KYUNGPOOK Math. J. 55(2015), 885-892
http://dx.doi.org/10.5666/KMJ.2015.55.4.885
pISSN 1225-6951 eISSN 0454-8124

Finite Operators and Weyl Type Theorems for Quasi-$*$-n-Paranormal Operators

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Abstract. In this paper, we mainly obtain the following assertions: (1) If T is a quasi-$*$-n-paranormal operator, then T is finite and simply polaroid. (2) If T or T^* is a quasi-$*$-n-paranormal operator, then Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ and is not constant on each connected component of the open set U containing $\sigma(T)$. (3) If E is the Riesz idempotent for a nonzero isolated point λ of the spectrum of a quasi-$*$-n-paranormal operator, then E is self-adjoint and $EH = N(T-\lambda) = N(T-\lambda)^*$.

1. Introduction

Let H be an infinite dimensional separable Hilbert space, denote by $B(H)$ the algebra of all bounded linear operators on H, write $N(T)$, $R(T)$ and $\sigma(T)$ for the null space, range space and the spectrum of $T \in B(H)$, respectively.

In recent years, some operators have been introduced as natural extensions of hyponormal operators. For example: let n be positive integer.
(1) T is $*$-paranormal if $||T^2x|| \geq ||T^*x||^2$ for unit vector x.(see [9])
(2) T is $*$-n-paranormal if $||T^{1+n}x||^{\frac{1}{1+n}} \geq ||T^*x||$ for unit vector x.(see [6])
(3) T is n-paranormal if $||T^{1+n}x||^{\frac{1}{1+n}} \geq ||Tx||$ for unit vector x.(see [17])
(4) T is normaloid if $||T^n|| = ||T||^n$ for $n \in \mathbb{N}$.(see [3])

In this paper, we generalize $*$-n-paranormal operators to quasi-$*$-n-paranormal operators as follows.

Definition 1.1. For a positive integer n, T is said to *a quasi-$*$-n-paranormal operator* if
$$||T^{2+n}x||^{\frac{1}{1+n}}||Tx||^{\frac{n}{1+n}} \geq ||T^*Tx|| \quad \text{for every } x \in H.$$

* Corresponding Author.
Received December 15, 2014; accepted May 13, 2015.
2010 Mathematics Subject Classification: 47A10, 47B20.
Key words and phrases: Quasi-$*$-n-paranormal operator, finite, polaroid, Weyl's theorem, Riesz idempotent.
This work was supported by the NNSF of China (11401180).

Using the same method as that in Lemma 1.1 [19] we have

Lemma 1.2. *T is quasi-$*$-n-paranormal operators if and only if*

$$T^*(T^{*1+n}T^{1+n} - (n+1)\mu^n TT^* + n\mu^{1+n})T \geq 0 \quad for\ any\ \mu > 0.$$

Let $K = \oplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. Given positive operators A and B on H, we define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

By straightforward computations, the following assertions hold:
(i) $T_{A,B}$ is $*$-n-paranormal iff $B^{2n+2} - (n+1)\mu^n A^2 + n\mu^{n+1} \geq 0$ for any $\mu > 0$.
(ii) $T_{A,B}$ is quasi-$*$-n-paranormal iff $A(B^{2n+2} - (n+1)\mu^n A^2 + n\mu^{n+1})A \geq 0$ for any $\mu > 0$.
So that we say $T_{A,B}$ has a very useful characterization by which one can distinguish $*$-n-paranormal operators from quasi-$*$-n-paranormal operators.

Example 1.3. A non-$*$-2-paranormal and quasi-$*$-2-paranormal operator.

Proof. Take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$B^6 - 3\mu^2 A^2 + 2\mu^3 = \begin{pmatrix} 233 - 12\mu + 2\mu^3 & 144 \\ 144 & 89 + 2\mu^3 \end{pmatrix}.$$

If $\mu = 1$, then$B^6 - 3\mu^2 A^2 + 2\mu^3 \ngeq 0$, so $T_{A,B}$ is not a $*$-2-paranormal operator.

On the other hand, we have

$$A(B^6 - 3\mu^2 A^2 + 2\mu^3)A = \begin{pmatrix} 4(233 - 12\mu + 2\mu^3) & 0 \\ 0 & 0 \end{pmatrix} \geq 0 \quad for\ any\ \mu > 0.$$

Hence $T_{A,B}$ is a quasi-$*$-2-paranormal operator. □

2. Finite Operators

An operator $T \in B(H)$ is said to be finite [16] if

$$(2.1) \qquad ||I - (TX - XT)|| \geq 1$$

for all $X \in B(H)$, where I is the identity operator. Williams has shown that the class of finite operators contains every normal, hyponormal operators. In [7], Williams' results are generalized to a more class of operators containing the classes of normal and hyponormal operators. The inequality (2.1) is the starting point of the topic of commutator approximation.

