KYUNGPOOK Math. J. 55(2015), 837-858
http://dx.doi.org/10.5666/KMJ.2015.55.4.837
pISSN 1225-6951 eISSN 0454-8124

Cellularity of a Larger Class of Diagram Algebras

N. Karimilla Bi
Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005, Tamilnadu, India
e-mail : karimilla.riasm@gmail.com

Abstract. In this paper, we realize the algebra of $\mathbb{Z}_2$ relations, signed partition algebras and partition algebras as tabular algebras and prove the cellularity of these algebras using the method of [2]. Using the results of Graham and Lehrer in [1], we give the modular representations of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras.

1. Introduction

The study of the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are important because as they are subalgebras of partition algebras which arose naturally as potts model in statistical mechanics. In this paper, we establish the cellularity of the algebra of $\mathbb{Z}_2$-relations and signed partition algebras and hence deduce the modular representations of these algebras. The algebra of $\mathbb{Z}_2$-relations and signed partition algebras are different from the $\mathbb{Z}_2$-colored partition algebra introduced in [4] and Tanabe algebras introduced in [8] which are explained in section 3.

2. Preliminaries

In this section, we some of the Definitions and theorems required for the development of this paper with references.

Definition 2.1.([9]) Let the group $\mathbb{Z}_2$ act on the set X. Then the action of $\mathbb{Z}_2$ on X can be extended to an action of $\mathbb{Z}_2$ on $R(X)$, where $R(X)$ denote the set of all equivalence relations on X, given by

$$g.d = \{(gp, gq) \mid (p, q) \in d\}$$

Received October 3, 2013; revised December 18, 2013; accepted January 29, 2014.
2010 Mathematics Subject Classification: 16Z05.
Key words and phrases: Cellularity, partition algebras, algebra of $\mathbb{Z}_2$-relations and signed partition algebras.
This work was supported by UGC-BSR Meritorious Fellowship.

where $d \in R_X$ and $g \in \mathbb{Z}_2$. (It is easy to see that the relation $g.d$ is again an equivalence relation).

An equivalence relation d on X is said to be a $\mathbb{Z}_2$-stable equivalence relation if $p \sim q$ in d implies that $gp \sim gq$ in d for all g in $\mathbb{Z}_2$. We denote $[k]$ for the set $\{1, 2, \ldots, k\}$. We shall only consider the case when $\mathbb{Z}_2$ acts freely on X; Let $X := [k] \times \mathbb{Z}_2$ and the action is defined by $g.(i, x) = (i, gx)$ for all $1 \leq i \leq k$. Let $R_k^{\mathbb{Z}_2}$ be the set of all $\mathbb{Z}_2$-stable equivalence relations on X.

Notation 2.2.([9]) $R_k^{\mathbb{Z}_2}$ denotes the set of all $\mathbb{Z}_2$-stable equivalence relations on $\{1, 2, \cdots, k\} \times \mathbb{Z}_2$.
Each element $d \in R_k^{\mathbb{Z}_2}$ can be represented as a simple graph on a row of $2k$ vertices.

(i) The vertices $(1, e), (1, g), \cdots, (k, e), (k, g)$ are arranged from left to right in a single row.

(ii) If $(i, g) \sim (j, g') \in R_k^{\mathbb{Z}_2}$ then $((i, g), (j, g'))$ is joined by a line $\forall g, g' \in \mathbb{Z}_2$.

We say that the two graphs are equivalent if they give rise to the same set partition of the $2k$ vertices $\{(1, e), (1, g), \cdots, (k, e), (k, g)\}$. We may regard each element d in $R_{2k}^{\mathbb{Z}_2}$ as a $2k$-partition diagram by arranging the $4k$ vertices $(i, g), i \in [2k], g \in \mathbb{Z}_2$ of d in two rows in such a way that (i, g) is in the top(bottom) row of d if $1 \leq i \leq k\big(k+1 \leq i \leq 2k\big)$ $\forall\ g \in \mathbb{Z}_2$ and put $(k+i, g) = (i', g), 1 \leq i \leq k$, for all $g \in \mathbb{Z}_2$ in the bottom row of d and if $(i, g) \sim (j, g')$ then $(i, g), (j, g')$ is joined by a line $\forall\ g, g' \in \mathbb{Z}_2$.

The diagrams d^+ and d^- are obtained from the diagram d by restricting the vertex set to $\{(1, e), (1, g), \ldots, (k, e), (k, g)\}$ and $\{(1', e), (1', g), \ldots, (k', e), (k', g)\}$ respectively.The diagrams d^+ and d^- are also $\mathbb{Z}_2$-stable equivalence relation and $d^+, d^- \in R_k^{\mathbb{Z}_2}$.

Definition 2.3.([9]) Let $d \in R_{2k}^{\mathbb{Z}_2}$. Then the equation

$$R^d = \{(i, j) \mid \text{ there exists } g, h \in \mathbb{Z}_2 \text{ such that } ((i, g), (j, h)) \in d\}$$

defines an equivalence relation on $[2k]$.

Remark 2.4.([9]) For $d \in R_{2k}^{\mathbb{Z}_2}$ and for every $\mathbb{Z}_2$-stable equivalence class or a connected component C in R^d there exists a unique subgroup denoted by H_C^d where

(i) $H_C^d = \{e\}$ if $(i, e) \nsim (i, g) \quad \forall i \in C$, C is called an $\{e\}$-class or $\{e\}$-component and the $\{e\}$ component C will always occur as a pair and

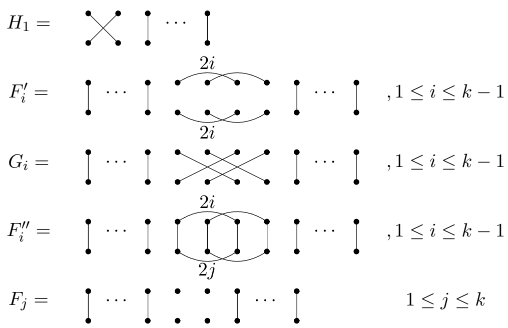
(ii) $H_C^d = \mathbb{Z}_2$ if $(i, e) \sim (i, g) \quad \forall i \in C$, C is called $\mathbb{Z}_2$-class or $\mathbb{Z}_2$-component and the number of vertices in the $\mathbb{Z}_2$-component C will always be even.

Definition 2.5.([9]) The linear span of $R_{2k}^{\mathbb{Z}_2}$ is a subalgebra of $\mathbb{P}_{2k}(x)$. We denote this subalgebra by $A_k^{\mathbb{Z}_2}(x)$, called the **algebra of $\mathbb{Z}_2$-relations**.

Definition 2.6.([9]) Let d be a $2k$-partition diagram. A connected component C of d which contains vertices in both the rows, is called a through class of d and $\sharp^p(d)$ denotes the number of through classes of d, called propagating number. Any

connected component C of d which contains vertices in only one row (either a top row or bottom row) is called a horizontal edge.
For $0 \leq 2s_1 + s_2 \leq 2k$, define $I^{2k}_{2s_1+s_2}$ to be the set of all $2k$-partition diagrams such that $\sharp^p(d) = 2s_1 + s_2$ for all $d \in I^{2k}_{2s_1+s_2}$.
i.e., d has s_1 number of pairs of $\{e\}$ through classes and s_2 number of $\mathbb{Z}_2$ through classes.
Let I_s be the linear space spanned by $\bigcup\limits_{2s_1+s_2 \leq s} I^{2k}_{2s_1+s_2}$.

Definition 2.7.([5], Definition 3.1.1) Let the **signed partition algebra** $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ be the subalgebra of $\mathbb{P}_{2k}(x)$ generated by



The subalgebra of the signed partition algebra generated by $F'_i, G_i, F''_i, F_j, 1 \leq i \leq k-1, 1 \leq j \leq k$ is isomorphic on to the partition algebra $\mathbb{P}_{2k}(x^2)$.

Definition 2.8.([5]) Let $d \in R^{\mathbb{Z}_2}_{2k}$. For $0 \leq r = 2s_1 + s_2 \leq 2k-1, 0 \leq s_1, s_2 \leq k-1$,
$\widetilde{I}^{2k}_{2s_1+s_2} = \Big\{ d \in I^{2k}_{2s_1+s_2} \mid s_1 + s_2 + H_e(d^+) + H_{\mathbb{Z}_2}(d^+) \leq k-1 \text{ and } s_1 + s_2 + H_e(d^-) + H_{\mathbb{Z}_2}(d^-) \leq k-1 \Big\}$,
where

(i) $s_1 = \natural \left\{ C : C \text{ is a through class of } R^d \text{ such that } H^d_C = \{e\} \right\}$,

(ii) $s_2 = \natural \left\{ C : C \text{ is a through class of } R^d \text{ such that } H^d_C = \mathbb{Z}_2 \right\}$,

(iii) $H_e(d^+)\,(H_e(d^-))$ is the number of $\{e\}$ horizontal edges C in the top(bottom) row of R^d such that $H^d_C = \{e\}$ and $|C| \geq 2$,

(iv) $H_{\mathbb{Z}_2}(d^+)\,(H_{\mathbb{Z}_2}(d^-))$ is the number of $\mathbb{Z}_2$ horizontal edges C in the top(bottom) row of R^d such that $H^d_C = \mathbb{Z}_2$.

(v) $\sharp^p(R^d) = s_1 + s_2$.

Put, $\widetilde{I}^{2k}_r = \bigcup\limits_{2s_1+s_2 \leq r} \widetilde{I}^{2k}_{2s_1+s_2}$.

Definition 2.9.([5]) When $s_1 = k, r = 2k, \widetilde{I}^{2k}_r = I^{2k}_{2k}$.

Let $\widetilde{I}_{2k} = \bigcup_{r=0}^{2k} \widetilde{I}_r^{2k}$. The linear span of $\widetilde{I}_{2k}$ is denoted by $\mathscr{H}$.

Theorem 2.10.([5], Theorem 3.1.4 and Theorem 3.1.5)

(i) *$\mathscr{H}$ is a finite-dimensional subalgebra of $A_k^{\mathbb{Z}_2}(x)$ where $\mathscr{H}$ is as in* Definition 2.9.

