq 0, \quad \forall v \in C\} \neq \phi$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \tag{1.7}$$

for every $n = 1, 2, \cdots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in [0,1]. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some

$a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

Recently, Kangtunyakarn [6] further studied the problem of finding a common element in a finite family of the set of solutions of variational inequalities and the sets of fixed points of nonexpansive and strictly pseudo-contractive mappings. More precisely, he proved the following theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \cdots, N$, let $A_i : C \to E$ be α_i-strongly accretive and L_i-Lipschitz continuous with $\overline{\alpha} = \min_{1 \le i \le N} \alpha_i$ and $\overline{L} = \max_{1 \le i \le N} L_i$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $K^2 \le \eta$, where K is the 2-uniformly smooth constant of E. Assume that $\mathcal{F} = F(T) \cap F(S) \cap \cap_{i=1}^N S(C, A_i) \neq \phi$, where $S(C, A_i) = \{u \in C : \langle A_i u, j(v - u)\rangle \ge 0, \forall v \in C\}$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) S x_n, \\ y_n = b_n x_n + (1 - b_n) T z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda \sum_{i=1}^N a_i A_i) y_n, \quad \forall n \ge 1, \end{cases}$$

where $a_i \in [0, 1]$ for all $i = 1, 2, \cdots, N$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in N$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < a \le \beta_n, \gamma_n, c_n, b_n \le b < 1$ *for some* $a, b > 0$, $\forall n \in N$ *and* $\sum_{i=1}^N a_i = 1$;

(iii) $0 \le \lambda K^2 \le \frac{\overline{\alpha}}{\overline{L}^2}$;

(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n|, \sum_{i=1}^\infty |\beta_{n+1} - \beta_n|, \sum_{n=1}^\infty |b_{n+1} - b_n|, \sum_{n=1}^\infty |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

It is main purpose in this paper to develop algorithms for a general system of variational inequalities, a nonexpansive mapping and an η-strictly pseudo-contractive mapping. Strong convergence theorem is given in a uniformly convex and 2-uniformly smooth Banach space. Our results improve and develop previously discussed variational inequalities and related algorithms (see [1,6] and the references therein).

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \quad \|x\| = \|x^*\|\}, \quad \forall x \in E.$$

If $E = H$ is a Hilbert space, then $J = I$, the identity mapping.

Let $U = \{x \in E : \|x\| = 1\}$. E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \|\frac{x + y}{2}\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to \infty} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. In this case, E is said to be smooth. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is well known that every uniformly smooth Banach space is smooth and if E is smooth, then J is single-valued which is denoted by j. Also, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of E as follows:

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| - \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\}.$$

It is well known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with with $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

Remark 2.1. Takahashi et al. [11] remind us of the fact that no Banach space is q-uniformly smooth for $q > 2$. So, in this paper, we focus on a 2-uniformly smooth Banach space.

Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to denote the fixed point set of the mapping S.

S is called η-strictly pseudo-contractive if there exists a constant $\eta \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta\|(I - T)x - (I - T)y\|^2 \tag{2.2}$$

for every $x, y \in C$ and for some $j(x-y) \in J(x-y)$. It is clear that (2.2) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta\|(I - T)x - (I - T)y\|^2 \tag{2.3}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

A mapping $A : C \to E$ is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be α-strongly accretive if there exist $j(x-y) \in J(x-y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2, \quad \forall x, y \in C.$$

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

In what follows we shall make use of the following lemmas.

Lemma 2.1.([13]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.2.([4]) *Let C be a nonempty closed convex subset of a smooth Banach space and let $T : C \to C$ be a strictly pseudo-contractive mapping. Then the fixed point set $F(T)$ is a closed and convex subset of E.*

Lemma 2.3.([9]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4.([5]) *Let E be a uniformly convex Banach space and $B_r = \{x \in E : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\| \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x-y\|)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.5.([2]) *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_i : i \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose*

$\cap_{i=1}^{\infty} F(T_i) \neq \phi$. Let $\{\xi_n\}$ be a sequence of positive numbers with $\sum_{i=1}^{\infty} \xi_i = 1$. Then a mapping T on C defined by $Tx = \sum_{i=1}^{\infty} \xi_i T_i x$ for $x \in C$ is well-defined, nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$ holds.

Lemma 2.6.([14]) *Assume that $\{\alpha_n\}$ is a sequence of nonexpansive real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.7.([16]) *Let C be a closed convex subset of a real uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that for any given $u \in C$, $Q_{F(T)}u = \lim_{t\to 0} x_t$ and*

$$\limsup_{n\to\infty} \langle u - Q_{F(T)}u, j(x_n - Q_{F(T)}u)\rangle \leq 0,$$

where $x_t = tu + (1-t)Tx_t$ for every $t \in (0,1)$.