Let $T \in B(H)$, we say that the approximate reduced spectrum of T, $\sigma_{ar}(T)$, is the set of scalars λ for which there exists a normed sequence $\{x_n\}$ in H satisfying

$$(T - \lambda I)x_n \to 0, \ (T - \lambda I)^* x_n \to 0.$$

In this section we present a new class of finite operators.

Lemma 2.1.([7]) *Let $T \in B(H)$. Then $\partial W(T) \cap \sigma(T) \subset \sigma_{ar}(T)$, where $W(T)$ is the numerical range of the operator T.*

Lemma 2.2.([7]) *If $\sigma_{ar}(T) \neq \phi$, then T is finite.*

Lemma 2.3. *If T is a quasi-$*$-n-paranormal operator, then T is normaloid.*

Proof. One can see from the definition of quasi-$*$-n-paranormal operator that

$$||T^{n+2}x|| ||Tx||^n \geq ||T^*Tx||^{n+1}$$

for every $x \in H$. If x is replaced by $T^k x$, then

$$||T^{n+1+k}x|| ||T^k x||^n \geq ||T^*T^k x||^{n+1}$$

holds for any integer $k \geq 1$, which admits that

$$||T^{n+1+k}|| ||T^k||^n \geq ||T^*T^k||^{n+1}. \tag{2.2}$$

Now suppose that$||T^k|| = ||T||^k$ for some $k \geq 1$ (which holds tautologically for $k = 1$). Then

$$\begin{aligned} ||T||^{(k-1)(n+1)} ||T^{n+1+k}|| ||T||^{kn} &\geq ||T^{*(k-1)}||^{n+1} ||T^{n+1+k}|| ||T^k||^n \\ &\geq ||T^{*(k-1)}||^{n+1} ||T^*T^k||^{n+1} \\ &\geq ||T^{*k}T^k||^{n+1} \\ &= ||T^k||^{2(n+1)} \\ &= ||T||^{2k(n+1)}, \end{aligned}$$

and hence

$$||T^{k+(n+1)}|| = ||T||^{k+(n+1)}.$$

Consequently, by induction, $||T^{1+(n+1)j}|| = ||T||^{1+(n+1)j}$ for every $j \geq 1$. This yields a subsequence $\{T^{n_j}\}$ of $\{T^n\}$, say $T^{n_j} = T^{1+(n+1)j}$, such that $\lim_j ||T^{n_j}||^{\frac{1}{n_j}} = \lim_j (||T||^{n_j})^{\frac{1}{n_j}} = ||T||$. Notice that $\{||T^n||^{\frac{1}{n}}\}$ is a convergent sequence that converges to $r(T)$, where $r(T)$ is the spectral radius of T, it follows that $r(T) = ||T||$. Therefore T is normaloid. □

Now, we establish an interesting property of quasi-$*$-n-paranormal operators.

Theorem 2.4. *Let $T \in B(H)$. If T is a quasi-$*$-n-paranormal operator, then T is finite.*

Proof. The hypothesis implies that T is normaloid by Lemma 2.3, and so is spectraloid, that is $\omega(T) = r(T)$, where $\omega(T)$ is the numerical radius of T. Then there exists $\lambda \in \sigma(T) \subset \overline{W(T)}$ such that $|\lambda| = \omega(T)$, where $W(T)$ is the numerical range of T. Thus $\lambda \in \partial W(T)$, which implies that $\partial W(T) \cap \sigma(T) \neq \emptyset$. Then the required result follows from Lemma 2.1 and Lemma 2.2. □

3. Weyl Type Theorems

An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim H/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \operatorname{acc} \sigma(T)$, where we write $\operatorname{acc} K$ for the set of all accumulation points of $K \subset \mathbb{C}$. If we write $\operatorname{iso} K = K \setminus \operatorname{acc} K$, then we note

$$\pi_{00}(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T).$$

In [14] the authors obtained that Weyl's theorem holds for $*$-paranormal operators. In [11] the authors obtained that Weyl's theorem holds for quasi-$*$-paranormal operators. In this section, we prove that Weyl's theorem holds for quasi-$*$-n-paranormal operators.

Lemma 3.1.([18]) *If T is a quasi-$*$-n-paranormal operator and $R(T)$ is not dense, then T has the matrix representation as follows:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

where T_1 is $*$-n-paranormal operator.