(ii) *The signed partition algebra $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ and $\mathscr{H}$ are equal.*

Theorem 2.11.([5], Theorem 3.1.7)

(i) *The dimension of $A_k^{\mathbb{Z}_2}(x)$ is*

$$\sum n_\lambda \prod_{i=1}^{t} \left(2^{\lambda_i - 1} + 1\right)$$

where the sum is over the partition $\lambda = (\lambda_1, \cdots, \lambda_t) \vdash 2k$ and n_λ is the number of diagrams $d \in \mathbb{P}_k(x)$ such that $\|d\| = \lambda = (\lambda_1, \cdots, \lambda_t)$ be the partition of $2k$, corresponding to the set partition d, where λ_i is the cardinality of the equivalence class.

(ii) *The dimension of the signed partition algebra $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ is*

$$k!\ 2^k + \sum [(2^r - 1)/2^r]^s \prod_{i \geq 1} (2^{\lambda_i - 1} + 1)$$

where the sum is over the partition diagrams d in $\mathbb{P}_k(x), \|d\| = \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t) \to 2k, r = k - \sharp^p(d), s = 0$ if $|d^+| \neq k$ and $|d^-| \neq k, s = 1$ if and only if $|d^+| = k$ or $|d^-| = k$ and $s = 2$ if $d \notin S_k, |d^+| = |d^-| = k$.

Example 2.12.

(i) For $k = 1, 2, \cdots$, Dimensions of $A_k^{\mathbb{Z}_2}(x)$ are $7, 164, \cdots$

(ii) For $k = 1, 2, 3, \cdots$, Dimensions of $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ are $3, 85, 5055, \cdots$

Lemma 2.13. *Let I_{2k}^{2k} be as in* Definition 2.9 *then $I_{2k}^{2k} \simeq \mathbb{Z}_2 \wr \mathfrak{S}_k$.*

Proof. Let $d \in I_{2k}^{2k}$, then $\sharp(d) = 2k$ and $\sharp(R^d) = k$ and R^d is a permutation in $\mathfrak{S}_k$.

Define,

$$f(i) = \begin{cases} \overline{1}, & \text{if } (i, e) \sim (i', g); \\ \overline{0}, & \text{if } (i, e) \sim (i', e). \end{cases}$$

Thus, $d = (f, R^d) \in \mathbb{Z}_2 \wr \mathfrak{S}_k$. □

Theorem 2.14.([7], Theorem 3.26) *Let R be a commutative ring with unity. Let $\Lambda_{s_1} = \{(\lambda_1, \lambda_2) \mid \lambda_1 \vdash k_1, \lambda_2 \vdash k_2, k_1 + k_2 = s_1\}$ and $\Lambda_{s_2} = \{\mu \mid \mu \vdash s_2\}$. For $(\lambda_1, \lambda_2) \in \Lambda_{s_1}$, and $\mu \in \Lambda_{s_2}$ define $M^{(\lambda_1, \lambda_2)}$ and M^μ be the set of all standard tableaux of shape (λ_1, λ_2) and μ respectively.*

(i) *The algebra* $\mathscr{H} = R[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ *is a free R-module with basis* $\mathscr{M} = \big\{m^{\lambda}_{s_\lambda,t_\lambda} \mid s_\lambda$ *and* t_λ *are standard tableaux of shape* λ *for some bi-partition* λ *of* k *in* $M^{(\lambda_1,\lambda_2)}$ *and* $\lambda = (\lambda_1, \lambda_2)\big\}$ *where* $m^{\lambda}_{s_\lambda,t_\lambda}$ *is as in* Definition 3.14 *of* [7].

Moreover, $\mathscr{M}$ *is a cellular basis for* $\mathscr{H}$.

(ii) *The algebra* $\mathscr{H}' = R[\mathfrak{S}_{s_2}]$ *is a free R-module with basis* $\mathscr{M}' = \big\{m^{\mu}_{s_\mu,t_\mu} \mid s_\mu$ *and* t_μ *are standard tableaux of shape* μ *for some partition* μ *of* k *in* $M^{\mu}\big\}$ *where* $m^{\mu}_{s_\mu,t_\mu}$ *is as in* Definition 3.14 *of* [7].

Moreover, $\mathscr{M}$ *is a cellular basis for* $\mathscr{H}'$.

Also, $\mathscr{M}$ *is a cellular basis for* $\mathscr{H}' \otimes K(x)$, *where* K *is a field.*

Theorem 2.15. *Let* $R\left[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}\right]$ *be the R-algebra, then by* Theorem 2.14, $R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}] \simeq R[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}] \otimes R[\mathfrak{S}_{s_2}]$ *is a cellular algebra with a cell datum* $(\Lambda_{s_1,s_2}, M^{((\lambda_1,\lambda_2),\mu)}, C^{((\lambda_1,\lambda_2),\mu)}, *)$ *given as follows:*

(i) $\Lambda_{s_1,s_2} := \{((\lambda_1,\lambda_2),\mu) \mid |\lambda_1|+|\lambda_2| = s_1, \mu \vdash s_2\} \cup \{((\lambda_1,\lambda_2),\Phi) \mid |\lambda_1|+|\lambda_2| = s_1\} \cup \{((\Phi,\Phi),\mu) \ |\mu \vdash s_2\} \cup \{\Phi\}$ (*ordered lexicographically*) *is a partially ordered set.*

(ii) $M^{((\lambda_1,\lambda_2),\mu)} := \{((s_{\lambda_1}, s_{\lambda_2}), s_\mu) \mid s_{\lambda_1}, s_{\lambda_2}$ *and* s_μ *are the standard tableaus of shape* λ_1, λ_2 *and* μ *respectively*$\}$ *such that*

$$C^{((\lambda_1,\lambda_2),\mu)} : \coprod_{\lambda,\mu\in\Lambda} M^{((\lambda_1,\lambda_2),\mu)} \times M^{((\lambda_1,\lambda_2),\mu)} \to (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$$

is an injective map with image an R basis of $(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.

(iii) *If* $\lambda = (\lambda_1, \lambda_2)$ *and* $S = ((s_{\lambda_1}, s_{\lambda_2}), s_\mu), T = ((t_{\lambda_1}, t_{\lambda_2}), t_\mu) \in M^{((\lambda_1,\lambda_2),\mu)}$, *write*

$$C(S,T) = m^{\lambda}_{s_\lambda t_\lambda} m^{\mu}_{s_\mu t_\mu}$$

where $m^{\lambda}_{s_\lambda t_\lambda}$ *and* $m^{\mu}_{s_\mu t_\mu}$ *are as in* Theorem 2.14. $*$ *is the anti-automorphism of* $(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *such that* $((f,\sigma_1),\sigma_2)^* = ((f,\sigma_1)^*, \sigma_2^*) = ((f,\sigma_1)^{-1}, \sigma_2^{-1})$ $\forall\, ((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *and* $\left(C^{((\lambda_1,\lambda_2),\mu)}(S,T)\right)^* = C^{((\lambda_1,\lambda_2),\mu)}(T,S)$.

3. Differences between the Algebras

In this section, we illustrate that the algebra of $\mathbb{Z}_2$-relations $A^{\mathbb{Z}_2}_k(x)$ and signed partition algebras $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ are different from the $\mathbb{Z}_2$-colored partition algebra $P_k(x;\mathbb{Z}_2)$ introduced in [4] and Tanabe algebras $T_{k,m}(x)$ introduced in [8].

Example 3.1. This example clearly illustrates that the signed partition algebras are different from $\mathbb{Z}_2$-colored partition algebra introduced in [4].

R^d	Diagrams in $\overrightarrow{A}_2^{\mathbb{Z}_2}$	Diagrams in G-colored partition algebras by Bloss for $G = \mathbb{Z}_2$
	fig 1, fig 2, fig 3, fig 4, fig 5, fig 6, fig 7, fig 8, fig 9	fig 1′, fig 2′, fig 3′, fig 4′, fig 5′, fig 6′, fig 7′, fig 8′
	fig 10, fig 11, fig 12, fig 13, fig 14	fig 9′, fig 10′, fig 11′, fig 12′
	fig 15, fig 16, fig 17, fig 18, fig 19	fig 13′, fig 14′, fig 15′, fig 16′
	fig 20, fig 21, fig 22, fig 23, fig 24	fig 17′, fig 18′, fig 19′, fig 20′
	fig 25, fig 26, fig 27, fig 28, fig 29	fig 21′, fig 22′, fig 23′, fig 24′
	fig 30, fig 31, fig 32, fig 33	fig 25′, fig 26′, fig 27′, fig 28′
	fig 34, fig 35, fig 36, fig 37	fig 29′, fig 30′, fig 31′, fig 32′
	fig 38, fig 39, fig 40, fig 41, fig 42, fig 43, fig 44, fig 45, fig 46	fig 33′, fig 34′, fig 35′, fig 36′

	fig 47, fig 48, fig 49	fig 37′, fig 38′
	fig 50, fig 51, fig 52	fig 39′, fig 40′
	fig 53, fig 54, fig 55	fig 41′, fig 42′
	fig 56, fig 57, fig 58	fig 43′, fig 44′
	fig 59, fig 60, fig 61, fig 62, fig 63, fig 64, fig 65, fig 66, fig 67	fig 45′, fig 46′
	fig 68, fig 69, fig 70, fig 71, fig 72, fig 73, fig 74, fig 75, fig 76	fig 47′, fig 48′
	fig 77, fig 78, fig 79, fig 80, fig 81, fig 82, fig 83, fig 84, fig 85	fig 49′

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Note 1. In the algebra of $\mathbb{Z}_2$-relations and signed partition algebras, the set of all diagrams having no horizontal edges and each through class contains two vertices is isomorphic to the hyperoctahedral group of type B_n whereas in Tanabe algebras, the set of all diagrams having no horizontal edges and each through class contains two vertices is isomorphic to the symmetric group.

Thus, the representations of algebra of $\mathbb{Z}_2$ relations and signed partition algebras are determined by the representations of hyperoctahedral group of type B_n whereas the representations of Tanabe algebras are determined by the representations of symmetric group.