Lemma 2.8.([7]) *Let C be a closed convex subset of a smooth Banach space E, let D be a nonempty subset of C and Q be a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Lemma 2.9. *For given $(x^*, y^*) \in C \times C$, where $y^* = Q_C(x^* - \mu \sum_{i=1}^{N} \mu_i B_i x^*)$, (x^*, y^*) is a solution of problem (1.1) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = Q_C[Q_C(x - \mu \sum_{i=1}^{N} \mu_i B_i x) - \lambda \sum_{i=1}^{N} \lambda_i A_i Q_C(x - \mu \sum_{i=1}^{N} \mu_i B_i x)], \quad \forall x \in C,$$

where $\lambda_i, \mu_i > 0$ $(i = 1, 2, \cdots, N)$ are constants and Q_C is a sunny nonexpansive retraction from E onto C.

Proof. We can rewrite (1.1) as

$$\text{(2.4)} \qquad \begin{cases} \langle x^* - (y^* - \lambda \sum_{i=1}^{N} \lambda_i A_i y^*), j(x - x^*)\rangle \geq 0, & x \in C, \\ \langle y^* - (x^* - \mu \sum_{i=1}^{N} \mu_i B_i x^*), j(x - y^*) \geq 0, & \forall x \in C. \end{cases}$$

By Lemma 2.8, we can check (2.4) is equivalent to

$$\begin{cases} x^* = Q_C(y^* - \lambda \sum_{i=1}^N \lambda_i A_i y^*), \\ y^* = Q_C(x^* - \mu \sum_{i=1}^N \mu_i B_i x^*). \end{cases}$$

This completes the proof. □

3. Main Results

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \cdots, N$, let $A_i, B_i : C \to E$ be α_i-strongly accretive, L_i-Lipschitz continuous and β_i-strongly accretive, M_i-Lipschitz continuous with $\alpha = \min_{1\le i\le N} \alpha_i$, $L = \max_{1\le i\le N} L_i$, $\beta = \min_{1\le i\le N} \beta_i$ and $M = \max_{1\le i\le N} M_i$, respectively. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $K^2 \le \eta$. Assume that $\mathcal{F} = F(T) \cap F(S) \cap F(D) \ne \phi$, where D is defined as Lemma 2.9. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) S x_n, \\ y_n = b_n x_n + (1 - b_n) T z_n, \\ u_n = Q_C(I - \mu \sum_{i=1}^N \mu_i B_i) y_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda \sum_{i=1}^N \lambda_i A_i) u_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where, $\lambda_i, \mu_i \in [0, 1]$ for all $i = 1, 2, \cdots, N$ with $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \lambda_i = 1$, $\lambda \in (0, \frac{\alpha}{K^2L^2}]$, $\mu \in (0, \frac{\beta}{K^2L^2}]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{b_n\}$, $\{c_n\} \subset (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = 0$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\sum_{n=1}^\infty |b_{n+1} - b_n| < \infty$ *and* $\sum_{n=1}^\infty |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\overline{x} = Q_{\mathcal{F}} u$ and $(\overline{x}, \overline{y})$ is a solution of the problem (1.1), where $\overline{y} = Q_C(\overline{x} - \mu \sum_{i=1}^N \mu_i B_i \overline{x})$ and $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we show that $\mathcal{F} = F(T) \cap F(S) \cap F(D)$ is closed and convex. We know from Lemma 2.2 and Theorem 4.5.3 of [10] that $F(T)$, $F(S)$ are closed and convex.

Next, we show that $F(D)$ is closed and convex. Indeed, from the α_i-strongly accretivity and the L_i-Lipschitz continuity of A_i $(i = 1, 2, \cdots, N)$, we have

$$\begin{aligned}\langle\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y, j(x-y)\rangle &= \sum_{i=1}^{N}\lambda_i\langle A_i x - A_i y, j(x-y)\rangle \\ &\geq \sum_{i=1}^{N}\lambda_i\alpha_i\|x-y\|^2 \\ &\geq \sum_{i=1}^{N}\lambda_i\frac{\alpha_i}{L_i^2}\|A_i x - A_i y\|^2 \\ &\geq \frac{\alpha}{L^2}\sum_{i=1}^{N}\lambda_i\|A_i x - A_i y\|^2 \\ &\geq \frac{\alpha}{L^2}\|\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y\|^2. \end{aligned} \tag{3.2}$$

