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Quasi-Valuation Maps on BCK/BCI-Algebras

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Abstract. The notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras is introduced, and then several properties are investigated. Relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal is established. In a BCI-algebra, a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra is provided, and conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal are discussed. Using the notion of a quasi-valuation map based on an ideal, (pseudo) metric spaces are constructed, and we show that the binary operation $*$ in BCK-algebras is uniformly continuous.

1. Introduction

Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science

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is twofold; as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty (see [10] for generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life.

BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki (see [2],[3],[4],[5]) and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Neggers and Kim [9] introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented diagraphs. In [8], Neggers et al. discussed the ideal theory in d-algebras. Neggers et al. [7] introduced the concept of d-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition they discussed a method of fuzzification of a wide class of algebraic systems onto $[0, 1]$ along with some consequences.

In this paper, we introduce the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then we investigate several properties. We provide relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. In a BCI-algebra, we give a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra, and find conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, we construct (pseudo) metric spaces, and we show that the binary operation $*$ in BCK-algebras is uniformly continuous.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x*y)*(x*z))*(z*y) = 0)$,

(II) $(\forall x, y \in X)\ ((x*(x*y))*y = 0)$,

(III) $(\forall x \in X)\ (x*x = 0)$,

(IV) $(\forall x, y \in X)\ (x*y = 0, y*x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0*x = 0)$,

Table 1: $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $(\forall x \in X)\ (x*0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x*y = 0 \ \Rightarrow\ (x*z)*(y*z) = 0,\ (z*y)*(z*x) = 0)$,

(a3) $(\forall x, y, z \in X)\ ((x*y)*z = (x*z)*y)$,

(a4) $(\forall x, y, z \in X)\ (((x*z)*(y*z))*(x*y) = 0)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y = 0$.

A nonempty subset S of a BCK/BCI-algerba X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b1) $0 \in A$,

(b2) $(\forall x, y \in X)\ (x*y \in A,\ y \in A \ \Rightarrow\ x \in A)$.

We refer the reader to the books [1, 6] for further information regarding BCK/BCI-algebras.

3. Quasi-Valuation Maps on BCK/BCI-Algebras

In what follows let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 3.1. By a *quasi-valuation map* of X based on a subalgebra (briefly *S-quasi-valuation map* of X), we mean a mapping $f : X \to \mathbb{R}$ which satisfies the following condition:

$$(\forall x, y \in X)\ (f(x*y) \geq f(x) + f(y)). \tag{3.1}$$

Example 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the $*$-operation given by Table 1. Let f be a real-valued function on X defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ 0 & -1 & -3 & -2 \end{pmatrix}.$$

Then f is an S-quasi-valuation map of X.

Example 3.3. Let $\mathbb{N}$ be the set of all natural numbers. Define an operation $*$ on $\mathbb{N}$ by

$$(\forall a, b \in \mathbb{N})\ \left(a*b = \tfrac{a}{(a,b)}\right)$$

where (a,b) is the greatest common divisor of a and b. Then $(\mathbb{N};*,1)$ is a BCK-algebra (see [1]). Define a real-valued function f_n on $\mathbb{N}$ by

$$f_n(x) = \begin{cases} t_1 & \text{if } x = 1, 2, 3, \cdots, n, \\ t_2 & \text{if } x = n+1, n+2, n+3, \cdots \end{cases}$$

for all $x \in \mathbb{N}$, where t_1 and t_2 are real numbers with $t_2 < t_1 \leq 0$. Then f_n is an S-quasi-valuation map of $\mathbb{N}$.

Proposition 3.4. *For any S-quasi-valuation map f of a BCK-algebra X, we have*

(c1) $(\forall x \in X)\ (f(x) \leq 0)$.

Proof. For any $x \in X$, we have $f(0) = f(0*x) \geq f(0) + f(x)$, and so $f(x) \leq 0$. □

For any real-valued function f on X, we consider the following conditions:

(c2) $f(0) = 0$.