4. The Algebra of $\mathbb{Z}_2$-Relations and Signed Partition Algebras

In this section, we realize the algebra of $\mathbb{Z}_2$-relations and signed partition algebras as tabular algebras introduced in [2].

Notation 4.1. Let $d \in I^{2k}_{2s_1+s_2}$ $\left(\widetilde{d} \in \widetilde{I}^{2k}_{2s_1+s_2}\right)$, be as in Definition 2.5(2.7).

(i) The vertex having least integer value in a connected component of $d(\widetilde{d})$ is called the minimal vertex of the connected component.

(ii) $|d|(|\widetilde{d}|)$ denotes the number of connected components in $d(\widetilde{d})$.

Definition 4.2. Define,

(i) $M[(r,(s_1,s_2))] = \Big\{(d,P) \mid d \in R^{\mathbb{Z}_2}_k, P \in R^{\mathbb{Z}_2}_{k'}$ and $d \setminus P \in R^{\mathbb{Z}_2}_{k-k'}, |d| \geq 2s_1 + s_2, P$ is a $\mathbb{Z}_2-$stable subset of the set of all connected components of d with $|P| = 2s_1+s_2$ where $r = 2s_1+s_2, P = \bigcup\limits_{i=1}^{s_1}(P^e_i \cup P^g_i) \bigcup\limits_{j=1}^{s_2} P^{\mathbb{Z}_2}_j$ such that $H^d_{R^{P_i^{\{e\}}}} = \{e\}, 1 \leq i \leq s_1, H^d_{R^{P_j^{\mathbb{Z}_2}}} = \mathbb{Z}_2, 1 \leq j \leq s_2\Big\}$.

(ii) $\widetilde{M}[(r,(s_1,s_2))] = \Big\{(\widetilde{d},\widetilde{P}) \mid \widetilde{d} \in R^{\mathbb{Z}_2}_k, \widetilde{P} \in R^{\mathbb{Z}_2}_{k'}$ and $\widetilde{d} \setminus \widetilde{P} \in R^{\mathbb{Z}_2}_{k-k'}, |d| \geq 2s_1 + s_2, \widetilde{P}$ is a $\mathbb{Z}_2-$stable subset of the set of all connected components of $\widetilde{d}$ with $|\widetilde{P}| = 2s_1+s_2$ where $r = 2s_1+s_2, \widetilde{P} = \bigcup\limits_{i=1}^{s_1}(\widetilde{P}^e_i \cup \widetilde{P}^g_i) \bigcup\limits_{j=1}^{s_2} \widetilde{P}^{\mathbb{Z}_2}_j$ such that $H^d_{R^{\tilde{P}_i^{\{e\}}}} = \{e\}, 1 \leq i \leq s_1$, $H^d_{R^{\tilde{P}_j^{\mathbb{Z}_2}}} = \mathbb{Z}_2, 1 \leq j \leq s_2$ and $2r_1(r_2)$ is the number of $\{e\}(\mathbb{Z}_2)$ connected components in $\widetilde{d} \setminus \widetilde{P}, s_1+s_2+r_1+r_2 \leq k-1$ if $s_1+s_2+r_1+r_2 = k$ then $s_1 = k$ or $r_1 \neq 0\Big\}$.

We shall now introduce an ordering for the connected components in P. Suppose that $P = P^e_1 \cup P^g_1 \cup \cdots \cup P^e_{s_1} \cup P^g_{s_1} \cup P^{\mathbb{Z}_2}_1 \cup \cdots \cup P^{\mathbb{Z}_2}_{s_2}$ then $R^P = R^{P_1^{\{e\}}} \cup \cdots \cup R^{P_{s_1}^{\{e\}}} \cup R^{P_1^{\mathbb{Z}_2}} \cup \cdots \cup R^{P_{s_2}^{\mathbb{Z}_2}}$.

Let $a_{11}, \cdots, a_{1s_1}$ be the minimal vertices of the connected components $R^{P_1^{\{e\}}}, \cdots,$ $R^{P_{s_1}^{\{e\}}}$ in R^P and $b_{11}, \cdots, b_{1s_2}$ be the minimal vertices of the connected components $R^{P_1^{\mathbb{Z}_2}}, \cdots, R^{P_{s_2}^{\mathbb{Z}_2}}$ in R^P then

$$P_i^e < P_j^e \text{ and } P_i^g < P_j^g \text{ if and only if } R^{P_i^{\{e\}}} < R^{P_j^{\{e\}}} \text{ if and only if } a_{1i} < a_{1j} \in R^P$$
$$\text{and } P_l^{\mathbb{Z}_2} < P_f^{\mathbb{Z}_2} \text{ if and only if } R^{P_l^{\mathbb{Z}_2}} < R^{P_f^{\mathbb{Z}_2}} \text{ if and only if } b_{1l} < b_{1f} \in R^P.$$

Similarly, we can introduce an ordering for the connected components in $\widetilde{P}$ as in P.

Lemma 4.3. *Let $M[(r,(s_1,s_2))]$ and $\widetilde{M}[(r,(s_1,s_2))]$ be as in Definition 4.2.*

(i) *Each $d \in I^{2k}_{2s_1+s_2}$ can be associated with a pair of elements $(d^+, P), (d^-, Q) \in M[(r,(s_1,s_2))]$ and an element $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where (d^+, P), $(d^-, Q) \in M[(r,(s_1,s_2))]$ and $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.*

(ii) *Each $\widetilde{d} \in \widetilde{I}^{2k}_{2s_1+s_2}$ can be associated with a pair of elements $(\widetilde{d}^+, \widetilde{P}), (\widetilde{d}^-, \widetilde{Q}) \in \widetilde{M}[(r,(s_1,s_2))]$ and an element $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where $(\widetilde{d}^+, \widetilde{P})$, $(\widetilde{d}^-, \widetilde{Q}) \in \widetilde{M}[(r,(s_1,s_2))]$ and $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.*

Proof. **Proof of (i) :** Let $d \in I^{2k}_{2s_1+s_2}$.
d^+, d^- are the diagrams obtained from the diagram d by restricting the vertex set to $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ and $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ respectively.
Identifying $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ with $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ by sending

$$(i',e) \mapsto (i,e) \text{ and } (i',g) \mapsto (i,g).$$

Thus, $d^+, d^- \in R_k^{\mathbb{Z}_2}$.
Let S_d be the set of all through classes of d. Let P denote the set of all connected components obtained from S_d by restricting the vertex set to $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$.
i.e., $S_d \cap d^+ = P$.
Thus, $|P| = 2s_1 + s_2$.
Similarly, let Q denote the set of all connected components obtained from S_d by restricting the vertex set to $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$. i.e., $S_d \cap d^- = Q$.
Identify $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ with $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ by sending

$$(i',e) \mapsto (i,e) \text{ and } (i',g) \mapsto (i,g).$$

Thus, $|Q| = 2s_1 + s_2$.
Write

$$P = P_1^e \cup P_1^g \cup \cdots P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots P_{s_2}^{\mathbb{Z}_2} \text{ and}$$
$$Q = Q_1^e \cup Q_1^g \cup \cdots \cup Q_{s_1}^e \cup Q_{s_1}^g \cup Q_1^{\mathbb{Z}_2} \cup \cdots Q_{s_2}^{\mathbb{Z}_2}$$

Define an element (f,σ_1) as follows:
If there is a connected component $X \in S_d$ containing P_i^e and $Q_j^{g'}, g' \in \mathbb{Z}_2$ then, define $\sigma_1(i) = j$ and

$$f(i) = \begin{cases} \overline{1}, & \text{if } g' = g; \\ \overline{0}, & \text{if } g' = e. \end{cases}$$

Thus, $(f,\sigma_1) \in \mathbb{Z}_2 \wr \mathfrak{S}_{s_1}$.
Similarly, define σ_2 as follows:

If there is a connected component $Y \in S_d$ containing $P_l^{\mathbb{Z}_2}$ and $Q_m^{\mathbb{Z}_2}$ then, define $\sigma_2(l) = m$.
Thus, $\sigma_2 \in \mathfrak{S}_{s_2}$ which implies that $((f, \sigma_1), \sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.

Proof of (ii) : By Definition 2.8, $s_1 + s_2 + H_e(\widetilde{d}^+) + H_{\mathbb{Z}_2}(\widetilde{d}^+) \leq k - 1$ and $s_1 + s_2 + H_e(\widetilde{d}^-) + H_{\mathbb{Z}_2}(\widetilde{d}^-) \leq k - 1$ and the proof of (ii) is same as proof of (i). □

Lemma 4.4.

(i) *For every pair* $(d', P), (d'', Q) \in M[(r, (s_1, s_2))]$ *and an element* $((f, \sigma_1), \sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *there is a unique diagram* $d \in I^{2k}_{2s_1+s_2}$ *where* $d^+ = (d', P), d^- = (d'', Q)$ *such that there is a unique connected component of* d *containing* P_i^e *and* $Q^{g'}_{\sigma_1(i)}$ *and* $P_j^{\mathbb{Z}_2}$ *and* $Q^{\mathbb{Z}_2}_{\sigma_2(j)}$.

(ii) *For every pair* $(\widetilde{d}', \widetilde{P}), (\widetilde{d}'', \widetilde{Q}) \in \widetilde{M}[(r, (s_1, s_2))]$ *and an element* $((\widetilde{f}, \widetilde{\sigma}_1), \widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *there is a unique diagram* $\widetilde{d} \in \widetilde{I}^{2k}_{2s_1+s_2}$ *where* $\widetilde{d}^+ = (\widetilde{d}', \widetilde{P}), \widetilde{d}^- = (\widetilde{d}'', \widetilde{Q})$ *such that there is a unique connected component of* $\widetilde{d}$ *containing* $\widetilde{P}_i^e$ *and* $\widetilde{Q}^{g'}_{\sigma_1(i)}$ *and* $\widetilde{P}_j^{\mathbb{Z}_2}$ *and* $\widetilde{Q}^{\mathbb{Z}_2}_{\sigma_2(j)}$.