It follows from (3.2) and Lemma 2.1 that for all $x, y \in C$, we have

$$\begin{aligned}&\|(I-\lambda\sum_{i=1}^{N}\lambda_i A_i)x - (I-\lambda\sum_{i=1}^{N}\lambda_i A_i)y\|^2 \\ &= \|x-y-\lambda(\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y)\|^2 \\ &\leq \|x-y\|^2 - 2\lambda\langle\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y, j(x-y)\rangle \\ &\quad + 2K^2\lambda^2\|\sum_{i=1}^{N}\lambda_i A_i x - -\sum_{i=1}^{n}\lambda_i A_i y\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \|x-y\|^2 - 2\lambda\frac{\alpha}{L^2}\|\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y\|^2 \\ &\quad + 2K^2\lambda^2\|\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y\|^2 \\ &= \|x-y\|^2 - 2\lambda(\frac{\alpha}{L^2} - K^2\lambda)\|\sum_{i=1}^{N}\lambda_i A_i x - \sum_{i=1}^{N}\lambda_i A_i y\|^2 \\ &\leq \|x-y\|^2. \end{aligned} \tag{3.3}$$

This shows that $I - \lambda\sum_{i=1}^{N}\lambda_i A_i$ is a nonexpansive mapping. So is $I - \mu\sum_{i=1}^{N}\mu_i B_i$.

From Lemma 2.9, we can see that

$$\begin{aligned} D &= Q_C[Q_C(I-\mu\sum_{i=1}^{N}\mu_i B_i) - \lambda\sum_{i=1}^{N}\lambda_i A_i Q_C(I-\mu\sum_{i=1}^{N}\mu_i B_i)] \\ &= Q_C(I-\lambda\sum_{i=1}^{N}\lambda_i A_i)Q_C(I-\mu\sum_{i=1}^{N}\mu_i B_i). \end{aligned}$$

So, D is nonexpansive. This shows that $\mathcal{F} = F(T)\cap F(S)\cap F(D)$ is closed and convex.

Letting $x^* \in \mathcal{F}$, we obtain from Lemma 2.9 that

$$x^* = Q_C[Q_C(x^*-\mu\sum_{i=1}^{N}\mu_i B_i x^*) - \lambda\sum_{i=1}^{N}\lambda_i A_i Q_C(x^*-\mu\sum_{i=1}^{N}\mu_i B_i x^*)].$$

Putting $y^* = Q_C(x^*-\mu\sum_{i=1}^{N}\mu_i B_i x^*)$, we see that $x^* = Q_C(y^*-\lambda\sum_{i=1}^{N}\lambda_i A_i y^*)$. Since S is an η-strictly pseudo-contractive mapping, we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|x_n - x^* + (1-c_n)(Sx_n - x_n)\|^2 \\ &\le \|x_n - x^*\|^2 + 2(1-c_n)\langle Sx_n - x_n, j(x_n - x^*)\rangle \\ &\quad + 2K^2(1-c_n)^2\|Sx_n - x_n\|^2 \\ &= \|x_n - x^*\|^2 - 2(1-c_n)\langle (I-S)x_n - (I-S)x^*, j(x_n - x^*)\rangle \\ &\quad + 2K^2(1-c_n)^2\|(I-S)x_n\|^2 \\ &\le \|x_n - x^*\|^2 - 2(1-c_n)\eta\|(I-S)x_n - (I-S)x^*\|^2 \\ &\quad + 2K^2(1-c_n)^2\|(I-S)x_n\|^2 \\ &= \|x_n - x^*\|^2 - 2(1-c_n)^2(\eta - K^2)\|(I-S)x_n\|^2 \\ &\le \|x_n - x^*\|^2. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\begin{aligned} \|Q_C(u_n-\lambda\sum_{i=1}^{N}\lambda_i A_i u_n) - x^*\| &= \|Q_C(u_n-\lambda\sum_{i=1}^{N}\lambda_i A_i u_n) - Q_C(y^*-\lambda\sum_{i=1}^{N}\lambda_i A_i y^*)\| \\ &\le \|u_n - y^*\| \\ &= \|Q_C(y_n-\mu\sum_{i=1}^{N}\mu_i B_i y_n) - Q_C(x^*-\mu\sum_{i=1}^{N}\mu_i B_i x^*)\| \\ &\le \|y_n - x^*\| \\ &= \|b_n(x_n - x^*) + (1-b_n)(Tz_n - x^*)\| \\ &\le b_n\|x_n - x^*\| + (1-b_n)\|z_n - x^*\| \\ &\le b_n\|x_n - x^*\| + (1-b_n)\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n u+\beta_n x_n+\gamma_n Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_n - x^*\| \\ &\le \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|Q_C(u_n-\lambda\sum_{i=1}^N\lambda_i A_i u_n)-x^*\| \\ &\le \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\| \\ &= \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\| \\ &\le \max\{\|u-x^*\|,\|x_1-x^*\|\},\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. So are $\{u_n\}$,$\{y_n\}$, $\{z_n\}$, $\{Sx_n\}$ and $\{Tz_n\}$. From (3.1), we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|c_{n+1}x_{n+1}+(1-c_{n+1})Sx_{n+1}-c_n x_n-(1-c_n)Sx_n\| \\ &= \|c_{n+1}(x_{n+1}-x_n)+(c_{n+1}-c_n)x_n \\ &\quad +(1-c_{n+1})(Sx_{n+1}-Sx_n)+(c_n-c_{n+1})Sx_n\| \\ &\le \|c_{n+1}(x_{n+1}-x_n)+(1-c_{n+1})(Sx_{n+1}-Sx_n)\| \\ &\quad +|c_{n+1}-c_n|\|x_n\|+|c_n-c_{n+1}|\|Sx_n\|. \end{aligned} \tag{3.6}$$

