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On Commutativity of σ-Prime Γ-Rings

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Abstract. Let U be a σ-square closed Lie ideal of a 2-torsion free σ-prime Γ-ring M. Let $d \neq 1$ be an automorphism of M such that $[u, d(u)]_\alpha \in Z(M)$ on U, $d\sigma = \sigma d$ on U, and there exists u_0 in $Sa_\sigma(M)$ with $M\Gamma u_0 \subseteq U$. Then, $U \subseteq Z(M)$. By applying this result, we generalize the results of Oukhtite and Salhi respect to Γ-rings. Finally, for a non-zero derivation of a 2-torsion free σ-prime Γ-ring, we obtain suitable conditions under which the Γ-ring must be commutative.

1. Introduction

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ (sending $(a, \alpha, b) \mapsto a\alpha b$) which satisfies the conditions

(1) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Then, M is a Γ-ring in the sense of Barnes [1].

Example 1. Every associative ring is a Γ-ring. In this case Γ is a single element set.

Example 2. Let $R = M_{m\times n}(D)$, the set of all matrices of type $m\times n$ over a division ring D and $\mathit{\Gamma} = M_{n\times m}(D)$, the set of all matrices of type $n \times m$ over a division ring D. Now, we define the map $R \times \mathit{\Gamma} \times R \longrightarrow R$ by $(A, B, C) \longmapsto ABC$, for all $A, C \in R$ and $B \in \mathit{\Gamma}$. Then, R is a $\mathit{\Gamma}$-ring.

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Let R be a Γ-ring. A subset of I of R is a left ideal of R if I is an additive subgroup of R and $R\Gamma I = \{r\alpha a | r \in R, \alpha \in \Gamma, a \in I\}$ is contained in I. Similarly, we can define a right ideal of R. If I is both a left and a right ideal, then I is a two-sided ideal, or simply an ideal of R.

Throughout the paper, we consider M to be the Γ-ring with center $Z(M)$. A Γ-ring M is called a 2-torsion free if $2a = 0$ implies $a = 0$, for all $a \in M$. And M is called commutative if $a\alpha b = b\alpha a$ holds, for all $a, b \in M$ and $\alpha \in \Gamma$. We use the notation $[a, b]_\alpha$ for the commutator a and b with respect to α and defined by $[a, b]_\alpha = a\alpha b - b\alpha a$. By the commutator, it is clear that M is commutative if and only if $[a, b]_\alpha = 0$, for all $a, b \in M$ and $\alpha \in \Gamma$. In our present paper we assume a condition

$$a\alpha b\beta c = a\beta b\alpha c, \tag{1.1}$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. According to this condition the basic commutator identities reduce to the form

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta \text{ and } [a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, which are extensively used in our paper. The concepts of a homomorphism and an automorphism were first introduced by Barnes [1] in Γ-rings in the following manner. Let M and N both be Γ-rings, and φ a map of M into N. Then, φ is called a homomorphism if $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$. If $M = N$, then φ is called an automorphism of M. The notion of derivation and Jordan derivation of a Γ-ring have been introduced by Sapanci and Nakajima [16]. Then in view of some significant results due to Jordan left derivation of a classical ring obtained Jun and Kim in [7], some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Ceven in [2]. Afterwards Halder and Paul [4] extended this remarkable result in square closed Lie ideals of a Γ-ring. Let M be a Γ-ring, An additive mapping $d : M \to M$ is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ is satisfies for all $a, b \in M$, $\alpha \in \Gamma$. Lie and Jordan structure of simple Γ-rings have been studied by Paul and Uddin in [13, 14]. They generalized some significant results made by Herstien [6] in Γ-rings. Following, Paul and Uddin studied on simple Γ-rings with involution in [15]. Haldar and Paul [3] worked on σ-prime Γ-rings by means of a non-zero derivation and obtained some commutativity results of a 2-torsion free σ-prime Γ-ring. Let M be a Γ-ring. An additive subgroup U of M is called a Lie ideal of M if $[u, m]_\alpha \in U$, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. A Lie ideal is said to be a square closed Lie ideal of M if $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. An additive mapping $\sigma : M \to M$ is called an involution if $\sigma(a) = a$ and $\sigma(a\alpha b) = \sigma(b)\alpha\sigma(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$. An ideal I of M is called $\sigma\sigma$-ideal if $\sigma^2(I) = I$. A Lie ideal U of M is called a σ-Lie ideal if $\sigma(U) = U$. A Lie ideal U is called σ-square closed if it is square closed together with it is a σ-Lie ideal. Let us assume that M be a Γ-ring having an involution σ. Now, we define the set $Sa_\sigma(M) = \{m \in M \ : \ \sigma(m) = \pm m\}$. For this definition we have seen that