Proof. Since T is a quasi-$*$-n-paranormal operator and T does not have dense range, we can represent T as the following 2×2 operator matrix with respect to the decomposition $H = \overline{R(T)} \oplus N(T^*)$,

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}.$$

One can see from the definition of quasi-$*$-n-paranormal operator that

$$T^*(T^{*1+n}T^{1+n} - (n+1)\mu^n TT^* + n\mu^{1+n})T \geq 0 \quad \text{for any} \quad \mu > 0.$$

Then, for any $\mu > 0$ and all $x \in \overline{R(T)}$, we have

$$((T_1^{*1+n}T_1^{1+n} - (n+1)\mu^n(T_1T_1^* + T_2T_2^*) + n\mu^{1+n})x, x) \geq 0,$$

which yields that

$$((T_1^{*1+n}T_1^{1+n} - (n+1)\mu^n T_1T_1^* + n\mu^{1+n})x, x) \geq 0 \quad \text{for any} \quad \mu > 0.$$

Therefore, T_1 is a $*$-n-paranormal operator. □

Recall that $T \in B(H)$ has the single valued extension property(abbrev. SVEP), if for every open set U of $\mathbb{C}$, the only analytic solution $f : U \to H$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U.

Theorem 3.2. *If T is a quasi-$*$-n-paranormal operator, then T has SVEP.*

Proof. If the range of T is dense, then T is $*$-n-paranormal operator. Hence T has SVEP by [19, Corollary 1]. Assume that the range of T is not dense. By Lemma 3.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*).$$

Assume $(T - z)f(z) = 0$. Put $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T)} \oplus N(T^*)$. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & -z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2f_2(z) \\ -zf_2(z) \end{pmatrix} = 0.$$

Since $f_2(z) = 0$, $(T_1 - z)f_1(z) = 0$. And T_1 is $*$-n-paranormal operator, T_1 has SVEP by [19, Corollary 1]. Hence $f_1(z) = 0$. Consequently, T has SVEP. □

Theorem 3.3. *If T is a quasi-$*$-n-paranormal operator with spectrum $\sigma(T) \subseteq \partial D$, where D denotes the unite disc, then T is unitary.*

Proof. Since T is a quasi-$*$-n-paranormal operator, for all $x \in H$,

$$\begin{aligned} ||Tx||^{2n+2} &= (Tx, Tx)^{n+1} \\ &\leq ||T^*Tx||^{n+1}||x||^{n+1} \\ &\leq ||T^{n+2}x||||Tx||^n||x||^{n+1}, \end{aligned}$$

implies that $||Tx||^{n+2} \leq ||T^{n+2}x||||x||^{n+1}$, for all $x \in H$. Hence T is a $n+1$ paranormal operator. Thus T is unitary by [14, Theorem 1]. □

Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T, respectively. In general, if T is polaroid then it is isoloid. However, the converse is not true.

The quasinilpotent part $H_0(T-\lambda)$ and the analytic core $K(T-\lambda)$ are defined by $H_0(T-\lambda) = \{x \in H : \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}$ and $K(T-\lambda) = \{x \in H :$ there exists a sequence $\{x_n\} \subseteq H$ and $c > 0$ for which $x = x_0, (T-\lambda)x_{n+1} = x_n$ and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}\}$. We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyperinvariant subspaces of $T-\lambda$ such that $N(T-\lambda)^n \subseteq H_0(T-\lambda)$ for all $n \in \mathbb{N}$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$; also, if $\lambda \in \text{iso}\,\sigma(T)$, then $H = H_0(T-\lambda) \oplus K(T-\lambda)$, where $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed [1, Theorem 3.76].

Theorem 3.4. *If T is a quasi-$*$-n-paranormal operator, then T is simply polaroid.*

Proof. Let $\lambda \in \text{iso}\,\sigma(T)$, T is quasi-$*$-n-paranormal operators. Then

$$H = H_0(T-\lambda) \oplus K(T-\lambda),$$

where $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed, $\sigma(T_1) := \sigma(T|_{H_0(T-\lambda)}) = \{\lambda\}$ and $\sigma(T|_{K(T-\lambda)}) = \sigma(T)\backslash\{\lambda\}$. If $\lambda = 0$, then, T being normaloid, $T_1 = 0$ and $H_0(T) = N(T)$. If instead $\lambda \neq 0$, we may assume that $\lambda = 1$. Applying Theorem 3.3 it follows that T_1 is unitary. Thus by [5, Theorem 1.5.14] $T_1 = I|_{H_0(T-1)}$, which implies that $H_0(T-1) = N(T-1)$. Consequently, in either case, we have that $H_0(T-\lambda) = N(T-\lambda)$. So that T is simply polaroid follows from the implications

$$\begin{aligned} & H = N(T-\lambda) \oplus K(T-\lambda) \\ \Rightarrow\ & (T-\lambda)H = 0 \oplus (T-\lambda)K(T-\lambda) = K(T-\lambda) \\ \Rightarrow\ & H = N(T-\lambda) \oplus R(T-\lambda). \end{aligned}$$