Since S is an η-strictly pseudo-contractive mapping, we have

$$\begin{aligned}&\|c_{n+1}(x_{n+1}-x_n)+(1-c_{n+1})(Sx_{n+1}-Sx_n)\|^2 \\ &= \|x_{n+1}-x_n-(1-c_{n+1})((I-S)x_{n+1}-(I-S)x_n)\|^2 \\ &\le \|x_{n+1}-x_n\|^2-2(1-c_{n+1})\langle (I-S)x_{n+1}-(I-S)x_n, j(x_{n+1}-x_n)\rangle \\ &\quad +2K^2(1-c_{n+1})^2\|(I-S)x_{n+1}-(I-S)x_n\|^2 \\ &\le \|x_{n+1}-x_n\|^2-2(1-c_{n+1})\eta\|(I-S)x_{n+1}-(I-S)x_n\|^2 \\ &\quad +2K^2(1-c_{n+1})^2\|(I-S)x_{n+1}-(I-S)x_n\|^2 \\ &\le \|x_{n+1}-x_n\|^2-2(1-c_{n+1})^2(\eta-K^2)\|(I-S)x_{n+1}-(I-S)x_n\|^2 \\ &\le \|x_{n+1}-x_n\|^2. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$
\begin{aligned}
&\|Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_{n+1} - Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_n\| \\
&\le \|u_{n+1}-u_n\| \\
&= \|Q_C(I-\mu\sum_{i=1}^N \mu_i B_i)y_{n+1} - Q_C(I-\mu\sum_{i=1}^N \mu_i B_i)y_n\| \\
&\le \|y_{n+1}-y_n\| \\
&= \|b_{n+1}x_{n+1}+(1-b_{n+1})Tz_{n+1}-b_nx_n-(1-b_n)Tz_n\| \\
&\le b_{n+1}\|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|x_n\|+(1-b_{n+1})\|Tz_{n+1}-Tz_n\| \\
&\quad +|b_{n+1}-b_n|\|Tz_n\| \\
&\le b_{n+1}\|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|x_n\|+(1-b_{n+1})\|z_{n+1}-z_n\| \\
&\quad +|b_{n+1}-b_n|\|Tz_n\| \\
&\le b_{n+1}\|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|x_n\|+(1-b_{n+1})[\|c_{n+1}(x_{n+1}-x_n) \\
&\quad +(1-c_{n+1})(Sx_{n+1}-Sx_n)\|+|c_{n+1}-c_n|\|x_n\|+|c_n-c_{n+1}|\|Sx_n\|] \\
&\quad +|b_{n+1}-b_n|\|Tz_n\| \\
&\le b_{n+1}\|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|x_n\|+(1-b_{n+1})\|x_{n+1}-x_n\| \\
&\quad +|c_{n+1}-c_n|\|x_n\|+|c_n-c_{n+1}|\|Sx_n\|+|b_{n+1}-b_n|\|Tz_n\| \\
&= \|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|x_n\|+|c_{n+1}-c_n|\|x_n\| \\
&\quad +|c_n-c_{n+1}|\|Sx_n\|+|b_{n+1}-b_n|\|Tz_n\|. \qquad (3.8)
\end{aligned}
$$