$Sa_\sigma(M)$ contains both symmetric and skew symmetric elements of M. Recall that M is said to be a σ-prime if $a\Gamma M\Gamma b = a\Gamma M\Gamma\sigma(b) = 0$ implies that $a = 0$ or $b = 0$. From about fifty years, many authors worked on Γ-rings and extended the results of the classical rings theories to Γ-ring theories. For these purposes, we study on Γ-rings for generalizing the results of rings in Γ-rings. In classical ring theories, a number of research works has done by Oukhtite and Salhi [11] and they extended the results of prime rings to σ-prime rings. Khan, Arora and Khan [8] obtained some commutativity results of σ-prime rings with a non-zero derivation. In the present paper, we generalize the results of [11] in Γ-rings. Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (1.1) and let U be a σ-square closed ideal of M. If $d \neq 1$ is an automorphism of M such that $[u, d(u)]_\alpha \in Z(M)$ on U, $d\sigma = \sigma d$ on U, and there exists u_0 in $Sa_\sigma(M)$ with $M\Gamma u_0 \subseteq U$, then U is central on M. In view of this result, we generalize other results of [11] in Γ-rings. At last we obtain some suitable conditions of a 2-torsion free σ-prime Γ-ring in which it must be commutative by means of non-zero derivation of M.

2. Automorphisms Centralizing on σ-Square Closed Lie Ideals

Throughout this section M will denote a 2-torsion free σ-prime Γ-ring, where σ is an involution of M.

Lemma 2.1. *If d is a homomorphism of M in which $[U, d(U)]_\Gamma \subseteq Z(M)$ on a square closed Lie ideal U of M, then $[U, d(U)]_\Gamma = 0$.*

Proof. For all $u \in U$ and $\alpha \in \Gamma$, we have $[u, d(u)]_\alpha \in Z(M)$. By linearization we have $[u, d(v)]_\alpha + [v, d(u)]_\alpha \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$. Replacing v by $u\beta u$, we obtain

$$
\begin{aligned}
&[u, d(u\beta u)]_\alpha + [u\beta u, d(u)]_\alpha \\
&= [u, d(u)\beta d(u)]_\alpha + [u\beta u, d(u)]_\alpha \\
&= d(u)\beta[u, d(u)]_\alpha + [u, d(u)]_\alpha\beta d(u) + u\beta[u, d(u)]_\alpha + [u, d(u)]_\alpha\beta u \\
&= (d(u) + u)\beta[u, d(u)]_\alpha + d(u)\beta[u, d(u)]_\alpha + u\beta[u, d(u)]_\alpha \text{ (since } [u, d(u)]_\alpha \in Z(M)) \\
&= (d(u) + u)\beta[u, d(u)]_\alpha + (d(u) + u)\beta[u, d(u)]_\alpha \\
&= 2(d(u) + u)\beta[u, d(u)]_\alpha \in Z(M).
\end{aligned}
$$

For $m \in M$, we then get,

$$
\begin{aligned}
m\delta(d(u) + u)\beta[u, d(u)]_\alpha) &= (d(u) + u)\beta[u, d(u)]_\alpha\delta m \\
&= (d(u) + u)\beta m\delta[u, d(u)]_\alpha \\
&= (d(u) + u)\delta m\beta[u, d(u)]_\alpha \text{ (by (1.1).}
\end{aligned}
$$