Proof. **Proof of (i) :** Let $P = P_1^e \cup P_1^g \cup \cdots \cup P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots P_{s_2}^{\mathbb{Z}_2}$ and $Q = Q_1^e \cup Q_1^g \cup \cdots \cup Q_{s_1}^e \cup Q_{s_1}^g \cup Q_1^{\mathbb{Z}_2} \cup \cdots \cup Q_{s_2}^{\mathbb{Z}_2}$.
Let $\{a^e_{11}, \cdots, a^e_{1s_1}\}, \{a^g_{11}, \cdots, a^g_{1s_1}\}$ and $\{b^e_{11}, \cdots, b^e_{1s_2}\}$ be the minimal vertices of the connected components $\{R^{P_1^e}, \cdots, R^{P^e_{s_1}}\}, \{R^{P_1^g}, \cdots, R^{P^g_{s_1}}\}$ and $\{R^{P_1^{\mathbb{Z}_2}}, \cdots, R^{P^{\mathbb{Z}_2}_{s_2}}\}$ respectively.
Similarly, let $\{l^e_{11}, \cdots, l^e_{1s_1}\}, \{l^g_{11}, \cdots, l^g_{1s_1}\}$ and $\{f^e_{11}, \cdots, f^e_{1s_2}\}$ be the minimal vertices of the connected components $\{R^{Q_1^e}, \cdots, R^{Q^e_{s_1}}\}, \{R^{Q_1^g}, \cdots, R^{Q^g_{s_1}}\}$ and $\{R^{Q_1^{\mathbb{Z}_2}}, \cdots, R^{Q^{\mathbb{Z}_2}_{s_2}}\}$ respectively.
Let $d \in \widetilde{I}^{2k}_{2s_1+s_2}$ be obtained as follows:

(i) Draw (d', P) above (d'', Q).

(ii) Connect P_i^e to $Q^{g'}_{\sigma_1(i)}$ if $f(i) = g'$. Also, connect $P_j^{\mathbb{Z}_2}$ to $Q^{\mathbb{Z}_2}_{\sigma_2(j)}$.

(iii) All other connected components in (d', P) $((d'', Q))$ other than the connected components of $P(Q)$ will remain as horizontal edges or isolated points in the top(bottom) row of $d \in \widetilde{I}^{2k}_{2s_1+s_2}$, by our construction $d^+ = (d', P)$ and $d^- = (d'', Q)$.

Proof of (ii) : Proof of (ii) is similar to the proof of (i). □

Remark 4.5. By Lemma 4.3, any $d \in I_{2k}(\widetilde{d} \in \widetilde{I}_{2k})$, is denoted by
$$C^{((f,\sigma_1),\sigma_2)}_{(d^+,P),(d^-,Q)} \left(\widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d}^+,\widetilde{P}),(\widetilde{d}^-,\widetilde{Q})} \right).$$

Definition 4.6.

(i) Define a map $\phi^r_{s_1,s_2} : M[(r,(s_1,s_2))] \times M[(r,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$ as follows:

$$\phi^r_{s_1,s_2}((d',P),(d'',Q)) = x^{l(P\vee Q)}((f,\sigma_1),\sigma_2) \text{ and}$$

(ii) Define a map $\widetilde{\phi}^r_{s_1,s_2} : \widetilde{M}[(r,(s_1,s_2))] \times \widetilde{M}[(r,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$ as follows:
$$\widetilde{\phi}^r_{s_1,s_2}\left((\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})\right) = x^{l(\widetilde{P}\vee\widetilde{Q})}((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)$$

Case (i) : if

(a) No two connected components of $Q(\widetilde{Q})$ in $d''(\widetilde{d}'')$ have non-empty intersection with a common connected component of $d'(\widetilde{d}')$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$, or vice versa.

(b) No connected component of $Q(\widetilde{Q})$ has non-empty intersection only with the connected components excluding the connected components of $P(\widetilde{P})$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$. Similarly, no connected component in $P(\widetilde{P})$ has non-empty intersection only with a connected component excluding the connected components of $Q(\widetilde{Q})$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$.

where $l(P \vee Q)\left(l(\widetilde{P}\vee\widetilde{Q})\right)$ denotes the number of connected components in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ excluding the union of all the connected components of $P(\widetilde{P})$ and $Q(\widetilde{Q})$. The permutation $((f,\sigma_1),\sigma_2)\left(\left((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2\right)\right)$ is obtained as follows: If there is a unique connected component in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ containing $P^e_i(\widetilde{P}^e_i)$ and $Q^{g'}_j(\widetilde{Q}^{g'}_j)$ then, define $\sigma_1(i) = j(\widetilde{\sigma}_1(i) = j)$ and

$$f(i) = \widetilde{f}(i) = \begin{cases} \overline{1}, & \text{if } g' = g; \\ \overline{0}, & \text{if } g' = e. \end{cases}$$

Also, if there is a unique connected component in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ containing $P^{\mathbb{Z}_2}_l$ and $Q^{\mathbb{Z}_2}_f(\widetilde{P}^{\mathbb{Z}_2}_l$ and $\widetilde{Q}^{\mathbb{Z}_2}_f)$ then, define $\sigma_2(l) = f(\widetilde{\sigma}_2(l) = f)$.

Case (ii) : Otherwise, $\phi^r_{s_1,s_2}((d',P),(d'',Q)) = 0\left(\widetilde{\phi}^r_{s_1,s_2}\left((\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})\right) = 0\right)$.

Since the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are subalgebras of partition algebras the proof of Lemmas 4.7 and 4.8 follow as in [10].

Lemma 4.7.

(i) *Let $\mu,\nu \in I^{2k}_{2s_1+s_2}$ then $\sharp^p(\mu\nu) \leq 2s_1+s_2$. If $\sharp^p(\mu\nu) = 2s_1+s_2$ then*

$$\mu\nu = C^{r_\mu[(d,R),(d'',Q)]((f',\sigma_1'),\sigma_2')}_{((d,R),(d''',T))}$$

where $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}, \nu = C^{((f',\sigma_1'),\sigma_2')}_{(d'',Q),(d''',T)}, (d,R),(d',P),(d'',Q),(d''',T) \in M[(r,(s_1,s_2))]$, $((f,\sigma_1),\sigma_2), ((f',\sigma_1'),\sigma_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1})\times\mathfrak{S}_{s_2}$, $r_\mu[(d,R),(d'',Q)] = ((f,\sigma_1),\sigma_2)\phi^r_{s_1,s_2}[(d',P),(d'',Q)]$ and $r_\mu[(d,R),(d'',Q)]$ is independent of (d''',T) and $((f',\sigma_1'),\sigma_2')$.

(ii) *Let* $\widetilde{\mu}, \widetilde{\nu} \in \widetilde{I}^{2k}_{2s_1+s_2}$ *then* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) \leq 2s_1 + s_2$. *If* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) = 2s_1 + s_2$ *then*

$$\widetilde{\mu}\widetilde{\nu} = C^{r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})]((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{((\widetilde{d},\widetilde{R}),(\widetilde{d}''',\widetilde{T}))}$$

where $\widetilde{\mu} = C^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P})}, \widetilde{\nu} = \widetilde{C}^{((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T})}, (\widetilde{d},\widetilde{R}), (\widetilde{d}',\widetilde{P}), (\widetilde{d}'',\widetilde{Q}), (\widetilde{d}''',\widetilde{T}) \in \widetilde{M}[(r,(s_1,s_2))]$, $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2), ((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$, $r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})] = ((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)\widetilde{\phi}^r_{s_1,s_2}[(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})]$ *and* $r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})]$ *is independent of* $(\widetilde{d}''',\widetilde{T})$ *and* $((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')$.

Proof. If $\sharp^p(\mu\nu) = 2s_1 + s_2$, then the proof of (i) and (ii) follows from the definition of multiplication of partition algebras and Definition 4.6 and from Lemma 4.4 of [10]. □

Lemma 4.8.

(i) *Let* $\mu \in I^{2k}_{2s_1'+s_2'}, \nu \in I^{2k}_{2s_1+s_2}$ *then* $\sharp^p(\mu\nu) \leq 2s_1 + s_2$. *If* $\sharp^p(\mu\nu) = 2s_1 + s_2$ *then*

$$\mu\nu = x^{l(P\vee Q)} C^{r_\mu[(w,F),(d'',Q)]((f',\sigma_1'),\sigma_2')}_{(w,F),(d''',T)}$$

where $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}, \nu = C^{((f',\sigma_1'),\sigma_2')}_{(d'',Q),(d''',T)}, (d,R),(d',P) \in M[(r',(s_1',s_2'))]$, $(w,F),(d'',Q),(d''',T) \in M[(r,(s_1,s_2))]$, $((f,\sigma_1),\sigma_2) \in \left(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1'}\right) \times \mathfrak{S}_{s_2'}$, $((f',\sigma_1'),\sigma_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_\mu[(w,F),(d'',Q)]$ *is independent of* $((f',\sigma_1'),\sigma_2')$ *and* (d''',T).

(ii) *Let* $\widetilde{\mu} \in \widetilde{I}^{2k}_{2s_1'+s_2'}, \widetilde{\nu} \in \widetilde{I}^{2k}_{2s_1+s_2}$ *then* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) \leq 2s_1 + s_2$. *If* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) = 2s_1 + s_2$ *then*

$$\widetilde{\mu}\widetilde{\nu} = x^{l(\widetilde{P}\vee\widetilde{Q})} C^{r_{\widetilde{\mu}}[(\widetilde{w},\widetilde{F}),(\widetilde{d}'',\widetilde{Q})]((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{(\widetilde{w},\widetilde{F}),(\widetilde{d}''',\widetilde{T})}$$

where $\widetilde{\mu} = \widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P})}, \widetilde{\nu} = \widetilde{C}^{((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T})}, (\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P}) \in \widetilde{M}[(r',(s_1',s_2'))]$, $(\widetilde{w},\widetilde{F}),(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T}) \in \widetilde{M}[(r,(s_1,s_2))]$, $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in \left(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1'}\right) \times \mathfrak{S}_{s_2'}$, $((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_{\widetilde{\mu}}[(\widetilde{w},\widetilde{F}),(\widetilde{d}'',\widetilde{Q})]$ *is independent of* $((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')$ *and* $(\widetilde{d}''',\widetilde{T})$.

