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# Some Paranormed Difference Sequence Spaces Derived by Using Generalized Means

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Abstract. This paper presents some new paranormed sequence spaces $X(r, s, t, p; \Delta)$ where $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ defined by using generalized means and difference operator. It is shown that these are complete linear metric spaces under suitable paranorms. Furthermore, the $\alpha$-, $\beta$-, $\gamma$- duals of these sequence spaces are computed and also obtained necessary and sufficient conditions for some matrix transformations from $X(r, s, t, p; \Delta)$ to $X$. Finally, it is proved that the sequence space $l(r, s, t, p; \Delta)$ is rotund when $p_n > 1$ for all $n$ and has the Kadec-Klee property.

## 1. Introduction

The study of sequence spaces play an important role in several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory. Besides this, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results with the help of Schauder basis.

Let $w$ be the space of all real or complex sequences $x = (x_n)$, $n \in \mathbb{N}_0$. For an infinite matrix $A$ and a sequence space $\lambda$, the matrix domain of $A$, which is denoted by $\lambda_A$ and defined as $\lambda_A = \{x \in w : Ax \in \lambda\}$ [3]. Basic methods, which are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain $\lambda_A$. Recently, there is an approach of forming new sequence spaces by using matrix domain of a suitable

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matrix and characterize the matrix mappings between these sequence spaces.
Let $(p_k)_{k=0}^{\infty}$ be a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1, H\}$. The linear spaces $l_\infty(p)$, $c(p), c_0(p)$ and $l(p)$ are introduced and studied by Maddox [14], where

$$l_\infty(p) = \Big\{x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty\Big\},$$
$$c(p) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some scalar } l\Big\} \text{ and },$$
$$c_0(p) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\Big\},$$
$$l(p) = \Big\{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty\Big\}.$$

The linear space $c_0(p)$ is a complete linear metric space with respect to the paranorm $g(x) = \sup_{k\in\mathbb{N}_0} |x_k|^{\frac{p_k}{M}}$. But the spaces $c(p), l_\infty(p)$ are fail to be linear metric space because the continuity of scalar multiplication is not hold for them. The spaces $c(p)$ and $l_\infty(p)$ are complete linear metric space with respect to the paranorm $g(x)$ iff $\inf p_k > 0$ for all $k$. The space $l(p)$ is a complete linear metric space with the paranorm $\tilde{g}(x) = \left(\sum_{k=0}^{\infty} |x_k|^{p_k}\right)^{\frac{1}{M}}$.
Recently, several authors introduced new sequence spaces by using matrix domain. For example, Başar et al. [11] studied the space $bs(p) = [l_\infty(p)]_S$, where $S$ is the summation matrix. Altay and Başar [5] studied the sequence spaces $r^t(p)$ and $r^t_\infty(p)$, which consist of all sequences whose Riesz transform are in the spaces $l(p)$ and $l_\infty(p)$ respectively, i.e., $r^t(p) = [l(p)]_{R^t}$ and $r^t_\infty(p) = [l_\infty(p)]_{R^t}$. Altay and Başar also studied the sequence spaces $r^t_c(p) = [c(p)]_{R^t}$ and $r^t_0(p) = [c_0(p)]_{R^t}$ in [4]. Using weighted mean Altay and Başar have introduced and studied new paranormed sequence spaces in [6] and [7]. Some recent results related to duals and matrix transformations on sequence spaces can be found in [8] and [20].
Kizmaz [13] first introduced and studied the difference sequence space. Later on, many authors including Ahmad and Mursaleen [25], Çolak and Et [18], Başar and Altay[4] etc. studied new sequence spaces defined by using difference operator. Using Euler and difference operator, Karakaya and Polat introduced the paranormed sequence spaces $e^\alpha_0(p;\Delta), e^\alpha_c(p;\Delta)$ and $e^\alpha_\infty(p;\Delta)$ in [22]. Mursaleen and Noman [17] introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces.
In 2012, Demiriz and Çakan [21] introduced new paranormed difference sequence space $\lambda(u, v, p; \Delta)$ for $\lambda \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, combining weighted mean and difference operator, defined as

$$\lambda(u, v, p; \Delta) = \Big\{x \in w : (G(u, v).\Delta)x \in \lambda\Big\},$$

where the matrices $G(u,v) = (g_{nk})$ and $\Delta = (\delta_{nk})$ are given by

$$g_{nk} = \begin{cases} u_n v_k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n \end{cases} \quad \text{and } \delta_{nk} = \begin{cases} 0 & \text{if } 0 \le k < n-1, \\ (-1)^{n-k} & \text{if } n-1 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

By using matrix domain, one can write $c_0(u,v,p;\Delta) = [c_0(p)]_{G(u,v;\Delta)}$, $c(u,v,p;\Delta) = [c(p)]_{G(u,v;\Delta)}$, $l_\infty(u,v,p;\Delta) = [l_\infty(p)]_{G(u,v;\Delta)}$ and $l(u,v,p;\Delta) = [l(p)]_{G(u,v;\Delta)}$.
The aim of this present paper is to introduce and study new sequence space $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$. It is shown that these spaces are complete paranormed sequence spaces under some suitable paranorms. Some topological results and the $\alpha$-, $\beta$-, $\gamma$- duals of these spaces are obtained. A characterization of some matrix transformations between these new sequence spaces is established. It is also shown that the sequence space $l(r,s,t,p;\Delta)$ is rotund when $p_n > 1$ for all $n$ and has the Kadec-Klee property.

## 2. Preliminaries

Let $l_\infty, c$ and $c_0$ be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively, with norm $\|x\|_\infty = \sup_n |x_n|$. Let $bs$ and $cs$ be the sequence spaces of all bounded and convergent series respectively. We denote by $e = (1,1,\cdots)$ and $e_n$ for the sequence whose $n$-th term is 1 and others are zero and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers.
For any subsets $U$ and $V$ of $w$, the multiplier space $M(U,V)$ of $U$ and $V$ is defined as

$$M(U,V) = \{a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u \in U\}.$$

In particular,

$$U^\alpha = M(U,l_1),\ U^\beta = M(U,cs) \text{ and } U^\gamma = M(U,bs)$$

are called the $\alpha$-, $\beta$- and $\gamma$- duals of $U$ respectively [3].
Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries $a_{nk}$. We write $A_n$ as the sequence of the $n$-th row of $A$, i.e., $A_n = (a_{nk})_k$ for every $n$. For $x = (x_n) \in w$, the $A$-transform of $x$ is defined as the sequence $Ax = ((Ax)_n)$, where

$$A_n(x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

provided the series on the right side converges for each $n$. For any two sequence spaces $U$ and $V$, we denote by $(U,V)$, the class of all infinite matrices $A$ that map $U$ into $V$. Therefore $A \in (U,V)$ if and only if $Ax = ((Ax)_n) \in V$ for all $x \in U$. In other words, $A \in (U,V)$ if and only if $A_n \in U^\beta$ for all $n$ [3]. An infinite matrix $T = (t_{nk})_{n,k}$ is said to be triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \ne 0$, $n \in \mathbb{N}_0$.

**3. Sequence Space $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$**

In this section, we first begin with the notion of generalized means given by Mursaleen et al. [17].
We denote the sets $\mathcal{U}$ and $\mathcal{U}_0$ as

$$\mathcal{U} = \Big\{u = (u_n)_{n=0}^\infty \in w : u_n \neq 0 \ \text{ for all } \ n\Big\} \text{ and}$$
$$\mathcal{U}_0 = \Big\{u = (u_n)_{n=0}^\infty \in w : u_0 \neq 0\Big\}.$$

Let $r = (r_n), t = (t_n) \in \mathcal{U}$ and $s = (s_n) \in \mathcal{U}_0$. The sequence $y = (y_n)$ of generalized means of a sequence $x = (x_n)$ is defined by

$$y_n = \frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k x_k \qquad (n \in \mathbb{N}_0).$$

The infinite matrix $A(r,s,t)$ of generalized means is defined by

$$(A(r,s,t))_{nk} = \begin{cases} \frac{s_{n-k}t_k}{r_n} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Since $A(r,s,t)$ is a triangle, it has a unique inverse and the inverse is also a triangle [2]. Take $D_0^{(s)} = \frac{1}{s_0}$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}}\begin{vmatrix} s_1 & s_0 & 0 & 0\cdots & 0 \\ s_2 & s_1 & s_0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3}\cdots & s_1 \end{vmatrix} \qquad \text{for } n \ge 1.$$