(c3) $f(x) \geq f(x*y) + f(y)$ for all $x, y \in X$.

(c4) $f(x*y) \geq f(((x*y)*y)*z) + f(z)$ for all $x, y, z \in X$.

(c5) $f(x*y)*f((x*y)*y)$ for all $x, y \in X$.

(c6) $f((x*z)*(y*z)) \geq f((x*y)*z)$ for all $x, y, z \in X$.

Definition 3.5. By a *quasi-valuation map* of X based on an ideal (briefly I-*quasi-valuation map* of X), we mean a mapping $f : X \to \mathbb{R}$ which satisfies the conditions (c2) and (c3).

Example 3.6. Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 2. Then $(X;*,0)$ is a BCK-algerba (see [6]). Let f be a real-valued function on X defined by

$$f = \begin{pmatrix} 0 & a & b & c & d \\ 0 & -4 & -9 & 0 & -11 \end{pmatrix}.$$

It is easy to verify that f is an I-quasi-valuation map of X.

Example 3.7. Consider the adjoint BCI-algebra $(\mathbb{Z};-,0)$ of the additive group $(\mathbb{Z};+,0)$ of integers. Let f be a real-valued function on $\mathbb{Z}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2x-1 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Z}$. Routine calculations show that f is an I-quasi-valuation map of $\mathbb{Z}$.

Table 2: $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Theorem 3.8. *In a BCK-algebra, every I-quasi-valuation map is an S-quasi-valuation map.*

Proof. Let f be an I-quasi-valuation map on a BCK-algebra X. Then

$$\begin{aligned} 0 = f(0) &= f((0*y)*y) = f(((x*x)*y)*y) \\ &= f(((x*y)*x)*y) \leq f((x*y)*x) - f(y) \\ &\leq f(x*y) - f(x) - f(y), \end{aligned}$$

and so $f(x*y) \geq f(x) + f(y)$ for all $x, y \in X$. Therefore f is an S-quasi-valuation map of X. □

The following example shows that the converse of Theorem 3.8 may not be true.

Example 3.9. Consider the S-quasi-valuation map f in Example 3.2. Since $f(b) = -3 \ngeq f(b*a) + f(a)$, it is not an I-quasi-valuation map of X.

In general, Theorem 3.8 may not be true in a BCI-algebra as shown by the following example.

Example 3.10. The I-quasi-valuation map f which is given in Example 3.7 is not an S-quasi-valuation map of $\mathbb{Z}$ since $f(3-1) = 3 \ngeq f(3) + f(1)$.

We provide a condition for an I-quasi-valuation map of a BCI-algebra X to be an S-quasi-valuation map of X.

Theorem 3.11. *Let X be a BCI-algebra. If an I-quasi-valuation map f of X satisfies $f(0*x) \geq f(x)$ for all $x \in X$, then f is an S-quasi-valuation map of X.*

Proof. For all $x, y \in X$, we have

$$\begin{aligned} f(x*y) &\geq f((x*y)*x) + f(x) = f((x*x)*y) + f(x) \\ &= f(0*y) + f(x) \geq f(x) + f(y). \end{aligned}$$

Therefore f is an S-quasi-valuation map of X. □

Proposition 3.12. *For any I-quasi-valuation map f of X, we have the following assertions:*

(1) *f is order reversing.*

(2) *$f(x{*}y) + f(y{*}x) \leq 0$ for all $x, y \in X$.*

(3) *$f(x{*}y) \geq f(x{*}z) + f(z{*}y)$ for all $x, y, z \in X$.*

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Then $x{*}y = 0$, and so

$$f(x) \geq f(x{*}y) + f(y) = f(0) + f(y) = f(y).$$

Hence f is order reversing.

(2) Using (c3), we have $f(x{*}y) \leq f(x) - f(y)$ and $f(y{*}x) \leq f(y) - f(x)$ for all $x, y \in X$. It follows that $f(x{*}y) + f(y{*}x) \leq 0$.