Proof. **Proof of (i) :** If $\sharp^p\left(C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}\ C^{((f',\sigma_1'),\sigma_2')}_{(d'',Q),(d''',T)}\right) = 2s_1 + s_2$, then by Lemma 4.4 and [10] there exists $(w,F),(v,Q) \in M[(r,(s_1,s_2))], ((f'',\sigma_1''),\sigma_2'') \in \mathbb{Z}_2 \wr \mathfrak{S}_{s_1} \times \mathfrak{S}_{s_2}$.

$$C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}\ C^{((f',\sigma_1'),\sigma_2')}_{(d'',Q),(d''',T)} = C^{r_\mu[(w,F),(d'',Q)]((f',\sigma_1'),\sigma_2')}_{(w,F),(v,Q)}$$

where $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}$ $r_\mu[(w,F),(d'',Q)]((f',\sigma_1'),\sigma_2') = x^{l(P\vee Q)}((f'',\sigma_1''),\sigma_2'')$ and it is independent of $((f',\sigma_1'),\sigma_2')$ and (d''',T).

Proof of (ii) : Proof of (ii) is same as that of proof of (i). □

Definition 4.9. Put,

(i) $\Lambda = \{(r,(s_1,s_2)) \mid r = 2s_1+s_2, 0 \le s_1, s_2 \le k\}$ and

(ii) $\widetilde{\Lambda} = \{(r,(s_1,s_2)) \mid r = 2s_1+s_2, 0 \le s_1 \le k, 0 \le s_2 \le k-1\}$.

Define a relation '$\le$' on $\Lambda(\widetilde{\Lambda})$ as follows:

$$(r,(s_1,s_2)) \le (r',(s'_1,s'_2))$$

if and only if

(a) $r < r'$ or

(b) $r' = r$ and $s_1+s_2 < s'_1+s'_2$

Thus, $(\Lambda,\le)((\widetilde{\Lambda},\le))$ is a partially ordered set.

Note 2. Let $B^r(s_1,s_2) = B(s_1,s_2) = (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ and $\Gamma(s_1,s_2) = \mathscr{A}[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$, where $r = 2s_1+s_2$. The elements of $B(s_1,s_2)$ forms a basis of $\Gamma(s_1,s_2)$. Thus $(\Gamma(s_1,s_2), B(s_1,s_2))$ is a hyper group.

Definition 4.10. Let $M[(r,(s_1,s_2))]\left(\widetilde{M}[(r,(s_1,s_2))]\right)$ be as in Definition 4.2.

Define maps,

(i) $C : M[(r,(s_1,s_2))] \times B(s_1,s_2) \times M[(r,(s_1,s_2))] \to A_k^{\mathbb{Z}_2}$ as follows:

$$C\left[(d',P),((f,\sigma_1),\sigma_2),(d'',Q)\right] = d,$$

where $d = C^{((f,\sigma_1),\sigma_2)}_{(d',P),(d'',Q)}$, as in Remark 4.5 and $d \in I^{2k}_{2s_1+s_2}$.

By Lemma 4.4, it is clear that C is injective.

(ii) $\widetilde{C} : \widetilde{M}[(r,(s_1,s_2))] \times B(s_1,s_2) \times \widetilde{M}[(r,(s_1,s_2))] \to \overrightarrow{A}_k^{\mathbb{Z}_2}$, as follows:

$$\widetilde{C}\left[(\widetilde{d}',\widetilde{P}),((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2),(\widetilde{d}'',\widetilde{Q})\right] = \widetilde{d},$$

where $\widetilde{d} = \widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}$, as in Remark 4.5 and $\widetilde{d} \in \widetilde{I}^{2k}_{2s_1+s_2}$.

By Lemma 4.4, it is clear that $\widetilde{C}$ is injective.

Definition 4.11. Define,

(i) $* : A_k^{\mathbb{Z}_2} \to A_k^{\mathbb{Z}_2}$ as follows:

$$\left(C^{((f,\sigma_1),\sigma_2)}_{(d',P),(d'',Q)}\right)^* = \left(C^{((f,\sigma_1),\sigma_2)}_{(d',P),(d'',Q)}\right)^f = C^{((f,\sigma_1),\sigma_2)^{-1}}_{(d'',Q),(d',P)}$$

where f is the flip of the diagram and inverse mapping is the anti-automorphism of the hyper group $(\Gamma(s_1,s_2), B(s_1,s_2))$.

Clearly, $*$ is an involutary anti-automorphism of $A_k^{\mathbb{Z}_2}$.

(ii) $\widetilde{*} : \overrightarrow{A}_k^{\mathbb{Z}_2} \to \overrightarrow{A}_k^{\mathbb{Z}_2}$ as follows:

$$\left(\widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}\right)^{\widetilde{*}} = \left(\widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}\right)^f = \widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)^{-1}}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}',\widetilde{P})}$$

where f is the flip of the diagram and inverse mapping is the anti-automorphism of the hyper group $(\Gamma(s_1, s_2), B(s_1, s_2))$.

Clearly, $*$ is an involutary anti-automorphism of $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

Notation 4.12. If $b \in \Gamma(s_1, s_2)$ such that $b = \sum_{((f_i,\sigma_{1_i}),\sigma_{2_i}) \in B(s_1,s_2)} c_i((f_i, \sigma_{1_i}), \sigma_{2_i})$ for some scalars.

(i) We write $C^b_{(d',P),(d'',Q)} \in \mathscr{A}\left[A_k^{\mathbb{Z}_2}\right]$ as shorthand for $\sum_{((f_i,\sigma_{1_i}),\sigma_{2_i}) \in B(s_1,s_2)} c_i C^{((f_i,\sigma_{1_i}),\sigma_{2_i})}_{(d',P),(d'',Q)}$.

Also, write C_{s_1,s_2} for the image under C of $M[(r,(s_1,s_2))] \times B(s_1,s_2) \times M[(r,(s_1,s_2))]$.

(ii) We write $\widetilde{C}^b_{(\widetilde{d'},\widetilde{P}),(\widetilde{d''},\widetilde{Q})} \in \mathscr{A}\left[\overrightarrow{A}_k^{\mathbb{Z}_2}\right]$ as shorthand for $\sum_{((\widetilde{f}_i,\widetilde{\sigma}_{1_i}),\widetilde{\sigma}_{2_i}) \in B(s_1,s_2)} c_i \widetilde{C}^{((\widetilde{f}_i,\widetilde{\sigma}_{1_i}),\widetilde{\sigma}_{2_i})}_{(\widetilde{d'},\widetilde{P}),(\widetilde{d''},\widetilde{Q})}$.

Also, write $\widetilde{C}_{s_1,s_2}$ for the image under $\widetilde{C}$ of $\widetilde{M}[(r,(s_1,s_2))] \times B(s_1,s_2) \times \widetilde{M}[(r,(s_1,s_2))]$.

Theorem 4.13. *Let* $\mathscr{A} = \mathbb{C}(x)$.

(i) *An algebra of* $\mathbb{Z}_2$*-relations* $\mathscr{A}[A_k^{\mathbb{Z}_2}]$ *is a tabular algebra together with a table datum* $(\Lambda, \Gamma, B, M[(r,(s_1,s_2))], C, *)$ *where :*

(a) Λ *is a finite poset where* Λ *is as in* Definition 4.9. *For each* $(r,(s_1,s_2)) \in \Lambda$, $(\Gamma(s_1,s_2), B(s_1,s_2))$ *is a hypergroup over* $\mathbb{C}$ *and* $M[(r,(s_1,s_2))]$ *is a finite set. The map*

$$C : \coprod_{(r,(s_1,s_2)) \in \Lambda} (M[(r,(s_1,s_2))] \times B(s_1,s_2) \times M[(r,(s_1,s_2))]) \to A_k^{\mathbb{Z}_2}$$

is injective with image an $\mathscr{A}$*-basis of* $A_k^{\mathbb{Z}_2}$.

(b) $*$ *is an* $\mathscr{A}$*-linear involutary anti-automorphism of* $A_k^{\mathbb{Z}_2}$.

(c) *If* $(r,(s_1,s_2)) \in \Lambda, ((f,\sigma_1),\sigma_2) \in \Gamma(s_1,s_2)$ *and* $(d',P),(d'',Q) \in M[(r,(s_1,s_2))]$ *then for all* $a \in A_k^{\mathbb{Z}_2}$ *we have*

$$aC^{((f,\sigma_1),\sigma_2)}_{(d',P),(d'',Q)} \equiv \sum_{(d_i''',R_i) \in M[(r,(s_1,s_2))]} C^{r_a[(d_i''',R_i),(d',P)]((f,\sigma_1),\sigma_2)}_{(d_i''',R_i),(d'',Q)} \quad \mathrm{mod}\ A_k^{\mathbb{Z}_2}(< (r,(s_1,s_2)))$$

where $r_a[(d_i''',R_i),(d',P)]((f,\sigma_1),\sigma_2)$ *is independent of* (d'',Q) *and of* $((f,\sigma_1),\sigma_2)$.