Hence, $[m, u + d(u)]_\delta\beta[u, d(u)]_\alpha = 0$, for all $m \in M$ and $\delta \in \Gamma$. In particular, $0 = [u, u + d(u)]_\alpha\beta[u, d(u)]_\alpha = [u, d(u)]_\alpha\beta[u, d(u)]_\alpha$. Now, we have $y\gamma[u, d(u)]_\alpha\beta[u, d(u)]_\alpha = [u, d(u)]_\alpha\gamma y\beta[u, d(u)]_\alpha = 0$, since $[u, d(u)]_\alpha \in Z(M)$. Therefore,

$$[u, d(u)]_\alpha\gamma y\beta[u, d(u)]_\alpha\delta\sigma([u, d(u)]_\alpha) = 0$$

and since $[u,d(u)]_\alpha\delta\sigma([u,d(u)]_\alpha)$ is invariant under σ, the σ-primeness of M yields $[u,d(u)]_\alpha = 0$ or $[u,d(u)]_\alpha\delta\sigma([u,d(u)]_\alpha) = 0$. If $[u,d(u)]_\alpha\delta\sigma([u,d(u)]_\alpha) = 0$, then $[u,d(u)]_\alpha\beta y\delta\sigma([u,d(u)]_\alpha) = 0$, for all $y \in M$ and $\beta \in \Gamma$, because $[u,d(u)]_\alpha \in Z(M)$, and consequently $[u,d(u)]_\alpha\beta y\delta[u,d(u)]_\alpha = [u,d(u)]_\alpha\beta y\delta\sigma([u,d(u)]_\alpha = 0$. By using the σ-primeness of M, we get $[u,d(u)]_\alpha = 0$, for all $u \in U$ and $\alpha \in \Gamma$. From now we may assume that d is an automorphism such that $[u,d(u)]_\alpha \in Z(M)$ on a σ-square closed Lie ideal U which contains an element u_0 in $Sa_\sigma(M)$ such that $m\alpha u_0 \in U$, for all $u \in M$ and $\alpha \in \Gamma$. Since d has the property $[u,d(u)]_\alpha \in Z(M)$ on U, Lemma 2.1 implies $[u,d(u)]_\alpha = 0$, for all $u \in U$ and $\alpha \in \Gamma$. □

Lemma 2.2. *If $a,b \in M$ are such that $a\alpha U\beta b = a\alpha U\beta\sigma(b) = 0$, for all $\alpha,\beta \in \Gamma$, then $a = 0$ or $b = 0$.*

Proof. Suppose that $a \neq 0$. Then, there are two cases:

Case 1: $u_0 \in Z(M)$.

Let $m \in M$ and $\gamma \in \Gamma$. From $a\alpha m\gamma u_0\beta b = a\alpha m\gamma u_0\beta\sigma(b) = 0$, we obtain $a\alpha m\gamma u_0\beta b = a\alpha m\gamma u_0\beta\sigma(b) = a\alpha m\gamma\sigma(u_0\beta b) = 0$. Since M is σ-prime, $u_0\beta b = 0$. Then, $m\alpha u_0\beta b = 0$, for all $m \in M$ and $\alpha \in \Gamma$. Since $u_0 \in Z(M)$, then $u_0\alpha m\beta b = \sigma(u_0)\alpha m\beta b = 0$, proving $b \neq 0$.

Case 2: $u_0 \notin Z(M)$.

If $a\alpha[r,u_0]_\gamma = 0$, for all $r \in M$ and $\gamma \in \Gamma$, then

$$0 = a\alpha[m\delta r,u_0]_\gamma = a\alpha m\delta[r,u_0]_\gamma + a\alpha[m,u_0]_\gamma\delta r = a\alpha m\delta[r,u_0]_\gamma,$$

so that $a\alpha m\delta[r,u_0]_\gamma = 0 = a\alpha m\delta\sigma([r,u_0]_\gamma)$ proving $[r,u_0]_\gamma = 0$, for all $r \in M$ and $\gamma \in \Gamma$ which contradicts $u_0 \notin Z(M)$. Thus, there exists m in M such that $a\alpha[m,u_0]_gamma \neq 0$. From $a\alpha[m,u_0]_\gamma\delta r\beta b = a\alpha[m,u_0]_\gamma\delta r\beta\sigma(b) = 0$, it gives that $a\alpha[m,u_0]_\gamma\delta M\beta b = a\alpha[m,u_0]_\gamma\delta M\beta\sigma(b) = 0$ and since M is σ-prime, $b = 0$. □

Lemma 2.3. *Let d commutes with σ on U. If $u \in U \cap Sa_\sigma(M)$ satisfies $d(u) \neq u$, then $u \in Z(M)$.*