Then the inverse of $A(r,s,t)$ is the triangle $B = (b_{nk})_{n,k}$ which is defined as

$$b_{nk} = \begin{cases} (-1)^{n-k}\ \frac{D_{n-k}^{(s)}}{t_n} r_k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Throughout this paper, we consider $p = (p_k)$ is a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1, H\}$.
We now introduce a sequence space $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ as

$$X(r,s,t,p;\Delta) = \Big\{x = (x_k) \in w : \Big(\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k \Delta x_k\Big)_n \in X\Big\},$$

which is a combination of generalized means and difference operator $\Delta$, where $\Delta x_k = x_k - x_{k-1}$, $x_{-1} = 0$. By using matrix domain, we can write $X(r,s,t,p;\Delta) = X_{A(r,s,t;\Delta)} = \{x \in w : A(r,s,t;\Delta)x \in X\}$, where $A(r,s,t;\Delta) = A(r,s,t).\Delta$, product of two triangles $A(r,s,t)$ and $\Delta$. These sequence spaces include many well known sequence spaces studied by several earlier authors as follows:

I. if $r_n = \frac{1}{u_n}$, $t_n = v_n$ and $s_n = 1 \;\forall\; n$, then the sequence space $X(r, s, t, p; \Delta)$ reduces to $X(u, v, p; \Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ introduced and studied by Demiriz and Çakan [21].

II. if $t_n = 1$, $s_n = 1 \;\; \forall\; n$ and $r_n = n + 1$, then the sequence space $l(r, s, t, p; \Delta)$ reduces to the non absolute type sequence space $X_p(\Delta)$ studied by Başarir [16].

III. if $r_n = \frac{1}{n!}$, $t_n = \frac{\alpha^n}{n!}$, $s_n = \frac{(1-\alpha)^n}{n!}$, where $0 < \alpha < 1$, then the sequence space $X(r, s, t, p; \Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p)\}$ reduces to $e^\alpha_\infty(p; \Delta)$, $e^\alpha_c(p; \Delta)$ and $e^\alpha_0(p; \Delta)$ respectively introduced and studied by Karakaya and Polat [22].

IV. if $r_n = n+1$, $t_n = 1 + \alpha^n$, $0 < \alpha < 1$ and $s_n = 1$, $p_n = 1 \;\forall n$, then the sequence spaces $c(r, s, t, p; \Delta)$ and $c_0(r, s, t, p; \Delta)$ reduce to the sequence spaces $a^\alpha_c(\Delta)$ and $a^\alpha_0(\Delta)$ respectively studied by Aydin and Başar [9].

## 4. Main Results

Throughout the paper, we denote the sequence spaces $X(r, s, t, p; \Delta)$ as $l(r, s, t, p; \Delta)$, $c_0(r, s, t, p; \Delta)$, $c(r, s, t, p; \Delta)$ and $l_\infty(r, s, t, p; \Delta)$ for $X = l(p), c_0(p), c(p)$ and $l_\infty(p)$ respectively.

**Theorem 4.1.** (a) *The sequence space $l(r, s, t, p; \Delta)$ is a complete linear metric space paranormed by $\tilde{h}$ defined as*

$$\tilde{h}(x) = \left( \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right)^{\frac{1}{M}}.$$

(b) *The sequence space $X(r, s, t, p; \Delta)$ for $X \in \{l_\infty(p),\ c(p), c_0(p)\}$ is a complete linear metric space paranormed by $h$ defined as*

$$h(x) = \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{\frac{p_n}{M}}.$$

*Proof.* We prove the part (a) of this theorem. In a similar way, we can prove the part (b).
Let $x,\ y \in l(r, s, t, p; \Delta)$. Using Minkowski's inequality

$$\begin{aligned} \left( \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta (x_k + y_k) \right|^{p_n} \right)^{\frac{1}{M}} &\leq \left( \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right)^{\frac{1}{M}} \\ &\quad + \left( \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta y_k \right|^{p_n} \right)^{\frac{1}{M}} < \infty, \end{aligned} \tag{4.1}$$

so we have $x + y \in l(r, s, t, p; \Delta)$.
Let $\alpha$ be any scalar. Since $|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}$ for any scalar $\alpha$, we have $\tilde{h}(\alpha x) \leq \max\{1, |\alpha|\}\tilde{h}(x)$. Hence $\alpha x \in l(r, s, t, p; \Delta)$. It is trivial to show that $\tilde{h}(\theta) = 0$, $\tilde{h}(-x) = \tilde{h}(x)$ for all $x \in l(r, s, t, p; \Delta)$ and subadditivity of $\tilde{h}$, i.e., $\tilde{h}(x + y) \leq \tilde{h}(x) + \tilde{h}(y)$ follows from (4.1).
Next we show that the scalar multiplication is continuous. Let $(x^m)$ be a sequence in $l(r, s, t, p; \Delta)$, where $x^m = (x_k^m) = (x_0^m, x_1^m, x_2^m, \ldots) \in l(r, s, t, p; \Delta)$ for each $m \in \mathbb{N}_0$ such that $\tilde{h}(x^m - x) \to 0$ as $m \to \infty$ and $(\alpha_m)$ be a sequence of scalars such that $\alpha_m \to \alpha$ as $m \to \infty$. Then $\tilde{h}(x^m)$ is bounded that follows from the following inequality

$$\tilde{h}(x^m) \leq \tilde{h}(x) + \tilde{h}(x - x^m).$$

Now consider

$$\begin{aligned}
&\tilde{h}(\alpha_m x^m - \alpha x) \\
&= \Big(\sum_{n=0}^{\infty} \Big| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta(\alpha_m x_k^m - \alpha x_k) \Big|^{p_n}\Big)^{\frac{1}{M}} \\
&= \Big(\sum_{n=0}^{\infty} \Big| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta\big((\alpha_m - \alpha)(x_k^m - x_k) \\
&\qquad + \alpha(x_k^m - x_k) + (\alpha_m - \alpha) x_k\big) \Big|^{p_n}\Big)^{\frac{1}{M}} \\
&\leq \max\{1, |\alpha_m - \alpha|\}\tilde{h}(x^m - x) + |\alpha|\tilde{h}(x^m - x) + \Big(\sum_{n=0}^{\infty} |(\alpha_m - \alpha) y_n|^{p_n}\Big)^{\frac{1}{M}},
\end{aligned}$$

where $y = (y_n)$ is defined in Section 3. Since $\alpha_m \to \alpha$ as $m \to \infty$, so there is a natural number $m_0$ such that $|\alpha_m - \alpha| < 1$ for $m \geq m_0$. Then we have $|\alpha_m - \alpha|^{p_n} < 1$ for all $n$. Let $n_0 \in \mathbb{N}$. Now we have

$$\sum_{n=0}^{\infty} |(\alpha_m - \alpha) y_n|^{p_n} \leq \sum_{n=0}^{n_0} |(\alpha_m - \alpha) y_n|^{p_n} + \sum_{n=n_0+1}^{\infty} |y_n|^{p_n}.$$

Since $\sum_{n=0}^{\infty} |y_n|^{p_n} < \infty$, so for given $\epsilon > 0$ there exists $n_0$ such that $\sum_{n=n_0+1}^{\infty} |y_n|^{p_n} < \frac{\epsilon}{2}$. Since $\alpha_m \to \alpha$ as $m \to \infty$, so we have $\sum_{n=0}^{n_0} |(\alpha_m - \alpha) y_n|^{p_n} \to 0$ . Hence $\Big(\sum_{n=0}^{\infty} |(\alpha_m - \alpha) y_n|^{p_n}\Big)^{\frac{1}{M}} \to 0$ as $m \to \infty$. Therefore we have $\tilde{h}(\alpha_m x^m - \alpha x) \to 0$ as $m \to \infty$. This shows that the scalar multiplication is continuous. Hence $\tilde{h}$ is a paranorm on the space $l(r, s, t, p; \Delta)$.