Next, we claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Putting $t_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for each $n\ge 1$, we see that

$$x_{n+1}=(1-\beta_n)t_n+\beta_n x_n, \quad \forall n\ge 1. \tag{3.9}$$

From

$$
\begin{aligned}
&t_{n+1}-t_n \\
&= \frac{\alpha_{n+1}u+\gamma_{n+1}Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u+\gamma_n Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_{n+1})+\frac{\alpha_n}{1-\beta_n}(Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_n-u) \\
&\quad +Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_{n+1}-Q_C(I-\lambda\sum_{i=1}^N\lambda_iA_i)u_n,
\end{aligned}
$$

we have

$$\|t_{n+1}-t_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Q_c(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_n-u\| + \|Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_{n+1}-Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_n\|. \tag{3.10}$$

Substituting (3.8) into (3.10), we obtain

$$\begin{aligned}\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\| \le{}& \frac{\alpha_n}{1-\beta_{n+1}}\|u-Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_{n+1}\| \\ &+ \frac{\alpha_n}{1-\beta_n}\|Q_C(I-\lambda\sum_{i=1}^N \lambda_i A_i)u_n-u\| \\ &+ |b_{n+1}-b_n|\|x_n\| + |c_{n+1}-c_n|\|x_n\| \\ &+ |c_n-c_{n+1}|\|Sx_n\| + |b_{n+1}-b_n|\|Tz_n\|.\end{aligned}$$

It follows from the condition (C1)-(C3) that

$$\limsup_{n\to\infty}(\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|) \le 0.$$

From Lemma 2.3, we have

$$\lim_{n\to\infty}\|t_n-x_n\| = 0. \tag{3.11}$$

Thanks to (3.9), we see that

$$x_{n+1}-x_n = (1-\beta_n)(t_n-x_n),$$

which combines with (3.11) yields that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.12}$$

From (3.4), (3.5) and Lemma 2.4, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x^*\|^2 \\
&\quad - \beta_n \gamma_n g_1(\|Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x_n\|) \\
&\le \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\|^2 + \gamma_n \|b_n(x_n - x^*) + (1 - b_n)(Tz_n - x^*)\|^2 \\
&\quad - \beta_n \gamma_n g_1(\|Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x_n\|) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n [b_n \|x_n - x^*\|^2 + (1 - b_n)\|z_n - x^*\|^2 - b_n(1 - b_n) g_2(\|x_n - Tz_n\|)]
\end{aligned}
$$

$$
\begin{aligned}
&- \beta_n \gamma_n g_1(\|Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x_n\|) \\
\le\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&+ \gamma_n [b_n \|x_n - x^*\|^2 + (1 - b_n)\{\|x_n - x^*\|^2 - 2(1 - c_n)^2(\eta - K^2)\|Sx_n - x_n\|^2\} \\
&- b_n(1 - b_n) g_2(\|x_n - Tz_n\|)] - \beta_n \gamma_n g_1(\|Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i)u_n - x_n\|) \\
\le\ & \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - 2\gamma_n(1 - b_n)(1 - c_n)^2(\eta - K^2)\|Sx_n - x_n\|^2 \\
&- \gamma_n b_n(1 - b_n) g_2(\|x_n - Tz_n\|) - \beta_n \gamma_n g_1(\|Dy_n - x_n\|).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&2\gamma_n(1 - b_n)(1 - c_n)^2(\eta - K^2)\|Sx_n - x_n\|^2 + \gamma_n b_n(1 - b_n) g_2(\|x_n - Tz_n\|) \\
&+ \beta_n \gamma_n g_1(\|Dy_n - x_n\|) \\
&\le \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*)\|x_{n+1} - x_n\|.
\end{aligned}
$$

From (3.12), restrictions (C1) and (C2), we have

$$
\begin{aligned}
\lim_{n\to\infty} \|Sx_n - x_n\| &= \lim_{n\to\infty} g_1(\|Dy_n - x_n\|) \\
&= \lim_{n\to\infty} g_2(\|x_n - Tz_n\|) \\
&= 0. \qquad (3.13)
\end{aligned}
$$

From the properties of g_1 and g_2, we have

$$\lim_{n\to\infty}\|Dy_n - x_n\| = \lim_{n\to\infty}\|x_n - Tz_n\| = 0. \tag{3.14}$$

From (3.13) and the definition of z_n, we have

$$\begin{aligned}\lim_{n\to\infty}\|z_n - x_n\| &= \lim_{n\to\infty}\|c_n x_n + (1-c_n)Sx_n - x_n\|\\ &= \lim_{n\to\infty}(1-c_n)\|Sx_n - x_n\|\\ &= 0. \end{aligned}\tag{3.15}$$