□

Theorem 3.5. *Let T or T^* be a quasi-$*$-n-paranormal operator. Then Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ and is not constant on each connected component of the open set U containing $\sigma(T)$.*

Proof. From [2, Theorem 2.11], we have that T is polaroid if and only if T^* is polaroid. We use the fact that if T is polaroid and T or T^* has SVEP then both T and T^* satisfy Weyl's theorem in [2, Theorem 3.3]. Suppose that T or T^* is quasi-$*$-n-paranormal operator. By Theorem 3.2 and Theorem 3.4 we have that T satisfies Weyl's theorem. We show next that Weyl's theorem holds for $f(T)$. Since T is polaroid and has SVEP, then $f(T)$ is polaroid by [2, Lemma 3.11] and has SVEP by [1, Theorem 2.40]. Consequently, Weyl's theorem holds for $f(T)$. □

Corollary 3.6. *Let T or T^* be a quasi-$*$-n-paranormal operator. If F is an operator commuting with T and F^n has a finite rank for some $n \in \mathbb{N}$, then Weyl's theorem holds for $f(T) + F$, where f is an analytic function on $\sigma(T)$ and is not constant on each connected component of the open set U containing $\sigma(T)$.*

Proof. Suppose T or T^* is a quasi-$*$-n-paranormal operator. By Theorem 3.4 and Theorem 3.5, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Notice that T is isoloid then $f(T)$ is isoloid. The required result stems from [8, Theorem 2.4] . □

4. Riesz Idempotent

Let λ be an isolated point of the spectrum of T. Then the Riesz idempotent E of T with respect to λ is defined by $E = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk centered at λ which contains no other points of the spectrum of T. Stampfli [12] showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = N(T - \lambda)$. Recently, Jeon and Kim [4] and Uchiyama [15] obtained Stampfli's result for quasi-class A operators and paranormal operators, Tanahashi, Jeon, Kim and Uchiyama [13] obtained Stampfli's result for quasi-class (A, k) operators, Tanahashi and Uchiyama [14] obtained Stampfli's result for $*$-paranormal operators. In this paper, we extend this result to quasi-$*$-n-paranormal operators.

Lemma 4.1.([18]) *If T is a quasi-$*$-n-paranormal operator and $\lambda \neq 0$, then $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.*

Theorem 4.2. *If T is a quasi-$*$-n-paranormal operator, $0 \neq \lambda \in iso\ \sigma(T)$ and E is the Riesz idempotent of T with respect to λ, then E is self-adjoint and $EH = N(T - \lambda) = N(T - \lambda)^*$.*

Proof. If T is a quasi-$*$-n-paranormal operator and λ is a nonzero isolated point of $\sigma(T)$, then $EH = N(T - \lambda)$ by Theorem 3.4. Since $N(T - \lambda) \subseteq N(T - \lambda)^*$ by Lemma 4.1, it suffices to show that $N(T - \lambda)^* \subseteq N(T - \lambda)$. Since $N(T - \lambda)$ is a reducing subspace of T by Lemma 4.1 and the restriction of a quasi-$*$-n-paranormal operator to its reducing subspace is also a quasi-$*$-n-paranormal operator, T can be written as $T = \lambda \oplus T_1$ on $H = N(T - \lambda) \oplus (N(T - \lambda))^{\perp}$, where T_1 is quasi-$*$-n-paranormal operator with $N(T_1 - \lambda) = \{0\}$. Since $\lambda \in \sigma(T) = \{\lambda\} \cup \sigma(T_1)$ is isolated, only two cases occur: either $\lambda \notin \sigma(T_1)$, or λ is an isolated point of $\sigma(T_1)$ and this contradicts the fact that $N(T_1 - \lambda) = \{0\}$. Since $T_1 - \lambda$ is invertible as an operator on $(N(T - \lambda))^{\perp}$, we have $N(T - \lambda) = N(T - \lambda)^*$.

Next, we show that E is self-adjoint. Since E is the Riesz idempotent of T with respect to λ and T is a quasi-$*$-n-paranormal operator, it results from Theorem 3.4 that $R(E) = N(T - \lambda)$ and $N(E) = R(T - \lambda)$. Since $N(T - \lambda) \subseteq N(T - \lambda)^*$ by Lemma 4.1, then $N(T - \lambda)$ and $R(T - \lambda)$ are orthogonal. Hence $R(E)^{\perp} = N(E)$, and so E is self-adjoint. □

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