Now we show that the completeness of the space $l(r,s,t,p;\Delta)$ with respect to the paranorm $\tilde{h}$. Let $(x^m)$ be a Cauchy sequence in $l(r,s,t,p;\Delta)$. So for every $\epsilon > 0$ there is a $n_1 \in \mathbb{N}$ such that

$$\tilde{h}(x^m - x^l) < \tfrac{\epsilon}{2} \quad \text{for all } m, l \geq n_1.$$

Then by definition of $\tilde{h}$, we have for each $n$

$$\begin{aligned} &\left|(A(r,s,t;\Delta)x^m)_n - (A(r,s,t;\Delta)x^l)_n\right| \\ &\leq \left(\sum_{n=0}^{\infty}\left|(A(r,s,t;\Delta)x^m)_n - (A(r,s,t;\Delta)x^l)_n\right|^{p_n}\right)^{\frac{1}{M}} < \frac{\epsilon}{2} \end{aligned} \tag{4.2}$$

for all $m, l \geq n_1$, which implies that the sequence $((A(r,s,t;\Delta)x^m)_n)$ is a Cauchy sequence of scalars for each fixed $n$ and hence converges for each $n$. We write

$$\lim_{m\to\infty}(A(r,s,t;\Delta)x^m)_n = (A(r,s,t;\Delta)x)_n \quad (n \in \mathbb{N}_0).$$

Now taking $l \to \infty$ in (4.2), we obtain

$$\left(\sum_{n=0}^{\infty}\left|(A(r,s,t;\Delta)x^m)_n - (A(r,s,t;\Delta)x)_n\right|^{p_n}\right)^{\frac{1}{M}} < \epsilon$$

for all $m \geq n_1$ and each fixed $n$. Thus $(x^m)$ converges to $x$ in $l(r,s,t,p;\Delta)$ with respect to $\tilde{h}$.
To show $x \in l(r,s,t,p;\Delta)$, we take

$$\begin{aligned} \Big(\sum_{n=0}^{\infty}\Big|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\Big|^{p_n}\Big)^{\frac{1}{M}} &= \Big(\sum_{n=0}^{\infty}\Big|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta(x_k - x_k^m + x_k^m)\Big|^{p_n}\Big)^{\frac{1}{M}} \\ &\leq \Big(\sum_{n=0}^{\infty}\Big|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta(x_k - x_k^m)\Big|^{p_n}\Big)^{\frac{1}{M}} \\ &\quad + \Big(\sum_{n=0}^{\infty}\Big|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k^m\Big|^{p_n}\Big)^{\frac{1}{M}} \\ &= \tilde{h}(x - x^m) + \tilde{h}(x^m) < \infty \quad \text{for all } m \geq n_1. \end{aligned}$$

Therefore $x \in l(r,s,t,p;\Delta)$. This completes the proof. □

**Theorem 4.2.** *The sequence space* $X(r,s,t,p;\Delta)$ *for* $X \in \{l_\infty(p),\ c(p), c_0(p), l(p)\}$ *is linearly isomorphic to the space* $X \in \{l_\infty(p),\ c(p), c_0(p), l(p)\}$ *respectively, i.e.,* $l_\infty(r,s,t,p;\Delta) \cong l_\infty(p)$, $c(r,s,t,p;\Delta) \cong c(p)$, $c_0(r,s,t,p;\Delta) \cong c_0(p)$ *and* $l(r,s,t,p;\Delta) \cong l(p)$.

*Proof.* We prove this theorem only for the case when $X = l(p)$. For this, we need to show that there exists a bijective linear map from $l(r,s,t,p;\Delta)$ to $l(p)$. Now we define a map $T : l(r,s,t,p;\Delta) \to l(p)$ by $x \mapsto Tx = y = (y_n)$, where

$$y_n = \frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k\Delta x_k.$$

Since the difference operator $\Delta$ is linear, so the linearity of $T$ is trivial. It is easy to see that $Tx = 0$ implies $x = 0$. Thus $T$ is injective. To prove $T$ is surjective, let $y \in l(p)$. Since $y = (A(r,s,t).\Delta)x$, i.e.,

$$x = \big(A(r,s,t).\Delta\big)^{-1}y = \Delta^{-1}.A(r,s,t)^{-1}y,$$

we can get a sequence $x = (x_n)$ as

$$x_n = \sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^k\frac{D_k^{(s)}}{t_{k+j}}r_jy_j \qquad (n \in \mathbb{N}_0). \tag{4.3}$$

Then

$$\tilde{h}(x) = \Bigg(\sum_{n=0}^{\infty}\bigg|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\bigg|^{p_n}\Bigg)^{\frac{1}{M}} = \Big(\sum_{n=0}^{\infty}|y_n|^{p_n}\Big)^{\frac{1}{M}} = \tilde{g}(y) < \infty.$$

Thus $x \in l(r,s,t,p;\Delta)$ and this shows that $T$ is surjective. Hence $T$ is a linear bijection from $l(r,s,t,p;\Delta)$ to $l(p)$. Also $T$ is paranorm preserving. This completes the proof. □

### 4.1. $\alpha$-, $\beta$-, $\gamma$-duals of $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$

In 1999, K. G. Grosse-Erdmann [15] has characterized the matrix transformations between the sequence spaces of Maddox, namely, $l_\infty(p), c(p), c_0(p)$ and $l(p)$. To compute $\alpha$-, $\beta$-, $\gamma$-duals of $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ and to characterize the classes of matrix mappings between these spaces, we list the following conditions.

Let $L$, $N$ be any two natural numbers, $F$ denotes finite subset of $\mathbb{N}_0$ and $\alpha, \alpha_k$ are complex numbers. Let $p = (p_k)$, $q = (q_k)$ be bounded sequences of strictly positive real numbers and $A = (a_{nk})_{n,k}$ be an infinite matrix. We put $K_1 = \{k \in \mathbb{N}_0 : p_k \le 1\}$ and $K_2 = \{k \in \mathbb{N}_0 : p_k > 1\}$ and $p'_k = \frac{p_k}{p_k - 1}$ for $k \in K_2$.

$$\sup_F \sup_{k\in K_1}\Big|\sum_{n\in F}a_{nk}\Big|^{p_k} < \infty \tag{4.4}$$

$$\exists L\ \sup_F \sum_{k\in K_2}\Big|\sum_{n\in F}a_{nk}L^{-1}\Big|^{p'_k} < \infty \tag{4.5}$$

$$\lim_n |a_{nk}|^{q_n} = 0 \text{ for all } k \tag{4.6}$$

$$\forall L\ \sup_n \sup_{k\in K_1}\Big|a_{nk}L^{\frac{1}{q_n}}\Big|^{p_k} < \infty \tag{4.7}$$

$$\forall L\ \exists N\ \sup_n \sum_{k\in K_2}\Big|a_{nk}L^{\frac{1}{q_n}}N^{-1}\Big|^{p'_k} < \infty \tag{4.8}$$

(4.9) $$\sup_n \sup_{k\in K_1} |a_{nk}|^{p_k} < \infty$$

(4.10) $$\exists N \sup_n \sum_{k\in K_2} \left|a_{nk}N^{-1}\right|^{p_k'} < \infty$$

(4.11) $$\exists(\alpha_k)\ \lim_n |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k$$

(4.12) $$\exists(\alpha_k)\ \forall L\ \sup_n \sup_{k\in K_1} \left(|a_{nk} - \alpha_k|L^{\frac{1}{q_n}}\right)^{p_k} < \infty$$

(4.13) $$\exists(\alpha_k)\ \forall L\ \exists N\ \sup_n \sum_{k\in K_2} \left(|a_{nk} - \alpha_k|L^{\frac{1}{q_n}}N^{-1}\right)^{p_k'} < \infty$$

(4.14) $$\exists L\ \sup_n \sup_{k\in K_1} \left|a_{nk}L^{-\frac{1}{q_n}}\right|^{p_k} < \infty$$

(4.15) $$\exists L\ \sup_n \sum_{k\in K_2} \left|a_{nk}L^{-\frac{1}{q_n}}\right|^{p_k'} < \infty$$

(4.16) $$\exists N\ \sup_F \sum_n \left|\sum_{k\in F} a_{nk}N^{-\frac{1}{p_k}}\right| < \infty$$

(4.17) $$\forall L\ \exists N\ \sup_n L^{\frac{1}{q_n}} \sum_k |a_{nk}|N^{-\frac{1}{p_k}} < \infty$$

(4.18) $$\exists N\ \sup_n \sum_k |a_{nk}|N^{-\frac{1}{p_k}} < \infty$$

(4.19) $$\exists(\alpha_k)\ \forall L\ \exists N\ \sup_n L^{\frac{1}{q_n}} \sum_k |a_{nk} - \alpha_k|N^{-\frac{1}{p_k}} < \infty$$