From (3.14) and the definition of y_n, we have

$$\begin{aligned}\lim_{n\to\infty}\|y_n - x_n\| &= \lim_{n\to\infty}\|b_n x_n + (1-b_n)Tz_n - x_n\|\\ &= \lim_{n\to\infty}(1-b_n)\|x_n - Tz_n\|\\ &= 0. \end{aligned}\tag{3.16}$$

Since

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tz_n\| + \|Tz_n - x_n\|\\ &\le \|x_n - z_n\| + \|Tz_n - x_n\|,\end{aligned}$$

(3.14) and (3.15), we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.17}$$

Define the mapping $G : C \to C$ by

$$Gx = \alpha Q_C(I - \lambda\sum_{i=1}^{N}\lambda_i A_i)Q_C(I - \mu\sum_{i=1}^{N}\mu_i B_i)x + \beta Tx + \gamma Wx, \quad \forall x \in C,$$

where $Wx = cx + (1-c)Sx$ for all $x \in C$ and $\alpha, \beta, \gamma, c \in [0,1]$ with $\alpha + \beta + \gamma = 1$. We show that W is a nonexpansive mapping. Let $x, y \in C$. Then we have

$$\begin{aligned}\|Wx - Wy\|^2 &= \|(cx + (1-c)Sx) - (cy + (1-c)Sy)\|^2\\ &= \|x - y - (1-c)((I-S)x - (I-S)y)\|^2\\ &\le \|x-y\|^2 - 2(1-c)\langle (I-S)x - (I-S)y, j(x-y)\rangle\\ &\quad + 2K^2(1-c)^2\|(I-S)x - (I-S)y\|^2\\ &\le \|x-y\|^2 - 2(1-c)\eta\|(I-S)x - (I-S)y\|^2\\ &\quad + 2K^2(1-c)^2\|(I-S)x - (I-S)y\|^2\\ &\le \|x-1\|^2 - 2(1-c)^2(\eta - K^2)\|(I-S)x - (I-S)y\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

Therefore, W is a nonexpansive mapping. It is easy to see that the mapping G is nonexpansive. From Lemma 2.5 and the definition of G, we have

$$\begin{aligned} F(G) &= F(Q_C(I-\lambda\sum_{i=1}^{N}\lambda_i A_i)Q_C(I-\mu\sum_{i=1}^{N}\mu_i A_i))\cap F(T)\cap F(S) \\ &= F(D)\cap F(T)\cap F(S) \\ &= \mathcal{F}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|x_n - Gx_n\| &= \|x_n - (\alpha Dx_n + \beta Tx_n + \gamma Wx_n)\| \\ &\le \alpha\|x_n - Dx_n\| + \beta\|x_n - Tx_n\| + \gamma\|x_n - Wx_n\| \\ &\le \alpha\|x_n - Dx_n\| + \beta\|x_n - Tx_n\| + \gamma(1-c)\|x_n - Sx_n\| \\ &\le \alpha(\|x_n - Dy_n\| + \|Dy_n - Dx_n\|) + \beta\|x_n - Tx_n\| + \gamma(1-c)\|x_n - Sx_n\| \\ &\le \alpha(\|x_n - Dy_n\| + \|y_n - x_n\|) + \beta\|x_n - Tx_n\| + \gamma(1-c)\|x_n - Sx_n\|. \end{aligned}$$

From (3.13), (3.14), (3.16) and (3.17), we have

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.18}$$

Let z_t be the fixed point of the contraction $z \mapsto tu+(1-t)Gz$, where $t\in(0,1)$. That is, $z_t = tu + (1-t)Gz_t$. By Lemma 2.7, there exists a unique sunny nonexpansive retraction $Q_{F(G)} : C \to F(G)$ such that $\lim_{t\to 0} z_t = Q_{F(G)}u = Q_{\mathcal{F}}u = \overline{x}$ and

$$\limsup_{n\to\infty}\langle u-\overline{x}, j(x_n-\overline{x})\rangle \le 0. \tag{3.19}$$

Finally, we show that $\lim_{n\to\infty} x_n = \overline{x}$.