(3) Note that $(x{*}y){*}(x{*}z) \leq z{*}y$ for all $x, y, z \in X$. Using (c3) and (1), we have $f(x{*}y) \geq f((x{*}y){*}(x{*}z)) + f(x{*}z) \geq f(z{*}y) + f(x{*}z)$ for all $x, y, z \in X$. □

Proposition 3.13. *If f is an I-quasi-valuation map of X, then* (c5) *and* (c6) *are equivalent.*

Proof. Assume that (c5) is valid. Note that

$$((x{*}(y{*}z)){*}z){*}z = ((x{*}z){*}(y{*}z)){*}z \leq (x{*}y){*}z$$

for all $x, y, z \in X$. Since f is order reversing, it follows that

$$f(((x{*}(y{*}z)){*}z){*}z) \geq f((x{*}y){*}z)$$

so from (a3) and (c5) that

$$f((x{*}z){*}(y{*}z)) = f((x{*}(y{*}z)){*}z) \geq f(((x{*}(y{*}z)){*}z){*}z) \geq f((x{*}y){*}z).$$

Conversely, (c5) follows from (c6) by repalcying z with y in (c6) and using (III) and (a1). □

Proposition 3.14. *Every I-quasi-valuation map f of X satisfies the following implication:*

$$(\forall x, y, z \in X)\ ((x{*}y){*}z = 0 \ \Rightarrow\ f(x) \geq f(y) + f(z)). \tag{3.2}$$

Proof. Let $x, y, z \in X$ be such that $(x{*}y){*}z = 0$. Using (c3) and (c2), we have

$$f(x{*}y) \geq f((x{*}y){*}z) + f(z) = f(0) + f(z) = f(z)$$

and so $f(x) \geq f(x{*}y) + f(y) \geq f(y) + f(z)$. □

The following corollary can be easily proved by induction.

Corollary 3.15. *Let f be an I-quasi-valuation map of X. If*

$$(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0,$$

then $f(x) \geq \sum_{k=1}^{n} f(a_k)$.

We provide conditions for a real-valued function on X to be an I-quasi-valuation map of X.

Theorem 3.16. *If a real-valued function f on X satisfies the conditions* (c2) *and* (3.2)*, then f is an I-quasi-valuation map of X.*

Proof. Note that $(x*(x*y))*y = (x*y)*(x*y) = 0$ for all $x, y \in X$. It follows from (3.2) that $f(x) \geq f(x*y) + f(y)$ for all $x, y \in X$. Therefore f is an I-quasi-valuation map of X. □

Theorem 3.17. *If a function $f : X \to \mathbb{R}$ satisfies the conditions* (c2) *and* (c4)*, then f is an I-quasi-valuation map of X.*

Proof. Taking $y = 0$ and $z = y$ in (c4) and using (a1), we have

$$f(x) = f(x*0) \geq f(((x*0)*0)*y) + f(y) = f(x*y) + f(y).$$

Hence f is an I-quasi-valuation map of X. □

Combining Proposition 3.4 and Theorem 3.8, we know that in a BCK-algebra X, every I-quasi-valuation map f of X satisfies the inequality (c1). But the following example shows that an I-quasi-valuation map f on a BCI-algebra X does not satisfy the inequality (c1).

Example 3.18. The I-quasi-valuation map f in Example 3.7 does not satisfy the inequality (c1).