(4.20) $$\exists N\ \sup_n \left(\sum_k |a_{nk}|N^{-\frac{1}{p_k}}\right)^{q_n} < \infty$$

(4.21) $$\sum_n \left|\sum_k a_{nk}\right| < \infty$$

(4.22) $$\lim_n \left|\sum_k a_{nk}\right|^{q_n} = 0$$

(4.23) $$\exists\alpha\ \lim_n \left|\sum_k a_{nk} - \alpha\right|^{q_n} = 0$$

(4.24) $$\sup_n \left|\sum_k a_{nk}\right|^{q_n} < \infty$$

(4.25) $$\forall N\ \sup_F \sum_n \left|\sum_{k\in F} a_{nk}N^{\frac{1}{p_k}}\right| < \infty$$

(4.26) $$\forall N\ \lim_n \left(\sum_k |a_{nk}|N^{\frac{1}{p_k}}\right)^{q_n} = 0$$

(4.27) $$\forall N\ \sup_n \sum_k |a_{nk}|N^{\frac{1}{p_k}} < \infty$$

$$\exists(\alpha_k)\ \forall N\ \lim_n \left( \sum_k |a_{nk} - \alpha_k| N^{\frac{1}{p_k}} \right)^{q_n} = 0 \tag{4.28}$$

$$\forall N\ \sup_n \left( \sum_k |a_{nk}| N^{\frac{1}{p_k}} \right)^{q_n} < \infty. \tag{4.29}$$

**Lemma 4.1.**([15]) (i) $A \in (l(p), l_1)$ if and only if (4.4) and (4.5) hold.

(ii) $A \in (l(p), c_0(q))$ if and only if (4.6), (4.7) and (4.8) hold.

(iii) $A \in (l(p), c(q))$ if and only if (4.9), (4.10), (4.11), (4.12) and (4.13) hold.

(iv) $A \in (l(p), l_\infty(q))$ if and only if (4.14) and (4.15) hold.

**Lemma 4.2.**([15]) (i) $A \in (c_0(p), l_1)$ if and only if (4.16) holds.

(ii) $A \in (c_0(p), c_0(q))$ if and only if (4.6) and (4.17) hold.

(iii) $A \in (c_0(p), c(q))$ if and only if (4.11), (4.18) and (4.19) hold.

(iv) $A \in (c_0(p), l_\infty(q))$ if and only if (4.20) holds.

**Lemma 4.3.**([15]) (i) $A \in (c(p), l_1)$ if and only if (4.16) and (4.21) hold.

(ii) $A \in (c(p), c_0(q))$ if and only if (4.6) and (4.17) and (4.22) hold.

(iii) $A \in (c(p), c(q))$ if and only if (4.11), (4.18), (4.19) and (4.23) hold.

(iv) $A \in (c(p), l_\infty(q))$ if and only if (4.20) and (4.24) hold.

**Lemma 4.4.**([15]) (i) $A \in (l_\infty(p), l_1)$ if and only if (4.25) holds.

(ii) $A \in (l_\infty(p), c_0(q))$ if and only if (4.26) holds.

(iii) $A \in (l_\infty(p), c(q))$ if and only if (4.27) and (4.28) hold.

(iv) $A \in (l_\infty(p), l_\infty(q))$ if and only if (4.29) holds.

We consider the following sets to obtain $\alpha$-dual of the sequence space $X(r, s, t, p; \Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$.

$$H_1(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_F \sum_{n=0}^{\infty} \left| \sum_{k \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n L^{\frac{-1}{p_k}} \right| < \infty \right\}$$

$$H_2(p) = \left\{ a = (a_n) \in w : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n \right| < \infty \right\}$$

$$H_3(p) = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_F \sum_{n=0}^{\infty} \left| \sum_{k \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n L^{\frac{1}{p_k}} \right| < \infty \right\}$$

$$H_4(p) = \left\{ a = (a_n) \in w : \sup_F \sup_{k \in \mathbb{N}_0} \left| \sum_{n \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n \right|^{p_k} < \infty \right\}$$

$$H_5(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_F \sum_{k=0}^{\infty} \left| \sum_{n \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n L^{-1} \right|^{p'_k} < \infty \right\}.$$

**Theorem 4.3.** (a) *If* $p_k > 1$*, then* $[l(r,s,t,p;\Delta)]^\alpha = H_5(p)$ *and* $[l(r,s,t,p;\Delta)]^\alpha = H_4(p)$ *for* $0 < p_k \le 1$.

(b) *If* $0 < p_k \le H < \infty$*, then*
  (i) $[c_0(r,s,t,p;\Delta)]^\alpha = H_1(p)$,
  (ii) $[c(r,s,t,p;\Delta)]^\alpha = H_1(p) \cap H_2(p)$,
  (iii) $[l_\infty(r,s,t,p;\Delta)]^\alpha = H_3(p)$.

*Proof.* (a) Let $p_k > 1$ $\forall k$, $a = (a_n) \in w$, $x \in l(r,s,t,p;\Delta)$ and $y \in l(p)$. Then for each $n$, we have

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n y_k = (Cy)_n,$$

where the matrix $C = (c_{nk})_{n,k}$ is defined as

$$c_{nk} = \begin{cases} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}$$

and $x_n$ is given by (4.3). Thus for each $x \in l(r,s,t,p;\Delta)$, $(a_n x_n)_n \in l_1$ if and only if $Cy \in l_1$ where $y \in l(p)$. Therefore $a = (a_n) \in [l(r,s,t,p;\Delta)]^\alpha$ if and only if $C \in (l(p), l_1)$. By using Lemma 4.1 (i), we have

$$\sup_F \sum_{k=0}^{\infty} \left| \sum_{n \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n L^{-1} \right|^{p'_k} < \infty.$$

Hence $[l(r,s,t,p;\Delta)]^\alpha = H_5(p)$.
If $0 < p_k \le 1$ $\forall k$, then by using Lemma 4.1 (i), we have

$$\sup_F \sup_{k \in \mathbb{N}_0} \left| \sum_{n \in F} \sum_{j=0}^{n-k} (-1)^j \frac{D_j^{(s)}}{t_{j+k}} r_k a_n \right|^{p_k} < \infty.$$

Thus $[l(r,s,t,p;\Delta)]^\alpha = H_4(p)$.
(b) In a similar way, using Lemma 4.2 (i), Lemma 4.3 (i) and Lemma 4.4 (i), we obtain $[c_0(r,s,t,p;\Delta)]^\alpha = H_1(p)$, $[c(r,s,t,p;\Delta)]^\alpha = H_1(p)\cap H_2(p)$ and $[l_\infty(r,s,t,p;\Delta)]^\alpha = H_3(p)$ respectively. □

To compute $\gamma$-dual of the sequence space $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, we consider the following sets:

$$\Gamma_1(p) = \bigcup_{L\in\mathbb{N}} \Big\{a=(a_k)\in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} |e_{nk}| L^{\frac{-1}{p_k}} < \infty\Big\}$$

$$\Gamma_2(p) = \Big\{a=(a_k)\in w : \sup_{n\in\mathbb{N}_0} \Big|\sum_{k=0}^{\infty} e_{nk}\Big| < \infty\Big\}$$

$$\Gamma_3(p) = \bigcap_{L\in\mathbb{N}} \Big\{a=(a_k)\in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big|e_{nk}\Big| L^{\frac{1}{p_k}} < \infty\Big\}$$

$$\Gamma_4(p) = \bigcup_{L\in\mathbb{N}} \Big\{a=(a_k)\in w : \sup_{n\in\mathbb{N}_0}\sup_{k\in\mathbb{N}_0} |e_{nk}L^{-1}|^{p_k} < \infty\Big\}$$

$$\Gamma_5(p) = \bigcup_{L\in\mathbb{N}} \Big\{a=(a_k)\in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} |e_{nk}L^{-1}|^{p'_k} < \infty\Big\},$$

where the matrix $E=(e_{nk})$ is defined as
(4.30)

$$e_{nk} = \begin{cases} r_k\Big[\frac{a_k}{s_0 t_k} + \Big(\frac{D_0^{(s)}}{t_k} - \frac{D_1^{(s)}}{t_{k+1}}\Big)\sum_{j=k+1}^{n} a_j + \sum_{l=k+2}^{n} (-1)^{l-k}\frac{D_{l-k}^{(s)}}{t_l}\Big(\sum_{j=l}^{n} a_j\Big)\Big] \\ \quad \text{if } 0\le k\le n, \\ 0 \\ \quad \text{if } k>n. \end{cases}$$

Note: We mean $\sum_{n}^{k} = 0$ if $n > k$.