Proof. Let Suppose that $u \in USa_\sigma(M)$ with $d(u) \neq u$. From $[v,d(v)]_\alpha = 0$, for all $v \in U$ and $\alpha \in \Gamma$. By linearization, we obtain $[w,d(v)]_\alpha + [v,d(w)]_\alpha = 0$, for all $v,w \in U$ and $\alpha \in \Gamma$. This gives $[w,d(v)]_\alpha = [d(w),v]_\alpha$, for all $v,w \in U$ and $\alpha \in \Gamma$. In particular, $[u,d(2u\beta v)]_\alpha = [d(u),2u\beta v]_\alpha$ because $2u\beta v \in U$, for all $\beta \in \Gamma$. By the 2-torsion freeness of M, we obtain $d(u)\beta[u,d(v)]_\alpha = u\beta[d(u),v]_\alpha$, since $[u,d(u)]_\alpha = 0$. Thus, $d(u)\beta[d(u),v]_\alpha = u\beta[d(u),v]_\alpha$, since $[w,d(v)]_\alpha = [d(w),v]_\alpha$. This gives $(d(u)-u)\beta[d(u),v]_\alpha = 0$, for all $v \in U$ and $\alpha,\beta \in \Gamma$.

Replacing v by $2w\delta v$, we obtain $(d(u)-u)\beta[d(u),w\delta v]_\alpha = 0$, for all $v,w \in U$ and $\alpha,\beta,\delta \in \Gamma$. This yields $(d(u)-u)\beta w\delta[d(u),v]_\alpha + (d(u)-u)\beta[d(u),w]_\alpha\delta v = 0$, for all $v,w \in U$ and $\alpha,\beta,\delta \in \Gamma$. This gives

$$(d(u)-u)\beta U\delta[d(u),v]_\alpha = (d(u)-u)\beta U\delta\sigma([d(u),v]_\alpha) = 0,$$

for all $v,w \in U$ and $\alpha,\beta,\delta \in \Gamma$. Since $d(u) \neq u$, by Lemma 2.2, $[d(u),v]_\alpha = 0$, for all $v \in U$ and $\beta \in \Gamma$. Therefore,

$$\begin{aligned}[d(u), t\beta r\gamma u_0]_\alpha &= t\beta[d(u), r\gamma u_0]_\alpha + [d(u), t]_\alpha \beta r\gamma u_0 \\ &= [d(u), t]_\alpha \beta r\gamma u_0 = 0,\end{aligned}$$

for all $t, r \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Thus, $[d(u), t]_\alpha \beta M\gamma u_0 = 0$. This proves that $[d(u), t]_\alpha = 0$, so that $d(u) \in Z(M)$. Since d is an automorphism, $u \in Z(M)$. □

Theorem 2.4. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (1.1) having an automorphism $d \neq 1$ such that $[u, d(u)]_\alpha \in Z(M)$ on a σ-square closed Lie ideal U. If d commutes with σ on U and there exists u_0 in $Sa_\sigma(M)$ with $M\Gamma u_0 \subseteq U$, then $U \subseteq Z(M)$.*

Proof. Suppose that d is identity on U. Then, for all $t, m \in M$, $\alpha, \beta \in \Gamma$, we have $d(t\alpha m\beta u_0) = t\alpha m\beta u_0 = d(t)\alpha d(m\beta u_0) = d(t)\alpha m\beta u_0$. Thus, $(d(t) - t)\alpha m\beta u_0 = 0$, so that $(d(t) - t)\alpha M\beta u_0 = 0$. Since M is σ-prime this gives $d(t) = t$, for all $t \in M$ which is impossible. So, d is non-trivial on U. Since M is 2-torsion free, the fact that $u+\sigma(u)$ and $u-\sigma(u)$ are in $U \cap Sa_\sigma(M)$, for all u in U such that d is non-trivial on $U \cap Sa_\sigma(M)$. So, there must be an element u in $U \cap Sa_\sigma(M)$ such that $u \neq d(u)$ and by Lemma 2.3, $u \in Z(M)$. Let $0 \neq v$ be in $U \cap Sa_\sigma(M)$ and not be in $Z(M)$. Again using Lemma 2.3, we have $d(v) = v$. But $d(u\alpha v) = d(u)\alpha v = u\alpha v$ so that $(d(u)-u)\alpha v = 0$. Since $u \in Z(M)$, we have $(d(u)-u)\beta m\alpha v = (d(u)-u)\beta m\alpha\sigma(v) = 0$, for all $m \in M$ and $\alpha, \beta \in \Gamma$. This yields $v = 0$, since M is σ-prime. Therefore, for all v in $U \cap Sa_\sigma(M)$, v must be in $Z(M)$. Now, let u be in U. The fact $u - \sigma(u)$ and $u + \sigma(u)$ are elements in $U \cap Sa_\sigma(M)$ and hence $u - \sigma(u) \in Z(M)$ and $u + \sigma(u) \in Z(M)$. These two relations yield that $2u \in Z(M)$. Consequently, $u \in Z(M)$ and that proves that $U \subseteq Z(M)$. □