For any function $f : X \to \mathbb{R}$, consider the following set:

$$I_f := \{x \in X \mid f(x) = 0\}.$$

Theorem 3.19. *Let X be a BCK-algebra. If f is an I-quasi-valuation map of X, then the set I_f is an ideal of X.*

Proof. Obviously, $0 \in I_f$. Let $x, y \in X$ be such that $x*y \in I_f$ and $y \in I_f$. Then $f(x*y) = 0$ and $f(y) = 0$. It follows from (c3) that $f(x) \geq f(x*y) + f(y) = 0$. By Theorem 3.8 and Proposition 3.4, we have $f(x) \leq 0$. Therefore $f(x) = 0$. Hence $x \in I_f$, which shows that I_f is an ideal of X. □

The following examples show that the converse of Theorem 3.19 may not be true, that is, there exist a BCK-algebra X and a function $f : X \to \mathbb{R}$ such that

(1) f is not an I-quasi-valuation map of X,

Table 3: $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

(2) $I_f := \{x \in X \mid f(x) = 0\}$ is an ideal of X.

□

Example 3.20. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the $*$-operation given by Table 3. Let g be a real-valued function on X defined by

$$g = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & -8 & 0 & -6 \end{pmatrix}.$$

Then $I_g = \{0, a, c\}$ is an ideal of X. But g is not an I-quasi-valuation map of X since $g(b) = -8 < -6 = g(b*d) + g(d)$.

Question. In a BCI-algebra X, if f is an I-quasi-valuation map of X, then is the set I_f an ideal of X?

For a real-valued function f on a BCK-algebra X, define a mapping

$$d_f : X \times X \to \mathbb{R},\ (x, y) \mapsto -f(x*y) - f(y*x).$$

Lemma 3.21. *If a real-valued function f on a BCK-algebra X is an I-quasi-valuation map of X, then d_f is a pseudo-metric on X, and so (X, d_f) is a pseudo-metric space.*

We say d_f is the pseudo-metric induced by an I-quasi-valuation map f.

Proof. Let f be an I-quasi-valuation map of a BCK-algebra X. Using Theorem 3.8 and Proposition 3.4, we know that $d_f(x, y) \geq 0$ for all $x, y \in X$. Obviously,

By a *pseudo-metric* we mean a real-valued function $d : X \times X \to \mathbb{R}$ satisfying the following properties: $d(x, y) \geq 0$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$.

$d_f(x,x) = 0$ and $d_f(x,y) = d_f(y,x)$ for all $x, y \in X$. Let $x, y, z \in X$. Using Proposition 3.12(3), we have

$$\begin{aligned} d_f(x,y) + d_f(y,z) &= \big(-f(x{*}y) - f(y{*}x)\big) + \big(-f(y{*}z) - f(z{*}y)\big) \\ &= -\big(f(x{*}y) + f(y{*}z)\big) - \big(f(z{*}y) + f(y{*}x)\big) \\ &\geq -f(x{*}z) - f(z{*}x) = d_f(x,z). \end{aligned}$$

Therefore (X, d_f) is a pseudo-metric space. □

Theorem 3.22. *Let f be an I-quasi-valuation map of a BCK-algebra X. If f satisfies the following condition:*

$$(\forall x \in X)\ (x \neq 0 \ \Rightarrow\ f(x) \neq 0), \tag{3.3}$$

then (X, d_f) is a metric space.

Proof. If f is an I-quasi-valuation map of a BCK-algebra X, then (X, d_f) is a pseudo-metric space by Lemma 3.21. Let $x, y \in X$ be such that $d_f(x,y) = 0$. Then $0 = d_f(x,y) = -f(x{*}y) - f(y{*}x)$, and so $f(x{*}y) = 0$ and $f(y{*}x) = 0$. It follows from (3.3) that $x{*}y = 0$ and $y{*}x = 0$ so from (IV) that $x = y$. Therefore (X, d_f) is a metric space. □

Proposition 3.23. *Let f be an I-quasi-valuation map of a BCK-algebra X. Then every pseudo-metric d_f induced by f satisfies the following inequalities:*

(1) $d_f(x,y) \geq d_f(x{*}a, y{*}a)$,

(2) $d_f(x,y) \geq d_f(a{*}x, a{*}y)$,

(3) $d_f(x{*}y, a{*}b) \leq d_f(x{*}y, a{*}y) + d_f(a{*}y, a{*}b)$

for all $x, y, a, b \in X$.