**Theorem 4.4.** (a) *If* $p_k > 1$, *then* $[l(r,s,t,p;\Delta)]^\gamma = \Gamma_5(p)$ *and* $[l(r,s,t,p;\Delta)]^\gamma = \Gamma_4(p)$ *if* $0 < p_k \le 1$.

(b) *If* $0 < p_k \le H < \infty$, *then*
(i) $[c_0(r,s,t,p;\Delta)]^\gamma = \Gamma_1(p)$,

(ii) $[c(r,s,t,p;\Delta)]^\gamma = \Gamma_1(p)\cap\Gamma_2(p)$,

(iii) $[l_\infty(r,s,t,p;\Delta)]^\gamma = \Gamma_3(p)$.

*Proof.* (a) Let $p_k > 1$ $\forall k$, $a=(a_k)\in w$, $x\in l(r,s,t,p;\Delta)$ and $y\in l(p)$. Then by using (4.3), we have

$$
\begin{aligned}
&\sum_{k=0}^{n} a_k x_k \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{k-j}(-1)^l \frac{D_l^{(s)} r_j y_j a_k}{t_{l+j}} \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{l=0}^{k-j}(-1)^l \frac{D_l^{(s)} r_j y_j a_k}{t_{l+j}} + \sum_{j=0}^{n}\sum_{l=0}^{n-j}(-1)^l \frac{D_l^{(s)} r_j y_j a_n}{t_{l+j}} \\
&= \Big[\frac{D_0^{(s)}}{t_0} a_0 + \Big(\frac{D_0^{(s)}}{t_0} - \frac{D_1^{(s)}}{t_1}\Big)\sum_{j=1}^{n} a_j + \sum_{l=2}^{n}(-1)^l \frac{D_l^{(s)}}{t_l}\Big(\sum_{j=l}^{n} a_j\Big)\Big] r_0 y_0 \\
&\quad + \Big[\frac{D_0^{(s)}}{t_1} a_1 + \Big(\frac{D_0^{(s)}}{t_1} - \frac{D_1^{(s)}}{t_2}\Big)\sum_{j=2}^{n} a_j + \sum_{l=3}^{n}(-1)^{l-1} \frac{D_{l-1}^{(s)}}{t_l}\Big(\sum_{j=l}^{n} a_j\Big)\Big] r_1 y_1 \\
&\quad + \cdots + \frac{r_n a_n}{t_n} D_0^{(s)} y_n \\
&= \sum_{k=0}^{n} r_k \Big[\frac{a_k}{s_0 t_k} + \Big(\frac{D_0^{(s)}}{t_k} - \frac{D_1^{(s)}}{t_{k+1}}\Big)\sum_{j=k+1}^{n} a_j + \sum_{l=k+2}^{n}(-1)^{l-k} \frac{D_{l-k}^{(s)}}{t_l}\Big(\sum_{j=l}^{n} a_j\Big)\Big] y_k \\
&= (Ey)_n,
\end{aligned}
\tag{4.31}
$$

where the matrix $E$ is defined in (4.30).
Thus $a \in \big[l(r,s,t,p;\Delta)\big]^{\gamma}$ if and only if $ax = (a_k x_k) \in bs$, where $x \in l(r,s,t,p;\Delta)$ if and only if $\Big(\sum_{k=0}^{n} a_k x_k\Big)_n \in l_\infty$, i.e., $Ey \in l_\infty$, where $y \in l(p)$. Hence by using Lemma 4.1 (iv) with $q_n = 1\ \forall n$, we have

$$\sup_{n\in\mathbb{N}_0}\sum_{k=0}^{\infty}|e_{nk}L^{-1}|^{p'_k} < \infty, \text{ for some } L \in \mathbb{N}.$$

Hence $\big[l(r,s,t,p;\Delta)\big]^{\gamma} = \Gamma_5(p)$.
If $0 < p_k \le 1\ \forall k$, then using Lemma 4.1 (iv), we have

$$\sup_{n\in\mathbb{N}_0}\sup_{k\in\mathbb{N}_0}|e_{nk}L^{-1}|^{p_k} < \infty \text{ for some } L \in \mathbb{N}.$$

Thus $[l(r,s,t,p;\Delta)]^{\gamma} = \Gamma_4(p)$.
(b) In a similar way, using Lemma 4.2 (iv), Lemma 4.3 (iv) and Lemma 4.4 (iv), we obtain $[c_0(r,s,t,p;\Delta)]^{\gamma} = \Gamma_1(p)$, $[c(r,s,t,p;\Delta)]^{\gamma} = \Gamma_1(p)\cap\Gamma_2(p)$ and $[l_\infty(r,s,t,p;\Delta)]^{\gamma} = \Gamma_3(p)$ respectively. □

To find $\beta$-dual of $X(r,s,t,p;\Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, we define the following sets:

$$B_1 = \Big\{a = (a_n) \in w : \sum_{j=k+1}^{\infty} a_j \text{ exists for all } k \in \mathbb{N}_0\Big\},$$

$$B_2 = \Big\{a = (a_n) \in w : \sum_{j=k+2}^{\infty} (-1)^{j-k} \frac{D_{j-k}^{(s)}}{t_j} \sum_{l=j}^{\infty} a_l \text{ exists for all } k \in \mathbb{N}_0\Big\},$$

$$B_3 = \Big\{a = (a_n) \in w : \Big(\frac{r_k a_k}{t_k}\Big) \in l_\infty(p)\Big\},$$

$$B_4 = \bigcup_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big|e_{nk} L^{-1}\Big|^{p_k{}'} < \infty\Big\},$$

$$B_5 = \Big\{a = (a_n) \in w : \sup_{n,k\in\mathbb{N}_0} |e_{nk}|^{p_k} < \infty\Big\},$$

$$B_6 = \Big\{a = (a_n) \in w : \exists(\alpha_k) \lim_{n\to\infty} e_{nk} = \alpha_k \ \forall\ k\Big\},$$

$$B_7 = \bigcap_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \exists(\alpha_k) \sup_{n,k\in\mathbb{N}_0} \Big(|e_{nk} - \alpha_k| L\Big)^{p_k} < \infty\Big\},$$

$$B_8 = \bigcap_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \exists(\alpha_k) \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big(|e_{nk} - \alpha_k| L\Big)^{p'_k} < \infty\Big\},$$

$$B_9 = \bigcup_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \exists(\alpha_k) \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big|e_{nk} - \alpha_k\Big| L^{\frac{-1}{p_k}} < \infty\Big\},$$

$$B_{10} = \bigcup_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big|e_{nk}\Big| L^{\frac{-1}{p_k}} < \infty\Big\},$$

$$B_{11} = \Big\{a = (a_n) \in w : \exists\alpha \lim_{n} \Big|\sum_{k=0}^{\infty} e_{nk} - \alpha\Big| = 0\Big\},$$

$$B_{12} = \bigcap_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big|e_{nk}\Big| L^{\frac{1}{p_k}} < \infty\Big\},$$

$$B_{13} = \bigcap_{L\in\mathbb{N}} \Big\{a = (a_n) \in w : \exists(\alpha_k) \lim_{n} \sum_{k=0}^{\infty} |e_{nk} - \alpha_k| L^{\frac{1}{p_k}} = 0\Big\}.$$

**Theorem 4.5.** (a) *If $p_k > 1$ for all $k$, then $[l(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_4 \bigcap B_6 \bigcap B_8$ and if $0 < p_k \le 1$ for all $k$, then $[l(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_5 \bigcap B_6 \bigcap B_7$.*

(b) *Let $p_k > 0$ for all $k$. Then*

(i) $[c_0(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_6 \bigcap B_9 \bigcap B_{10}$.

(ii) $[c(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_6 \bigcap B_9 \bigcap B_{10} \bigcap B_{11}$.

(iii) $[l_\infty(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_{12} \bigcap B_{13}$.