Observe that

$$\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &= \|\alpha_n u+\beta_n x_n+\gamma_n Q_C(I-\lambda\sum_{i=1}^N\lambda_i A_i)u_n-\overline{x}\|^2\\
&= \langle \alpha_n u+\beta_n x_n+\gamma_n Q_C(I-\lambda\sum_{i=1}^N\lambda_i A_i)u_n-\overline{x}, j(x_{n+1}-\overline{x})\rangle\\
&= \alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle+\beta_n\langle x_n-\overline{x}, j(x_{n+1}-\overline{x})\rangle\\
&\quad +\gamma_n\langle Q_C(I-\lambda\sum_{i=1}^N\lambda_i A_i)u_n-\overline{x}, j(x_{n+1}-\overline{x})\rangle\\
&\le \alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle+\beta_n\|x_n-\overline{x}\|\|x_{n+1}-\overline{x}\|\\
&\quad +\gamma_n\|Q_C(I-\lambda\sum_{i=1}^N\lambda_i A_i)u_n-\overline{x}\|\|x_{n+1}-\overline{x}\|\\
&\le \alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle+\beta_n\|x_n-\overline{x}\|\|x_{n+1}-\overline{x}\|\\
&\quad +\gamma_n\|x_n-\overline{x}\|\|x_{n+1}-\overline{x}\|\\
&= \alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle+(1-\alpha_n)\|x_n-\overline{x}\|\|x_{n+1}-\overline{x}\|\\
&\le \alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle+\frac{1-\alpha_n}{2}(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2),
\end{aligned}$$

which implies that

$$\|x_{n+1}-\overline{x}\|^2\le(1-\alpha_n)\|x_n-\overline{x}\|^2+2\alpha_n\langle u-\overline{x}, j(x_{n+1}-\overline{x})\rangle.$$

From the restriction (C1), (3.19) and Lemma 2.6, we obtain that $\lim_{n\to\infty}\|x_n-\overline{x}\|=0$. This completes the proof. □

It is well known that the smooth constant $K=\frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem 3.1, we can the following result immediately.

Corollary 3.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H. For every $i=1,2,\cdots,N$, $A_i, B_i: C\to H$ be α_i-strongly monotone mapping and L_i-Lipschitz continuous and β_i-strongly monotone mapping and M_i-Lipschitz continuous with $\alpha=\min_{1\le i\le N}\alpha_i$, $L=\max_{1\le i\le N}L_i$, $\beta=\min_{1\le i\le N}\beta_i$ and $M=\max_{1\le i\le N}M_i$, respectively. Let $T: C\to C$ be a nonexpansive mapping and $S: C\to C$ be an η-strictly pseudo-contractive mapping with $\frac{1}{2}\le\eta$. Assume that $\mathcal{F}=F(T)\cap F(S)\cap F(D)\neq\phi$, where D is a mapping from C into itself defined by $D(x)=P_C[P_C(x-\mu\sum_{i=1}^N\mu_i B_i x)-\lambda\sum_{i=1}^N\lambda_i A_i P_C(x-\mu\sum_{i=1}^N\mu_i B_i x)]$ for every $x\in C$ and P_C is a metric projection of H onto C. Let $\{x_n\}$ be a sequence generated* *by* $u, x_1 \in C$ *and*

$$\begin{cases} z_n = c_n x_n + (1-c_n) S x_n, \\ y_n = b_n x_n + (1-b_n) T z_n, \\ u_n = P_C(I - \mu \sum_{i=1}^N \mu_i B_i) z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda \sum_{i=1}^N \lambda_i A_i) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\lambda_i, \mu_i \in [0,1]$ *for all* $i = 1, 2, \cdots, N$ *with* $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \lambda_i = 1$, $\lambda \in (0, \frac{2\alpha}{L^2}]$, $\mu \in (0, \frac{2\beta}{M^2}]$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$,$\{b_n\}$, $\{c_n\} \subset (0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\sum_{n=1}^\infty |b_{n+1} - b_n| < \infty$ *and* $\sum_{n=1}^\infty |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $\overline{x} = P_{\mathcal{F}} u$ *and* $(\overline{x}, \overline{y})$ *is a solution of the problem* (1.2)*, where* $\overline{y} = P_C(\overline{x} - \mu \sum_{i=1}^N \mu_i B_i \overline{x})$.

Remark 3.1. (1) Since L^p for all $p \geq 2$ is uniformly convex and 2-uniformly smooth, we see that Theorem 3.1 is applicable to L^p for all $p \geq 2$.

(2) Aoyama's algorithm (1.7) has weak convergence for solving the variational inequality (1.6) and Kangtunyakarn's algorithm (1.8) has strong convergence for solving the variational inequality (1.5). However, our explicit method (3.1) have strong convergence for solving the general system of variational inequalities (1.1).

4. Applications

The computation of common fixed points is important in the study of many real world problems including the inverse problems, the split feasibility problems and the convex feasibility problems in signal processing and image reconstruction (see [3, 12] and the references therein).