(ii) *An algebra of signed partition algebras* $\mathscr{A}[\overrightarrow{A}_k^{\mathbb{Z}_2}]$ *is a tabular algebra together with a table datum* $(\widetilde{\Lambda}, \Gamma, B, \widetilde{M}[(r,(s_1,s_2))], \widetilde{C}, \widetilde{*})$ *where :*

(a) $\widetilde{\Lambda}$ *is a finite poset where* $\widetilde{\Lambda}$ *is as in Definition 4.9. For each* $(r,(s_1,s_2)) \in \widetilde{\Lambda}$, $(\Gamma(s_1,s_2), B(s_1,s_2))$ *is a hypergroup over* $\mathbb{C}$ *and* $\widetilde{M}[(r,(s_1,s_2))]$ *is a finite set. The map*

$$\widetilde{C} : \coprod_{(r,(s_1,s_2))\in\widetilde{\Lambda}} (\widetilde{M}[(r,(s_1,s_2))] \times B(s_1,s_2) \times \widetilde{M}[(r,(s_1,s_2))]) \to \overrightarrow{A}_k^{\mathbb{Z}_2}$$

is injective with image an $\mathscr{A}$*-basis of* $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(b) $\widetilde{*}$ *is an* $\mathscr{A}$*-linear involutory anti-automorphism of* $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(c) *If* $(r,(s_1,s_2)) \in \widetilde{\Lambda}, ((f,\sigma_1),\sigma_2) \in \Gamma(s_1,s_2)$ *and* $\left((\widetilde{d'},\widetilde{P}),(\widetilde{d''},\widetilde{Q}) \in \widetilde{M}[(r,(s_1,s_2))]\right)$ *then for all* $\widetilde{a} \in \overrightarrow{A}_k^{\mathbb{Z}_2}$ *we have*

$$\widetilde{a}\widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d'},\widetilde{P}),(\widetilde{d''},\widetilde{Q})} \equiv \sum_{(\widetilde{d_i'''},\widetilde{R}_i)\in\widetilde{M}[(r,(s_1,s_2))]} \widetilde{C}^{r_{\widetilde{a}}[(\widetilde{d_i'''},\widetilde{R}_i),(\widetilde{d'},\widetilde{P})]((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d_i'''},\widetilde{R}_i),(\widetilde{d''},\widetilde{Q})} \quad \textit{mod } \overrightarrow{A}_k^{\mathbb{Z}_2}(<(r,(s_1,s_2)))$$

where $r_{\widetilde{a}}[(\widetilde{d_i'''},\widetilde{R}_i),(\widetilde{d'},\widetilde{P})]((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)$ *is independent of* $(\widetilde{d''},\widetilde{Q})$ *and of* $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)$.

Proof. The proof of (i)(a) and (ii)(a) follows Definitions 4.2, 4.9, 4.10 and note 2, proof of (i)(b) and (ii)(b) follows from Definition 4.11 and proof of (i)(c) and (ii)(c) follows from Lemmas 4.3, 4.4, 4.7 and 4.8. □

Corollary 4.14. *Let* $\mathscr{A} = \mathbb{C}(x)$*. A partition algebra of* $\mathbb{P}_{2k}(x^2)$ *is a tabular algebra together with a table datum* $(\Lambda, \Gamma, B, M[(r,(s_1,s_2))], C, *)$ *with* $f = id$ *and* $s_2 = 0$.

5. A Cellular Basis of the Algebra of $\mathbb{Z}_2$-Relations and Signed Partition Algebras

In this section, we compute a cellular basis for the algebra of $\mathbb{Z}_2$-relations and signed partition algebras by making use of the basis defined in Lemma 4.3 and also by using cellular bases of the group algebras $\mathscr{A}[\mathbb{Z}_2 \wr \mathfrak{S}_k]$ and $\mathscr{A}[\mathfrak{S}_k]$ given in [7].

Definition 5.1. Define,

(i) $\Lambda' := \{((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \mid (r,(s_1,s_2)) \in \Lambda\}$

(ii) $\widetilde{\Lambda}' := \left\{((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \mid (r,(s_1,s_2)) \in \widetilde{\Lambda}\right\}$

with the order given by

$$\left(r',(s_1',s_2'),((\lambda_1',\lambda_2'),\mu')\right) \geq \left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)$$

if and only if

(a) $r' \geq r$ or

(b) $r' = r$ and $(s_1',s_2') \geq (s_1,s_2)$ i.e., $s_1'+s_2' > s_1+s_2$

(c) $r' = r, (s_1',s_2') = (s_1,s_2)$ and $(\lambda_1',\lambda_2') \rhd (\lambda_1,\lambda_2)$.

(d) $r = r', (s_1',s_2') = (s_1,s_2)$, $(\lambda_1',\lambda_2') = (\lambda_1,\lambda_2)$ and $\mu' \rhd \mu$.

Definition 5.2. Let $[\lambda], [\mu]$ denote the trivial representation of λ, μ.

For $((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \Lambda'$ and $((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \widetilde{\Lambda}'$, define

$$M'\,[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)] := M[(r,(s_1,s_2))] \times M^{((\lambda_1,\lambda_2),\mu)}$$

$$\widetilde{M}'\,[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)] := \widetilde{M}[(r,(s_1,s_2))] \times M^{((\lambda_1,\lambda_2),\mu)}$$

where $M^{((\lambda_1,\lambda_2),\mu)} := \{((s_{\lambda_1}, s_{\lambda_2}), s_\mu) \mid s_{\lambda_1}, s_{\lambda_2}$ and s_μ are the standard tableaus of shape λ_1, λ_2 and μ respectively$\}$.

(a) if $s_1 \neq 0$ and $s_2 \neq 0$ then

(i) $M'\Big[\big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big] = \Big\{\Big((d',P),((t_{\lambda_1},t_{\lambda_2}),t_\mu)\Big) \;\Big|\; (d',P) \in M[(r,(s_1,s_2))], t_{\lambda_1}, t_{\lambda_2}$ and t_μ are the standard tableaux of shapes λ_1, λ_2 and μ respectively$\Big\}$,

(ii) $\widetilde{M}'\Big[\big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big] = \Big\{\Big((\widetilde{d}',\widetilde{P}),((t_{\lambda_1},t_{\lambda_2}),t_\mu)\Big) \;\Big|\; (\widetilde{d}',\widetilde{P}) \in M[(r,(s_1,s_2))], t_{\lambda_1}, t_{\lambda_2}$ and t_μ are the standard tableaux of shapes λ_1, λ_2 and μ respectively$\Big\}$,

(b) If $s_1 \neq 0$ and $s_2 = 0$ then

(i) $M'\Big[\big(r,(s_1,0),((\lambda_1,\lambda_2),\Phi)\big)\Big]$
$= \Big\{((d',P),(t_{\lambda_1},t_{\lambda_2})) \;\Big|\; (d',P) \in M[(r,(s_1,0))], t_{\lambda_1}$ and t_{λ_2} are the standard tableaux of shapes λ_1 and λ_2 respectively$\Big\}$,

(ii) $\widetilde{M}'\Big[\big(r,(s_1,0),((\lambda_1,\lambda_2),\Phi)\big)\Big]$
$= \Big\{((\widetilde{d}',\widetilde{P}),(t_{\lambda_1},t_{\lambda_2})) \;\Big|\; (\widetilde{d}',\widetilde{P}) \in \widetilde{M}[(r,(s_1,0))], t_{\lambda_1}$ and t_{λ_2} are the standard tableaux of shapes λ_1 and λ_2 respectively$\Big\}$,

(c) If $s_1 = 0$ and $s_2 \neq 0$ then

(i) $M'\Big[\big(r,(0,s_2),((\Phi,\Phi),\mu)\big)\Big]$
$= \Big\{((d',P),t_\mu) \;\Big|\; (d',P) \in M[(r,(0,s_2))], t_\mu$ is a standard tableau of shape $\mu\Big\}$,

(ii) $\widetilde{M}'\Big[\big(r,(0,s_2),((\Phi,\Phi),\mu)\big)\Big]$
$= \Big\{((\widetilde{d}',\widetilde{P}),t_\mu) \;\Big|\; (\widetilde{d}',\widetilde{P}) \in \widetilde{M}[(r,(0,s_2))], t_\mu$ is a standard tableau of shape $\mu\Big\}$,

(d) If $r = 0, s_1 = 0$ and $s_2 = 0$ then

(i) $M'\Big[\big(0,(0,0),((\Phi,\Phi),\Phi)\big)\Big] = \Big\{(d',\Phi) \;\Big|\; (d',\Phi) \in M[(0,(0,0))]\Big\}$

(ii) $\widetilde{M}'\Big[\big(0,(0,0),((\Phi,\Phi),\Phi)\big)\Big] = \Big\{(\widetilde{d}',\Phi) \;\Big|\; (\widetilde{d}',\Phi) \in M[(0,(0,0))]\Big\}$

where $s_1 = \natural\{C : C$ is a connected component of P such that $H_C^P = \{e\}\}$ and $s_2 = \natural\{C : C$ is a connected component of P such that $H_C^P = \mathbb{Z}_2\}$.

Definition 5.3. Let

(i) $C' : \coprod_{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\Lambda'} M'[(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))]\times M'[(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))] \to A_k^{\mathbb{Z}_2}$

be defined as

$$C'[((d',P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),((d'',Q),((t_{\lambda_1},t_{\lambda_2}),t_\mu))] = C^{m^\lambda_{s_\lambda,t_\lambda}m^\mu_{s_\mu t_\mu}}_{(d',P),(d'',Q)}$$

(ii) $\widetilde{C}' : \coprod_{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\widetilde{\Lambda}'} \widetilde{M}'[(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))]\times \widetilde{M}'[(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))] \to \overrightarrow{A}_k^{\mathbb{Z}_2}$

be defined as

$$\widetilde{C}'[((\widetilde{d}',\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),((\widetilde{d}'',\widetilde{Q}),((t_{\lambda_1},t_{\lambda_2}),t_\mu))] = C^{m^\lambda_{s_\lambda,t_\lambda}m^\mu_{s_\mu t_\mu}}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}$$

where $m^\lambda_{s_\lambda,t_\lambda}$ and $m^\mu_{s_\mu t_\mu}$ are cellular basis for the algebras $\mathscr{A}[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ and $\mathscr{A}[\mathfrak{S}_{s_2}]$ respectively.

Definition 5.4. Let $A_k^{\mathbb{Z}_2}\left(\overrightarrow{A}_k^{\mathbb{Z}_2}\right)$ be the $\mathscr{A}$- algebra defined in Definition 2.5(2.7).

(i) The algebra of $\mathbb{Z}_2$ relations $\mathscr{A}[A_k^{\mathbb{Z}_2}]$ is a cellular algebra with a cell datum $(\Lambda', M', C', *')$ given as follows:

(a) Λ' is a partially ordered set where Λ' is as in Definition 5.1.

(b) $*$ is the unique anti involution of $A_k^{\mathbb{Z}_2}$.