*Proof.* $(a)$ Let $p_k > 1$ for all $k$. We have from (4.31)

$$\sum_{k=0}^{n} a_k x_k = (Ey)_n,$$

where the matrix $E$ is defined in (4.30). Thus $a \in \left[l(r,s,t,p;\Delta)\right]^\beta$ if and only if $ax = (a_k x_k) \in cs$ where $x \in l(r,s,t,p;\Delta)$ if and only if $Ey \in c$ where $y \in l(p)$, i.e., $E \in (l(p), c)$. Hence by Lemma 4.1 (iii) with $q_n = 1\ \forall n$, we have

$$\exists L \in \mathbb{N}\ \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \left| e_{nk} L^{-1} \right|^{p_k{}'} < \infty,$$

$$\exists (\alpha_k)\ \lim_{n\to\infty} e_{nk} = \alpha_k \text{ for all } k,$$

$$\exists (\alpha_k)\ \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \Big( |e_{nk} - \alpha_k| L \Big)^{p_k'} < \infty \text{ for all } L.$$

Therefore $[l(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_4 \bigcap B_6 \bigcap B_8$.
If $0 < p_k \le 1\ \forall k$, then using Lemma 4.1 (iii) with $q_n = 1$, $\forall n$, we have

$$\sup_{n,k\in\mathbb{N}_0} |e_{nk}|^{p_k} < \infty,\ \exists (\alpha_k)\ \lim_{n\to\infty} e_{nk} = \alpha_k \text{ for all } k,$$

$$\forall L \in \mathbb{N}\ \exists (\alpha_k)\ \sup_{n,k\in\mathbb{N}_0} \Big( |e_{nk} - \alpha_k| L \Big)^{p_k} < \infty.$$

Thus $[l(r,s,t,p;\Delta)]^\beta = B_1 \bigcap B_2 \bigcap B_3 \bigcap B_5 \bigcap B_6 \bigcap B_7$.
$(b)$ In a similar way, using Lemma 4.2 (iii), Lemma 4.3 (iii) and Lemma 4.4 (iii), we can obtain $\beta$-duals of $c_0(r,s,t,p;\Delta)$, $c(r,s,t,p;\Delta)$ and $l_\infty(r,s,t,p;\Delta)$ respectively. □

### 4.2. Matrix mappings

**Theorem 4.6.** *Let $\tilde{E} = (\tilde{e}_{nk})$ be the matrix which is same as the matrix $E = (e_{nk})$ defined in* (4.30)*, where $a_j$ is replaced by $a_{nj}$ and $a_k$ by $a_{nk}$.*
(a) *Let $p_k > 1$ for all $k$, then $A \in (l(r,s,t,p;\Delta), l_\infty)$ if and only if there exists $L \in \mathbb{N}$ such that*

$$\sup_n \sum_k \left| \tilde{e}_{nk} L^{-1} \right|^{p_k'} < \infty \quad \text{and } (a_{nk})_k \in B_1 \bigcap B_2 \bigcap B_3 \bigcap B_4 \bigcap B_6 \bigcap B_8.$$

(b) *Let $0 < p_k \le 1$ for all $k$. Then $A \in (l(r,s,t,p;\Delta), l_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}_0} \left| \tilde{e}_{nk} \right|^{p_k} < \infty \quad \text{and } (a_{nk})_k \in B_1 \bigcap B_2 \bigcap B_3 \bigcap B_5 \bigcap B_6 \bigcap B_7.$$

*Proof.* (a) Let $p_k > 1$ for all $k$. Since $(a_{nk})_k \in \big[l(r,s,t,p;\Delta)\big]^\beta$ for each fixed $n$, $Ax$ exists for all $x \in l(r,s,t,p;\Delta)$. Now for each $n$, we have

$$\begin{aligned}&\sum_{k=0}^{m} a_{nk}x_k\\ &= \sum_{k=0}^{m} r_k\Big[\frac{a_{nk}}{s_0 t_k} + \Big(\frac{D_0^{(s)}}{t_k} - \frac{D_1^{(s)}}{t_{k+1}}\Big)\sum_{j=k+1}^{n} a_{nj} + \sum_{j=k+2}^{n}(-1)^{j-k}\frac{D_{j-k}^{(s)}}{t_j}\Big(\sum_{l=j}^{n} a_{nl}\Big)\Big]y_k\\ &= \sum_{k=0}^{m} \tilde{e}_{nk}y_k,\end{aligned}$$

Taking $m \to \infty$, we have

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{e}_{nk}y_k \quad \text{for all } n.$$

We know that for any $L > 0$ and any two complex numbers $a, b$

$$|ab| \le L(|aL^{-1}|^{p'} + |b|^p), \tag{4.32}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Using (4.32), we get

$$\sup_n\Big|\sum_{k=0}^{\infty} a_{nk}x_k\Big| \le \sup_n\sum_{k=0}^{\infty}|\tilde{e}_{nk}||y_k| \le L\Big[\sup_n\sum_{k=0}^{\infty}|\tilde{e}_{nk}L^{-1}|^{p_k{}'} + \sum_{k=0}^{\infty}|y_k|^{p_k}\Big] < \infty.$$

Thus $Ax \in l_\infty$. This proves that $A \in (l(r,s,t,p;\Delta), l_\infty)$.
Conversely, assume that $A \in (l(r,s,t,p;\Delta), l_\infty)$ and $p_k > 1$ for all $k$. Then $Ax$ exists for each $x \in l(r,s,t,p;\Delta)$, which implies that $(a_{nk})_k \in [l(r,s,t,p;\Delta)]^\beta$ for each $n$. Thus
$(a_{nk})_k \in B_1 \bigcap B_2 \bigcap B_3 \bigcap B_4 \bigcap B_6 \bigcap B_8$. Also from $\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty}\tilde{e}_{nk}y_k$, we have $\tilde{E} = (\tilde{e}_{nk}) \in (l(p), l_\infty)$, i.e., for some natural number $L$, $\sup_{n\in\mathbb{N}_0}\sum_{k=0}^{\infty}\big|\tilde{e}_{nk}L^{-1}\big|^{p_k'} < \infty$.
This completes the proof.
(b) We omit the proof of this part as it is similar to the previous one. □

**Theorem 4.7.** (a) *Let $p_k > 1$ for all $k$. Then $A \in (l(r,s,t,p;\Delta), l_1)$ if and only if*

$$\sup_F\sum_{k=0}^{\infty}\Big|\sum_{n\in F}\tilde{e}_{nk}L^{-1}\Big|^{p_k'} < \infty \quad \textit{for some } L \in \mathbb{N}$$

*and* $(a_{nk})_{k\in\mathbb{N}_0} \in B_1 \bigcap B_2 \bigcap B_3 \bigcap B_4 \bigcap B_6 \bigcap B_8$.
(b) *Let $0 < p_k \le 1$ for all $k$. Then $A \in (l(r,s,t,p;\Delta), l_1)$ if and only if*

$$\sup_F\sup_k\Big|\sum_{n\in F}\tilde{e}_{nk}\Big|^{p_k} < \infty$$

*and*

$$(a_{nk})_k \in B_1 \bigcap B_2 \bigcap B_3 \bigcap B_5 \bigcap B_6 \bigcap B_7.$$

*Proof.* We omit the proof as it follows in a similar way of Theorem 4.5. □

**Corollary 4.1.** (a) $A \in (c_0(r,s,t,p;\Delta), c_0(q))$ *if and only if* (4.6), (4.17) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c_0(r,s,t,p;\Delta)\right]^{\beta}$,
(b) $A \in (c_0(r,s,t,p;\Delta), c(q))$ *if and only if* (4.11), (4.18), (4.19) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c_0(r,s,t,p;\Delta)\right]^{\beta}$,
(c) $A \in (c_0(r,s,t,p;\Delta), l_\infty(q))$ *if and only if* (4.20) *holds with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c_0(r,s,t,p;\Delta)\right]^{\beta}$.

**Corollary 4.2.** (a) $A \in (c(r,s,t,p;\Delta), c_0(q))$ *if and only if* (4.6), (4.17), (4.22) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c(r,s,t,p;\Delta)\right]^{\beta}$,
(b) $A \in (c(r,s,t,p;\Delta), c(q))$ *if and only if* (4.11), (4.18), (4.19), (4.23) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c(r,s,t,p;\Delta)\right]^{\beta}$,
(c) $A \in (c(r,s,t,p;\Delta), l_\infty(q))$ *if and only if* (4.20), (4.24) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[c(r,s,t,p;\Delta)\right]^{\beta}$.