Proof. (1) Let $x, y, a \in X$. Since

$$((x{*}a){*}(y{*}a)){*}(x{*}y) = 0 \text{ and } ((y{*}a){*}(x{*}a)){*}(y{*}x) = 0,$$

it follows from Proposition 3.12(1) that $f(x{*}y) \leq f((x{*}a){*}(y{*}a))$ and $f(y{*}x) \leq f((y{*}a){*}(x{*}a))$ so that

$$\begin{aligned} d_f(x,y) &= -f(x{*}y) - f(y{*}x) \\ &\geq -f((x{*}a){*}(y{*}a)) - f((y{*}a){*}(x{*}a)) \\ &= d_f(x{*}a, y{*}a). \end{aligned}$$

(2) It is similar to the proof of (1).

(3) Using Proposition 3.12(3), we have

$$f((x{*}y){*}(a{*}b)) \geq f((x{*}y){*}(a{*}y)) + f((a{*}y){*}(a{*}b)),$$

$$f((a{*}b){*}(x{*}y)) \geq f((a{*}b){*}(a{*}y)) + f((a{*}y){*}(x{*}y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_f(x*y, a*b) &= -f((x*y)*(a*b)) - f((a*b)*(x*y)) \\ &\leq -\big(f((x*y)*(a*y)) + f((a*y)*(a*b))\big) \\ &\quad - \big(f((a*b)*(a*y)) + f((a*y)*(x*y))\big) \\ &= \big(-f((x*y)*(a*y)) - f((a*y)*(x*y))\big) \\ &\quad + \big(-f((a*b)*(a*y)) - f((a*y)*(a*b))\big) \\ &= d_f(x*y, a*y) + d_f(a*y, a*b) \end{aligned}$$

for all $x, y, a, b \in X$. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras. Define a binary operation $\odot$ on $X_1 \times X_2$ by

$$\big(\forall (x, y), (a, b) \in X_1 \times X_2\big) \ \big((x, y) \odot (a, b) = (x*_1 a, y*_2 b)\big).$$

Then $(X_1 \times X_2, \odot, (0_1, 0_2))$ is a BCK-algebra (see [6]).

Lemma 3.24. *For a real-valued function f on a BCK-algebra X, if d_f is a pseudo-metric on X, then $(X \times X, d_f^*)$ is a pseudo-metric space, where*

$$d_f^*\big((x, y), (a, b)\big) = \max\{d_f(x, a), d_f(y, b)\} \tag{3.4}$$

for all $(x, y), (a, b) \in X \times X$.

Proof. Suppose d_f is a pseudo-metric on X. Clearly, $d_f^*\big((x, y), (a, b)\big) \geq 0$ for all $(x, y), (a, b) \in X \times X$. For any $(x, y), (a, b) \in X \times X$, we have

$$d_f^*\big((x, y), (x, y)\big) = \max\{d_f(x, x), d_f(y, y)\} = 0$$

and

$$\begin{aligned} d_f^*\big((x, y), (a, b)\big) &= \max\{d_f(x, a), d_f(y, b)\} \\ &= \max\{d_f(a, x), d_f(b, y)\} \\ &= d_f^*\big((a, b), (x, y)\big). \end{aligned}$$

Now let $(x, y), (a, b), (u, v) \in X \times X$. Then

$$\begin{aligned} & d_f^*\big((x, y), (u, v)\big) + d_f^*\big((u, v), (a, b)\big) \\ &= \max\{d_f(x, u), d_f(y, v)\} + \max\{d_f(u, a), d_f(v, b)\} \\ &\geq \max\{d_f(x, u) + d_f(u, a), d_f(y, v) + d_f(v, b)\} \\ &\geq \max\{d_f(x, a), d_f(y, b)\} \\ &= d_f^*\big((x, y), (a, b)\big). \end{aligned}$$