Lemma 4.1.([15]) *Let* E *be a smooth Banach space and let* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that* $S_i : C \to C$ *is an* η_i*-strict pseudo-contractive mapping for each* $1 \leq i \leq N$ *such that* $\cap_{i=1}^N F(S_i) \neq \phi$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*. Then* $F(\sum_{i=1}^N S_i) = \cap_{i=1}^N F(S_i)$.

Theorem 4.1. *Let* E *be a uniformly convex and* 2*-uniformly smooth Banach space with the best smooth constant* K*,* C *be a nonempty closed convex subset of* E *and* Q_C *be a sunny nonexpansive retraction from* E *onto* C*. For every* $i = 1, 2, \cdots, N$*, let* $A_i, B_i : C \to E$ *be* α_i*-strongly accretive,* L_i*-Lipschitz continuous and* β_i*-strongly accretive,* M_i*-Lipschitz continuous with* $\alpha = \min_{1\leq i\leq N} \alpha_i$*,* $L = \max_{1\leq i\leq N} L_i$*,* $\beta = \min_{1\leq i\leq N} \beta_i$ *and* $M = \max_{1\leq i\leq N} M_i$*, respectively. Let* $T_i : C \to C$ *be a nonexpansive mapping and* $S_i : C \to C$ *be an* η_i*-strictly pseudo-contractive mapping with* $K^2 \leq \eta$ *for each* $1 \leq i \leq N$*, where* $\eta = \min\{\eta_i : 1 \leq i \leq N\}$*. Assume* *that* $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \cap F(D) \neq \phi$, *where* D *is defined as Lemma 2.9. Let* $\{x_n\}$ *be a sequence generated by* $u, x_1 \in C$ *and*

$$\begin{cases} z_n = c_n x_n + (1-c_n)\sum_{i=1}^{N} \zeta_i S_i x_n, \\ y_n = b_n x_n + (1-b_n)\sum_{i=1}^{N} \xi_i T_i z_n, \\ u_n = Q_C(I - \mu \sum_{i=1}^{N} \mu_i B_i) y_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda \sum_{i=1}^{N} \lambda_i A_i) u_n, \quad \forall n \geq 1, \end{cases}$$

where $\lambda_i, \mu_i, \xi_i, \zeta_i \in [0,1]$ *for all* $i = 1, 2, \cdots, N$ *with* $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \xi_i = \sum_{i=1}^{N} \zeta_i = 1$, $\lambda \in (0, \frac{\alpha}{K^2 L^2}]$, $\mu \in (0, \frac{\beta}{K^2 M^2}]$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{c_n\}$, $\{b_n\} \subset (0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$ *and* $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $\overline{x} = Q_{\mathcal{F}} u$ *and* $(\overline{x}, \overline{y})$ *is a solution of the problem* (1.1), *where* $\overline{y} = Q_C(\overline{x} - \mu \sum_{i=1}^{N} \mu_i B_i \overline{x})$ *and* $Q_{\mathcal{F}}$ *is a sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. Putting $S = \sum_{i=1}^{N} \zeta_i S_i$, we see that S is a η-strictly pseudo-contractive mapping, where $\eta = \min\{\eta_i : 1 \leq i \leq N\}$. Indeed, we have the following:

$$\begin{aligned} \langle Sx - Sy, j(x-y)\rangle &= \zeta_1 \langle S_1 x - S_1 y, j(x-y)\rangle + \zeta_2 \langle S_2 x - S_2 y, j(x-y)\rangle \\ &\quad + \cdots + \zeta_N \langle S_N x - S_N y, j(x-y)\rangle \end{aligned}$$

$$\begin{aligned} &\leq \zeta_1(\|x-y\|^2 - \eta_1 \|(I-S_1)x - (I-S_1)y\|^2) \\ &\quad + \zeta_2(\|x-y\|^2 - \eta_2 \|(I-S_2)x - (I-S_2)y\|^2) + \cdots \\ &\quad + \zeta_N(\|x-y\|^2 - \eta_N \|(I-S_N)x - (I-S_N)y\|^2) \\ &\leq \|x-y\|^2 - \eta(\zeta_1 \|(I-S_1)x - (I-S_1)y\|^2 \\ &\quad + \zeta_2 \|(I-S_2)x - (I-S_2)y\|^2 + \cdots \\ &\quad + \zeta_N \|(I-S_N)x - (I-S_N)y\|^2) \\ &\leq \|x-y\|^2 - \eta \|(I-S)x - (I-S)y\|^2. \end{aligned}$$

This proves that $S = \sum_{i=1}^{N} \zeta_i S_i$ is a η-strictly pseudo-contractive mapping. From Lemma 2.5, 4.1 and Theorem 3.1, we can conclude the desired results. □

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