**Corollary 4.3.** (a) $A \in (l_\infty(r,s,t,p;\Delta), c_0(q))$ *if and only if* (4.26) *holds with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[l_\infty(r,s,t,p;\Delta)\right]^{\beta}$,
(b) $A \in (l_\infty(r,s,t,p;\Delta), c(q))$ *if and only if* (4.27), (4.28) *hold with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[l_\infty(r,s,t,p;\Delta)\right]^{\beta}$,
(c) $A \in (l_\infty(r,s,t,p;\Delta), l_\infty(q))$ *if and only if* (4.29) *holds with* $\tilde{e}_{nk}$ *in place of* $a_{nk}$ *and* $(a_{nk}) \in \left[l_\infty(r,s,t,p;\Delta)\right]^{\beta}$.

## 5. Kadec-Klee Property and Rotundity of $l(r,s,t,p;\Delta)$

In many geometric properties of Banach spaces, Kadec-Klee property and rotundity play an important role in metric fixed point theory. These properties are extensively studied in Orlicz spaces (see [10], [12], [19]) and also studied in difference sequence spaces by Kananthai [1]. In this section, we discuss these properties in the sequence space $l(r,s,t,p;\Delta)$.

Throughout the paper, for any Banach space $(Y, \|.\|)$, we denote $S(Y)$ and $B(Y)$ as the unit sphere and closed unit ball respectively.

A point $x \in S(Y)$ is called an extreme point if $x = \frac{y+z}{2}$ implies $y = z$ for every $y, z \in S(Y)$. A Banach space $Y$ is said to be rotund (strictly convex) if every point of $S(Y)$ is an extreme point.

Let $X$ be a real vector space. A functional $\sigma : X \to [0,\infty]$ is called a modular if
(i) $\sigma(x) = 0$ if and only if $x = 0$,
(ii) $\sigma(-x) = \sigma(x)$,
(iii) $\sigma(\alpha x + \beta y) \le \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.
A modular $\sigma$ is said to be convex if

(iv) $\sigma(\alpha x+\beta y)\le \alpha\sigma(x)+\beta\sigma(y)$ for all $x,y\in X$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.
For any modular $\sigma$, the modular space $X_\sigma$ is defined by

$$X_\sigma=\{x\in X:\sigma(\lambda x)\to 0 \text{ as } \lambda\to 0+\}.$$

We define $X^*_\sigma=\{x\in X:\sigma(\lambda x)<\infty \text{ for some } \lambda>0\}$. It is clear that $X_\sigma\subseteq X^*_\sigma$. Orlicz [23] prove that if $\sigma$ is convex then $X_\sigma=X^*_\sigma$.
A modular $\sigma$ is said to be
(i) *right continuous* if $\lim\limits_{\lambda\to 1+}\sigma(\lambda x)=\sigma(x)$,
(ii) *left continuous* if $\lim\limits_{\lambda\to 1-}\sigma(\lambda x)=\sigma(x)$,
(iii) *continuous* if it is both left and right continuous.
A modular $\sigma$ is said to satisfy $\Delta_2$-condition [24], denoted by $\sigma\in\Delta_2$ if for any $\epsilon>0$, there exist constants $K\ge 2$ and $a>0$ such that $\sigma(2x)\le K\sigma(x)+\epsilon$ for all $x\in X_\sigma$ with $\sigma(x)\le a$.
If $\sigma$ satisfies $\Delta_2$-condition for any $a>0$ with $K\ge 2$ dependent on $a$, we say that $\sigma$ satisfies strong $\Delta_2$-condition, denoted by $\sigma\in\Delta_2^s$ [24].
Let $p_n>1$ for all $n\in\mathbb{N}_0$. Then for $x\in l(r,s,t,p;\Delta)$, we define

$$\sigma_p(x)=\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\right|^{p_n}\qquad (n\in\mathbb{N}_0).$$

By the convexity of the function $t\longmapsto |t|^{p_n}$ for each $n\in\mathbb{N}_0$, we have $\sigma_p$ is a convex modular on $l(r,s,t,p;\Delta)$.
We consider $l(r,s,t,p;\Delta)$ equipped with the so called Luxemburg norm given by

$$\|x\|=\inf\Big\{c>0:\sigma_p\Big(\frac{x}{c}\Big)\le 1\Big\}.$$

It is easy to observe that the space $l(r,s,t,p;\Delta)$ endowed with the norm $\|x\|$ forms a Banach space. A normed sequence space $X$ is said to be $K$-space if each coordinate mapping $P_k$ defined by $P_k(x)=x_k$ is continuous for each $k\in\mathbb{N}_0$. If $X$ is a Banach space as well as $K$-space, then it is called a BK space. Let $p_k\ge 1$ $\forall k\in\mathbb{N}_0$ and $M=\sup\limits_k p_k$. It is easy to show that $\sigma_p$ satisfies the strong $\Delta_2$-condition, i.e., $\sigma_p\in\Delta_2^s$.

**Proposition 5.1.** *For $x\in l(r,s,t,p;\Delta)$, the modular $\sigma_p$ on $l(r,s,t,p;\Delta)$ satisfies the following:*

(i) *if $0<\alpha\le 1$, then $\alpha^M\sigma_p(\frac{x}{\alpha})\le\sigma_p(x)$ and $\sigma_p(\alpha x)\le\sigma_p(x)$.*
(ii) *if $\alpha\ge 1$, then $\sigma_p(x)\le\alpha^M\sigma_p(\frac{x}{\alpha})$.*
(iii) *if $\alpha\ge 1$, then $\sigma_p(x)\le\alpha\sigma_p(x)\le\sigma_p(\alpha x)$.*

*Proof.* (i) We have

$$\sigma_p\Big(\frac{x}{\alpha}\Big)=\sum_{n=0}^{\infty}\left|\frac{1}{\alpha r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\right|^{p_n}\le\frac{1}{\alpha^M}\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\right|^{p_n}=\frac{1}{\alpha^M}\sigma_p(x),$$

i.e., $\alpha^M \sigma_p(\frac{x}{\alpha}) \leq \sigma_p(x)$ and using convexity of $\sigma_p$, we have $\sigma_p(\alpha x) \leq \sigma_p(x)$ for $0 < \alpha \leq 1$.
Statements (ii) and (iii) can be proved in a similar way. So, we omit the details.□

**Proposition 5.2.** *The modular $\sigma_p$ is continuous.*

*Proof.* Let $\lambda > 1$. From Proposition , we have

$$\sigma_p(x) \leq \lambda\sigma_p(x) \leq \sigma_p(\lambda x) \leq \lambda^M \sigma_p(x).$$

Taking $\lambda \to 1+$, we obtain $\lim_{\lambda\to 1+} \sigma_p(\lambda x) = \sigma_p(x)$. So $\sigma_p$ is right continuous.
If $0 < \lambda < 1$, then we have $\lambda^M \sigma_p(x) \leq \sigma_p(\lambda x) \leq \lambda\sigma_p(x)$. Taking $\lambda \to 1-$, we obtain $\lim_{\lambda\to 1-} \sigma_p(\lambda x) = \sigma_p(x)$. So $\sigma_p$ is left continuous. Thus $\sigma_p$ is continuous. □

Now we give some relationship between norm and modular.

**Proposition 5.3.** *For any $x \in l(r, s, t, p; \Delta)$, we have*

(i) *if $\|x\| < 1$ then $\sigma_p(x) \leq \|x\|$,*

(ii) *if $\|x\| > 1$ then $\sigma_p(x) \geq \|x\|$,*

(iii) *$\|x\| = 1$ if and only if $\sigma_p(x) = 1$,*

(iv) *$\|x\| < 1$ if and only if $\sigma_p(x) < 1$,*

(v) *$\|x\| > 1$ if and only if $\sigma_p(x) > 1$,*

(vi) *if $0 < \alpha < 1$ and $\|x\| > \alpha$ then $\sigma_p(x) > \alpha^M$,*

(vii) *if $\alpha \geq 1$ and $\|x\| < \alpha$ then $\sigma_p(x) < \alpha^M$.*

*Proof.* (i) Suppose $\|x\| < 1$. Let $u$ be a positive number such that $\|x\| < u < 1$. Then by the definition of norm $\|.\|$, we have $\sigma_p\Big(\frac{x}{u}\Big) \leq 1$. Using convexity of $\sigma_p$, we have $\sigma_p(x) = \sigma_p\Big(u\frac{x}{u}\Big) < u\sigma_p\Big(\frac{x}{u}\Big) \leq u$. Since $u$ is arbitrary, this implies that $\sigma_p(x) \leq \|x\|$.
(ii) Let $u$ be a positive number such that $\|x\| > u > 1$. Then $\sigma_p\Big(\frac{x}{u}\Big) > 1$ and $1 < \sigma_p\Big(\frac{x}{u}\Big) < \frac{1}{u}\sigma_p(x)$, i.e., $\sigma_p(x) > u$. Taking $u \to \|x\|_-$, we obtain $\sigma_p(x) \geq \|x\|$.
(iii) Since $\sigma_p \in \Delta_2^s$, so the proof follows from Corollary 2.2 in [24] and Proposition 5.2.
(iv) and (v) follows from (i) and (iii).
(vi) and (vii) follows from Proposition 5.1 (i) and (ii). □