Therefore $(X \times X, d_f^*)$ is a pseudo-metric space. □

Theorem 3.25. *Let $f : X \to \mathbb{R}$ be an I-quasi-valuation map of a BCK-algebra X satisfying the condition* (3.3)*. Then $(X \times X, d_f^*)$ is a metric space.*

Proof. Let $f : X \to \mathbb{R}$ be an I-quasi-valuation map of a BCK-algebra X satisfying the condition (3.3). Then d_f is a pseudo-metric on X by Lemma 3.21. It follows from Lemma 3.24 that $(X \times X, d_f^*)$ is a pseudo-metric space. Let $(x, y), (a, b) \in X \times X$ be such that $d_f^*\big((x, y), (a, b)\big) = 0$. Then

$$0 = d_f^*\big((x, y), (a, b)\big) = \max\{d_f(x, a), d_f(y, b)\},$$

and so $d_f(x, a) = 0 = d_f(y, b)$ since $d_f(x, y) \geq 0$ for all $(x, y) \in X \times X$. Hence

$$0 = d_f(x, a) = -f(x{*}a) - f(a{*}x)$$

and

$$0 = d_f(y, b) = -f(y{*}b) - f(b{*}y).$$

It follows that $f(x{*}a) = 0 = f(a{*}x)$ and $f(y{*}b) = 0 = f(b{*}y)$ so from (3.3) that $x{*}a = 0 = a{*}x$ and $y{*}b = 0 = b{*}y$. Using (IV), we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore $(X \times X, d_f^*)$ is a metric space. □

Theorem 3.26. *If $f : X \to \mathbb{R}$ is an I-quasi-valuation map of a BCK-algebra X satisfying the condition (3.3), then the operation $*$ in the BCK-algebra X is uniformly continuous.*

Proof. For any $\varepsilon > 0$, if $d_f^*\big((x, y), (a, b)\big) < \frac{\varepsilon}{2}$, then $d_f(x, a) < \frac{\varepsilon}{2}$ and $d_f(y, b) < \frac{\varepsilon}{2}$. Using Proposition 3.23, we have

$$\begin{aligned} d_f(x{*}y, a{*}b) &\leq d_f(x{*}y, a{*}y) + d_f(a{*}y, a{*}b) \\ &\leq d_f(x, a) + d_f(y, b) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore the operation $* : X \times X \to X$ is uniformly continuous. □

The following example illustrates Theorem 3.26.

Example 3.27. Let $X = \{0, a, b, c\}$ be a set with the $*$-operation given by Table 4. Then $(X; *, 0)$ is a BCK-algebra (see [6]). Let f be a real-valued function on X defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ 0 & -3 & -4 & -5 \end{pmatrix}.$$

Then f is an I-quasi-valuation map of X and satisfies the condition (3.3). Using Theorem 3.22, (X, d_f) is a metric space where d_f is given by

$$d_f = \begin{pmatrix} (0,0) & (0,a) & (0,b) & (0,c) & (a,a) & (a,b) & (a,c) & (b,b) & (b,c) & (c,c) \\ 0 & 3 & 4 & 5 & 0 & 7 & 8 & 0 & 9 & 0 \end{pmatrix}.$$

Also, $(X \times X, d_f^*)$ is a metric space where d_f^* is obtained by (3.4), for example,

$$d_f^*\big((a, b), (c, a)\big) = \max\{d_f(a, c), d_f(b, a)\} = \max\{8, 7\} = 8,$$

Table 4: $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$$d_f^*\big((0,b),(a,c)\big) = \max\{d_f(0,a), d_f(b,c)\} = \max\{3,9\} = 9,$$
$$d_f^*\big((c,a),(0,0)\big) = \max\{d_f(c,0), d_f(a,0)\} = \max\{5,3\} = 5,$$

and so on. Now, it is routine to verify that the operation $*$ in the BCK-algebra X

$$* : X \times X \to X,\ (x,y) \mapsto x*y$$

is uniformly continuous.

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