(c) 1. $aC'^{m^\lambda_{s_\lambda,t_\lambda}m^\mu_{s_\mu t_\mu}}_{(d',P),(d'',Q)}$

$$\equiv \sum_{S'\in M'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} r_a[(d''',P'''),(d',P)]C'^{m^\lambda_{s'_\lambda,t_\lambda}m^\mu_{s'_\mu,t_\mu}}_{(d''',P'''),(d'',Q)}$$

$$\mod A_k^{\mathbb{Z}_2}\Big(<\big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $r_a[(d''',P'''),(d',P)]$ is independent of (d'',Q).

2. $aC'_{(d,\Phi),(d',\Phi)} \equiv \sum_{(d'',\Phi)\in M'\left[(0,(0,0),((\Phi,\Phi))\right]} r_a[(d'',\Phi),(d,\Phi)]C'_{(d'',\Phi),(d,\Phi)}.$

(ii) The signed partition algebra is a cellular algebra $\mathscr{A}[\overrightarrow{A}_k^{\mathbb{Z}_2}]$ with a cell datum $(\widetilde{\Lambda}', \widetilde{M}', \widetilde{C}', \widetilde{*}')$ given as follows:

(a) $\widetilde{\Lambda}'$ is a partially ordered set where $\widetilde{\Lambda}'$ is as in Definition 5.1.

(b) $\widetilde{*}$ is the unique anti involution of $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(c) 1. $\widetilde{a}\widetilde{C'}^{m^{\lambda}_{s_\lambda,t_\lambda} m^{\mu}_{s_\mu t_\mu}}_{(\widetilde{d'},\widetilde{P}),(\widetilde{d''},\widetilde{Q})}$

$$\equiv \sum_{\widetilde{S'}\in\widetilde{M'}\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} r_{\widetilde{a}}[(\widetilde{d'''},\widetilde{P'''}),(\widetilde{d'},\widetilde{P})]\widetilde{C'}^{m^{\lambda}_{s'_\lambda,t_\lambda} m^{\mu}_{s'_\mu,t_\mu}}_{(\widetilde{d'''},\widetilde{P'''}),(\widetilde{d''},\widetilde{Q})}$$

$$\bmod \overrightarrow{A}_k^{\mathbb{Z}_2}\Big(< \big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $r_{\widetilde{a}}[(\widetilde{d'''},\widetilde{P'''}),(\widetilde{d'},\widetilde{P})]$ is independent of $(\widetilde{d''},\widetilde{Q})$.

2. $\widetilde{a}\widetilde{C'}_{(\widetilde{d},\Phi),(\widetilde{d'},\Phi)} \equiv \sum_{(\widetilde{d''},\Phi)\in\widetilde{M'}\left[(0,(0,0),((\Phi,\Phi))\right]} r_{\widetilde{a}}[(\widetilde{d''},\Phi),(\widetilde{d},\Phi)]\widetilde{C'}_{(\widetilde{d''},\Phi),(\widetilde{d},\Phi)}.$

Proof. The proof follows from Theorem 4.2.1 of [3], Lemma 4.7 and Theorem 4.13. □

Remark 5.5. From (1.8) of [1], $A_k^{\mathbb{Z}_2}\left(\overrightarrow{A}_k^{\mathbb{Z}_2}\right)$ is a cellular algebra over any field K with cell datum $(\Lambda',M',C',*')((\widetilde{\Lambda'},\widetilde{M'},\widetilde{C'},\widetilde{*'}))$ where $(\Lambda',M',C',*')((\widetilde{\Lambda'},\widetilde{M'},\widetilde{C'},\widetilde{*'}))$ is as in Theorem 5.4.

Corollary 5.6. Let $\mathbb{P}_{2k}(x^2)$ be the $\mathscr{A}$- algebra defined in Definition 2.8. Then $\mathbb{P}_{2k}(x^2)$ has a cell datum $(\Lambda',M',C',*')$ with $f = id$ and $s_2 = 0$.

6. Modular Representations of the Algebra of $\mathbb{Z}_2$-Relations and Signed Partition Algebras

In this section, we give a description of the complete set of irreducible modules for the algebra of $\mathbb{Z}_2$ relations $A_k^{\mathbb{Z}_2}$ and signed partition algebras $\overrightarrow{A}_k^{\mathbb{Z}_2}$ over any field.

Definition 6.1. Let $r = 2s_1 + s_2$. For $0 \le r \le 2k$ and $((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \Lambda'\left(((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \widetilde{\Lambda'}\right)$, put $\lambda = (\lambda_1,\lambda_2)$.

The left cell module $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\left(\widetilde{W}\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\right)$ for the cellular algebra $\mathscr{A}\left[A_k^{\mathbb{Z}_2}\right]\left(\mathscr{A}\left[\overrightarrow{A}_k^{\mathbb{Z}_2}\right]\right)$ is defined as follows:

(i) $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]$ is a free $\mathscr{A}$-module with basis $\left\{C_S^{C_{s_1,s_2}(s)} = C_S^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \;\middle|\; S=(d,P), s=((s_{\lambda_1},s_{\lambda_2}),s_\mu)\in M'\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\right\}$ and $A_k^{\mathbb{Z}_2}$-action is defined on the basis element by a

$$aC_S^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \equiv \sum_{(S',s')\in M'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} C_{S'}^{r_a(S',S)m^{\lambda}_{s'_\lambda} m^{\mu}_{s'_\mu}}$$

$$\bmod A_k^{\mathbb{Z}_2}\Big(< \big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (S',s') = ((d',P'),((s'_{\lambda_1},s'_{\lambda_2}),s'_\mu))$, $r_a(S',S)$ is as in 3(a)(i) and (b)(i) of Theorem 5.4.

(ii) $W[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ is a free $\mathscr{A}$-module with basis $\left\{\widetilde{C}_{\widetilde{S}}^{\widetilde{C}_{s_1,s_2}(s)} = \widetilde{C}_{\widetilde{S}}^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \mid \widetilde{S} = (\widetilde{d},\widetilde{P}), s = ((s_{\lambda_1},s_{\lambda_2}),s_\mu) \in \widetilde{M}'[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]\right\}$ and $\overrightarrow{A}_k^{\mathbb{Z}_2}$-action is defined on the basis element by $\widetilde{a}$

$$\widetilde{a}\widetilde{C}_{\widetilde{S}}^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \equiv \sum_{(\widetilde{S}',s')\in\widetilde{M}'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} \widetilde{C}_{\widetilde{S}'}^{r_{\widetilde{a}}(\widetilde{S}',\widetilde{S}) m^{\lambda}_{s'_\lambda} m^{\mu}_{s'_\mu}} \mod \overrightarrow{A}_k^{\mathbb{Z}_2}\Big(< \big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $(\widetilde{S},s) = ((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (\widetilde{S}',s') = ((\widetilde{d}',\widetilde{P}'),((s'_{\lambda_1},s'_{\lambda_2}),s'_\mu))$ $r_a(\widetilde{S}',\widetilde{S})$ is as in 3(a)(ii) and (b)(ii) of Theorem 5.4.

Lemma 6.2.

(i) $$C_{S,S}^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}} C_{T,T}^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}} \equiv \Phi_1((S,s),(T,t))\, C_{S,T}^{m^{\lambda}_{s_\lambda,t_\lambda} m^{\mu}_{s_\mu,t_\mu}} \quad mod \left[A_k^{\mathbb{Z}_2} < (r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$$

where

$\Phi_1((S,s),(T,t))$
$= x^{l(P\vee P')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)$ *when conditions* (a) *and* (b) *of* Definition 4.6 *are satisfied*
$= 0$ *Otherwise*

(ii) $$\widetilde{C}_{\widetilde{S},\widetilde{S}}^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}} \widetilde{C}_{\widetilde{T},\widetilde{T}}^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}} \equiv \Phi_1((\widetilde{S},s),(\widetilde{T},t))\, C_{\widetilde{S},\widetilde{T}}^{m^{\lambda}_{s_\lambda,t_\lambda} m^{\mu}_{s_\mu,t_\mu}} \quad mod \left[\overrightarrow{A}_k^{\mathbb{Z}_2} < (r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$$ *where*

$\Phi_1((\widetilde{S},s),(\widetilde{T},t))7$
$= x^{l(\widetilde{P}\vee\widetilde{P}')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)$ *when conditions* (a) *and* (b) *of* Definition 4.6 *are satisfied*
$= 0$ *Otherwise*

$(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (\widetilde{S},s) = ((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),$
$(T,t) = ((d',P'),((t_{\lambda_1},t_{\lambda_2}),t_\mu)), (\widetilde{T},t) = ((\widetilde{d}',\widetilde{P}'),((t_{\lambda_1},t_{\lambda_2}),t_\mu)), l(P\vee P')\left(l(\widetilde{P}\vee\widetilde{P}')\right)$ *denotes the number of connected components in* $d'.d''$ $\left(\widetilde{d}'.\widetilde{d}''\right)$ *excluding the union of all the connected components of* P *and* P' $\left(\widetilde{P}\text{ and }\widetilde{P}'\right)$,
$m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} \equiv \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda) m^{\lambda}_{s'_\lambda,t_\lambda}\, mod\ \mathscr{H}(<(\lambda_1,\lambda_2))$,
$m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} \equiv \phi^{\mu}_{\delta_2}(s_\mu,t_\mu) m^{\mu}_{s'_\mu,t_\mu}\, mod\ \mathscr{H}'(<\mu)$
as in Lemma 1.7 [1].