**Proposition 5.4.** *Let $(x^m)$ be any sequence of elements of $l(r, s, t, p; \Delta)$.*

(i) *If $\|x^m\| \to 1$ then $\sigma_p(x^m) \to 1$ as $m \to \infty$,*

(ii) *If $\|x^m\| \to 0$ if and only if $\sigma_p(x^m) \to 0$ as $m \to \infty$.*

*Proof.* (i) Suppose that $\|x^m\| \to 1$ as $m \to \infty$. Then for every $\epsilon \in (0, 1)$ there exists $N \in \mathbb{N}_0$ such that $1 - \epsilon < \|x^m\| < 1 + \epsilon$ for all $m \geq N$. Thus by Proposition 5.3 $(vi)$ and $(vii)$, we have $(1 - \epsilon)^M < \sigma_p(x^m) < (1 + \epsilon)^M$ for all $m \geq N$. Hence

$\sigma_p(x^m) \to 1$ as $m \to \infty$.
(ii) Since $\sigma_p \in \Delta_2^s$, so the proof follows from Lemma 2.3 in [24]. □

**Lemma 5.1.** *The space $l(r,s,t,p;\Delta)$ is a BK space.*

*Proof.* Since the space $l(r,s,t,p;\Delta)$ equipped with the Luxemberg norm $\|.\|$ is a Banach space, so it is enough to prove that $l(r,s,t,p;\Delta)$ is a $K$-space. Suppose $(x^m) \subset l(r,s,t,p;\Delta)$ such that $x^m \to 0$ as $m \to \infty$. By Proposition 5.4 (ii), we have $\sigma_p(x^m) \to 0$ as $m \to \infty$. This implies that

$$\left|\tfrac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k\Delta x_k^m\right|^{p_n} \to 0 \ \text{ as } \ m \to \infty \ \text{ and for each } \ n \in \mathbb{N}_0.$$

By induction, we have $x_k^m \to 0$ as $m \to \infty$ for each $k \in \mathbb{N}_0$. Hence the coordinate mappings $P_k(x^m) = x_k^m \to 0$ as $m \to \infty$ which implies that $P_k$'s are continuous for each $k$. □

**Lemma 5.2.** *Let $x \in l(r,s,t,p;\Delta)$ and $(x^m) \subset l(r,s,t,p;\Delta)$. If $\sigma_p(x^m) \to \sigma_p(x)$ and $x_k^m \to x_k$ as $m \to \infty$ for each $k$ then $x^m \to x$.*

*Proof.* Since $x \in l(r,s,t,p;\Delta)$, i.e., $\sigma_p(x) < \infty$, so for a given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k\Delta x_k\right|^{p_n} < \frac{\epsilon}{3}\frac{1}{2^{M+1}}. \tag{5.1}$$

Again $\sigma_p(x^m) \to \sigma_p(x)$ and $x_k^m \to x_k$ as $m \to \infty$ for each $k$, so there exists $m_0, n_0 \in \mathbb{N}$ such that for $m \ge m_0$

$$\begin{aligned}&\sigma_p(x^m) - \left(\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k\Delta x_k^m\right|^{p_n}\right)\\ &< \sigma_p(x) - \left(\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k\Delta x_k\right|^{p_n}\right) + \frac{\epsilon}{3}\frac{1}{2^M}\end{aligned} \tag{5.2}$$

and

$$\left(\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k(\Delta x_k^m - \Delta x_k)\right|^{p_n}\right) < \frac{\epsilon}{3}. \tag{5.3}$$

Thus for $m \ge m_0$, using equations (5.1), (5.2) and (5.3), we have

$$\begin{aligned}
&\sigma_p(x^m - x) \\
&= \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k (\Delta x_k^m - \Delta x_k) \right|^{p_n} \\
&= \sum_{n=0}^{n_0} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k (\Delta x_k^m - \Delta x_k) \right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k (\Delta x_k^m - \Delta x_k) \right|^{p_n} \\
&< \frac{\epsilon}{3} + 2^M \left\{ \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k^m \right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right\} \\
&= \frac{\epsilon}{3} + 2^M \left\{ \sigma_p(x^m) - \sum_{n=0}^{n_0} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k^m \right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right\} \\
&< \frac{\epsilon}{3} + 2^M \left\{ \sigma_p(x) - \sum_{n=0}^{n_0} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right. \\
&\quad \left. + \frac{\epsilon}{3.2^M} + \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right\} \\
&= \frac{\epsilon}{3} + 2^M \left\{ \sum_{n=n_0+1}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right|^{p_n} \right\} + \frac{\epsilon}{3} + 2^M . \frac{\epsilon}{3} \frac{1}{2^{M+1}} \\
&< \frac{\epsilon}{3} + 2^M . \frac{\epsilon}{3} \frac{1}{2^{M+1}} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}$$

This shows that $\sigma_p(x^m - x) \to 0$ as $m \to \infty$. Therefore by Proposition 5.4, we have $x^m \to x$ in norm. □

**Theorem 5.1.** *The space $l(r, s, t, p; \Delta)$ has the Kadec-Klee property.*

*Proof.* Let $x \in S(l(r, s, t, p; \Delta))$ and $(x^m)$ be a sequence in $l(r, s, t, p; \Delta)$ such that $\|x^m\| \to 1$ as $m \to \infty$ and $x^m \to x$ weakly as $m \to \infty$. Since $\|x\| = 1$ so by Proposition 5.3 (iii), we have $\sigma_p(x) = 1$ and it follows from Proposition 5.4 that $\sigma_p(x^m) \to \sigma_p(x)$ as $m \to \infty$. By Lemma 5.1, we conclude that the coordinate mapping $P_k : l(r, s, t, p; \Delta) \to \mathbb{R}$ is continuous, which follows that $x_k^m \to x_k$ as $m \to \infty$ for each $k$. Hence by Lemma 5.2, we obtain $x^m \to x$ as $m \to \infty$ in norm. □

**Theorem 5.2.** *The space $l(r, s, t, p; \Delta)$ is rotund if $p_n > 1$ for each $n$.*

*Proof.* Let $x \in S(l(r, s, t, p; \Delta))$ and $y, z \in B(l(r, s, t, p; \Delta))$ with $x = \frac{y+z}{2}$. We have to show that $y = z$. Since $\sigma_p(x) = 1$ and

$$1 = \sigma_p(x) = \sigma_p(\tfrac{y+z}{2}) \leq \tfrac{1}{2}(\sigma_p(y) + \sigma_p(z)) \leq 1,$$

we have $\sigma_p(x) = \frac{1}{2}(\sigma_p(y) + \sigma_p(z))$ and $\sigma_p(y) = 1$, $\sigma_p(z) = 1$.
Therefore, we have

$$\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta x_k\right|^{p_n} =$$
$$\frac{1}{2}\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta y_k\right|^{p_n} + \frac{1}{2}\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta z_k\right|^{p_n}.$$

Since $x = \frac{y+z}{2}$, we have from above

$$\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\frac{\Delta y_k + \Delta z_k}{2}\right|^{p_n} =$$
$$\frac{1}{2}\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta y_k\right|^{p_n} + \frac{1}{2}\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta z_k\right|^{p_n}.$$

By the strict convexity of the function $f(t) = |t|^{p_k}, p_k > 1$ for each $k$, from above, we obtain for each $n$

$$\frac{1}{2r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta y_k = \frac{1}{2r_n}\sum_{k=0}^{n}s_{n-k}t_k\Delta z_k.$$

By induction, we obtain $y_k = z_k$ for each $k \in \mathbb{N}_0$, i.e., $y = z$. Therefore the sequence space $l(r, s, t, p; \Delta)$ is rotund. □

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