Proof. **Proof of (i):** Consider the product

$$C_{S,S}^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}} C_{T,T}^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}} = x^{l(P\vee P')} C_{S,T}^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}(\delta_1,\delta_2) m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}}$$

where $\phi^{r}_{s_1,s_2}((d,P),(d',P')) = x^{l(P\vee P')}(\delta_1,\delta_2)$ is as in Definition 4.6,
We know that,

$$(6.1) m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}(\delta_1,\delta_2) m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu} = m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} = \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda) m^{\lambda}_{s'_\lambda,t_\lambda}\phi^{\mu}_{\delta_2}(s_\mu,t_\mu) m^{\mu}_{s'_\mu,t_\mu} = \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)\ m^{\lambda}_{s'_\lambda,t_\lambda} m^{\mu}_{s'_\mu,t_\mu}$$

where $m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} \equiv \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda) m^{\lambda}_{s'_\lambda,t_\lambda} \mod \mathscr{H}\left(<(\lambda_1,\lambda_2)\right)$,
$m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} \equiv \phi^{\mu}_{\delta_2}(s_\mu,t_\mu) m^{\mu}_{s'_\mu,t_\mu} \mod \mathscr{H}'(<\mu)$.
Substitute the above in the product $C^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}}_{S,S}\ C^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}}_{T,T}$ we get,

$$C^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}}_{S,S}\ C^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}}_{T,T} = x^{l(P\vee P')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu) C^{m^{\lambda}_{s'_\lambda,t_\lambda} m^{\mu}_{s'_\mu,t_\mu}}_{S,T} = \Phi_1((S,s),(T,t))\ C^{m^{\lambda}_{s'_\lambda,t_\lambda} m^{\mu}_{s'_\mu,t_\mu}}_{S,T}$$

where $\Phi_1((S,s),(T,t)) = x^{l(P\vee P')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)$.
Proof of (ii): Proof of (ii) is same as proof of (i). □

Definition 6.3. For $\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \in \Lambda'$ $\left(\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \in \widetilde{\Lambda}'\right)$, the bilinear map $\phi^{\lambda,\mu}_{s_1,s_2}$ $\left(\widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}\right)$ is defined as

(i) $\phi^{\lambda,\mu}_{s_1,s_2}\left(C^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}}_{(d,P)}, C^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}}_{(d',P')}\right) = \Phi_1((S,s),(T,t)),$
$(S,s),(T,t) \in M'\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$

(ii) $\widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}\left(\widetilde{C}^{m^{\lambda}_{s_\lambda,s_\lambda} m^{\mu}_{s_\mu,s_\mu}}_{(\widetilde{d},\widetilde{P})}, \widetilde{C}^{m^{\lambda}_{t_\lambda,t_\lambda} m^{\mu}_{t_\mu,t_\mu}}_{(\widetilde{d}',\widetilde{P}')}\right) = \Phi_1((\widetilde{S},s),(\widetilde{T},t)),$
$(\widetilde{S},s),(\widetilde{T},t) \in \widetilde{M}'\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$

where $\Phi_1((S,s),(T,t))$ $\left(\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t))\right)$ is as in Lemma 6.2.

Put

(i) $G^{\lambda,\mu}_{2s_1+s_2} = \left(\Phi_1((S,s),(T,t))\right)_{(S,s),(T,t)\in M'\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}$

where
$\Phi_1((S,s),(T,t))$
$= x^{l(P_i\vee P_j)}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)$ when conditions (a) and (b) of Definition 4.6 are satisfied
$= 0$ Otherwise

where $(S,s) = ((d_i,P_i),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (T,t) = ((d_j,P_j),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$

(ii) $\widetilde{G}^{\lambda,\mu}_{2s_1+s_2} = \left(\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t))\right)_{(\widetilde{S},s),(\widetilde{T},t)\in\widetilde{M}'\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}$

where

$\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t))$

$= x^{l(\widetilde{P}_i\vee\widetilde{P}_j)}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\ \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)$ when conditions (a) and (b) of Definition 4.6 are satisfied

$= 0$ Otherwise

where $(\widetilde{S},s) = ((\widetilde{d}_i,\widetilde{P}_i),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(\widetilde{T},t) = ((\widetilde{d}_j,\widetilde{P}_j),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$, $l(P_i\vee P_j)\left(l(\widetilde{P}_i\vee\widetilde{P}_j)\right)$ denotes the number of connected components in $d'.d''\left(\widetilde{d}'.\widetilde{d}''\right)$ excluding the union of all the connected components of P_i and P_j $\left(\widetilde{P}_i \text{ and } \widetilde{P}_j\right)$,
$m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} \equiv \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)m^{\lambda}_{s'_\lambda,t_\lambda} \bmod \mathscr{H}\,(<(\lambda_1,\lambda_2))$,
$m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} \equiv \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)m^{\mu}_{s'_\mu,t_\mu} \bmod \mathscr{H}'\,(<\mu)$ as in Lemma 1.7 [1].
$G^{\lambda,\mu}_{2s_1+s_2}\left(\widetilde{G}^{\lambda,\mu}_{2s_1+s_2}\right)$ is called the **Gram matrix of the cell module** $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\left(\widetilde{W}\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\right)$.

Definition 6.4. For $\big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\in\Lambda'\left(\big(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\in\widetilde{\Lambda}'\right)$, define

(i) $Rad\big(W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big) = \big\{x\in W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right] \mid \phi^{\lambda,\mu}_{s_1,s_2}(x,y) = 0\ \ \forall\ y\in W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big\}$,

(ii) $Rad\big(\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big) = \big\{x\in \widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right] \mid \widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}(x,y) = 0\ \ \forall\ y\in \widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big\}$,

where $(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(\widetilde{S},s) = ((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu))$, $(T,t) = ((d',P'),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$ and $(\widetilde{T},t) = ((\widetilde{d}',\widetilde{P}'),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$.

Notation 6.5. Let

(i) $\Lambda'_0 = \{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\Lambda' \mid \phi^{\lambda,\mu}_{s_1,s_2}\neq 0\}$.

(ii) $\widetilde{\Lambda}'_0 = \{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\widetilde{\Lambda}' \mid \widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}\neq 0\}$.

Theorem 6.6. *Let $\mathbb{K}(x)$ be a field. For $(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\Lambda'_0$ $\Big((r,(s_1,s_2),((\lambda_1,\lambda_2),\mu))\in\widetilde{\Lambda}'_0\Big)$, let*

(i) $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} = \dfrac{W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}{Rad\ \big(W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big)}$,

(ii) $\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} = \dfrac{\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}{Rad\ \big(\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\big)}$.

(a) $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0\right)$ *if and only if* $\lambda = (\lambda_1, \lambda_2)$ *is p- restricted and* μ *is p- restricted and it is absolutely irreducible over a field of characteristic p.*

(a)′ $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(b) $D^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0 \left(\widetilde{D}^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0\right)$ *if and only if* $\lambda = (\lambda_1, \lambda_2)$ *is p-restricted and it is absolutely irreducible over a field of characteristic p.*

(b)′ $D^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0 \left(\widetilde{D}^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(c) $D^{(r,(0,s_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(0,s_2),\mu)} \neq 0\right)$ *if and only if* μ *is p- restricted and it is absolutely irreducible over a field of characteristic p.*

(c)′ $D^{(r,(0,s_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(0,s_2),\mu)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(d) $D^{(0,\Phi)} \left(\widetilde{D}^{(0,\Phi)}\right)$ *is non-zero and it is absolutely irreducible over a field of characteristic 0.*

Proof. We shall show that $\Phi_1((S,s),(T,t)) \neq 0$ for some $(S,s),(T,t)$.

Consider $(S,s) = ((d,P),((s_{\lambda_1}, s_{\lambda_2}), s_\mu))$ and $(S',s') = ((d,P),((s'_{\lambda_1}, s'_{\lambda_2}), s'_\mu))$ then

$$\Phi_1((S,s),(S',s')) = x^{l(P\vee P)}\phi_1(s_\lambda, s'_\lambda)\phi_1(s_\mu, s'_\mu),$$

where $\lambda = (\lambda_1,\lambda_2), \phi_1(s_\lambda, s'_\lambda)$ and $\phi_1(s_\mu, s'_\mu)$ are the bilinear forms of the cell module W^λ and W^μ of the cellular algebras $k[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ and $K[\mathfrak{S}_{s_2}]$ respectively.

We know that $\phi_1(s_\lambda, s'_\lambda) \neq 0$ and $\phi_1(s_\mu, s_\mu) \neq 0$ for some s'_λ and s'_μ which implies that

$$\Phi_1((S,s),(T,t)) \neq 0$$

for some $(S,s) = \big((d,P),((s_{\lambda_1}, s_{\lambda_2}), s_\mu)\big), (T,t) = \big((d,Q),((t_{\lambda_1}, t_{\lambda_2}), t_\mu)\big)$.

Conversely, assume that $\Phi_1((S,s),(T,t)) \neq 0$ for some $(S,s),(T,t)$.

$$i.e., \Phi_1((S,s),(T,t)) = x^{l(P\vee Q)}\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\phi^\mu_{\delta_2}(s_\mu, t_\mu) \neq 0$$

which implies that

$$\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) \neq 0, \phi^\mu_{\delta_2}(s_\mu, t_\mu) \neq 0 \tag{6.2}$$

where $\phi^r_{s_1,s_2}((d,P),(d',Q)) = x^{l(P\vee Q)}(\delta_1,\delta_2)$ is as in Definition 4.6,
$m^\lambda_{s_\lambda,s_\lambda}\delta_1 m^\lambda_{t_\lambda,t_\lambda} \equiv \phi^\lambda_{\delta_1}(s_\lambda,t_\lambda)m^\lambda_{s_\lambda,t_\lambda} \mod \mathscr{H}(<(\lambda_1,\lambda_2))$ and
$m^\mu_{s_\mu,s_\mu}\delta_2 m^\mu_{t_\mu,t_\mu} \equiv \phi^\mu_{\delta_2}(s_\mu,t_\mu)m^\mu_{s_\mu,t_\mu} \mod \mathscr{H}'(<\mu)$.
Also we know that by proof of (ii) of proposition 2.4 in [1],

$$\phi^{\lambda}_{\delta_1}(s_\lambda, t_\lambda) = \sum r_{\delta^\lambda_1}(s'_\lambda, t_\lambda)\phi_1(s_\lambda, t_\lambda) \text{ and } \phi^{\mu}_{\delta_2}(s_\mu, t_\mu) = \sum r_{\delta^\mu_2}(s'_\mu, t_\mu)\phi_1(s_\mu, t_\mu)$$

By equation (6.2) we have,

$\phi_1(s_\lambda, t_\lambda) \neq 0$ and $\phi_1(s_\mu, t_\mu) \neq 0$ for some t_λ and t_μ.

Thus the proof of (a), (b), (c) follows from [7] and (7.6) of [6] and the absolute irreducibility follows Proposition 3.2 of [1]. □

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