Theorem 2.5. *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (1.1) having an automorphism $d \neq 1$ which commutes with on a non-zero σ-ideal I of M. If $[a, d(a)]_\alpha \in Z(M)$, for all $a \in I$ and $\alpha \in \Gamma$, then M is a commutative Γ-ring.*

Proof. We know that a σ-ideal is a σ-square closed Lie ideal. So, by Theorem 2.4, $I \subseteq Z(M)$. Now, if $a\alpha a = 0$, for all $a \in I$ and $\alpha \in \Gamma$, then $(\sigma(a)+a)\alpha(\sigma(a)+a) = 0$. As $\sigma(\sigma(a) + a) = \sigma(a) + a$, the fact that $(\sigma(a) + a)\alpha M\beta(\sigma(a) + a) = 0$, for all $\beta \in \Gamma$. By the σ-primeness of M, we have $\sigma(a) = -a$. But $a\alpha a = 0$ implies that $a\alpha M\beta a = 0$ so that $a = 0$ which contradicts $I \neq 0$. Thus, there exists an element $a \in I$ such that $a\alpha a \neq 0$, for all $\alpha \in \Gamma$. For all $m, n \in M$ and $\beta, \delta \in \Gamma$, we have

$$\begin{aligned}a\alpha a\beta m\delta n &= a\alpha(a\beta m)\delta n \\ &= a\alpha(m\beta a)\delta n \\ &= (a\alpha m)\beta a\delta n \\ &= m\alpha a\beta a\delta n \\ &= a\alpha n\beta m\delta a \\ &= a\alpha a\beta n\delta m.\end{aligned}$$

Hence, $a\alpha a\beta[m, n]_\delta = 0$. Thus, $a\alpha a\beta x\gamma[m, n]_\delta = 0$, for all $x \in M$ and $\gamma \in \Gamma$. Similarly, we have $a\alpha a\beta x([m, n]_\delta) = 0$, for all $x \in M$ and $\gamma \in \Gamma$. Since $a\alpha a \neq 0$, the σ-primeness of M yields $[m,n]_\delta = 0$, for all $m,n \in M$ and $\delta \in \Gamma$. This proves that M is commutative. □

3. Derivations in σ-Prime Γ-Rings

Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (1.1) and let $d \neq 0$ be a derivation on M. In this section, we develop suitable conditions under which the Γ-ring M must be commutative. For this purpose we frequently use the following lemma.

Lemma 3.1. *Let $I \neq 0$ be a σ-ideal of M. If $a, b \in M$ are such that $a\Gamma I\Gamma b = 0 = a\Gamma I\Gamma\sigma(b)$, then $a = 0$ or $b = 0$.*

Proof. Suppose that $a \neq 0$. Then, there exists some $x \in I$ such that $a\alpha x \neq 0$, for all $\alpha \in \Gamma$. Indeed, otherwise $a\alpha m\beta x = 0$ and $a\alpha m\beta\sigma(x) = 0$, for all $x \in I$, $m \in M$, $\beta \in \Gamma$. So, $a = 0$. Since $a\alpha I\Gamma M\beta b \subseteq a\alpha I\beta b = 0$ and $a\alpha I\Gamma M\beta\sigma(b) \subseteq a\alpha I\beta\sigma(b) = 0$, for all $\alpha, \beta \in \Gamma$. In particular, $a\alpha x\Gamma M\beta b = a\alpha x\Gamma M\beta\sigma(b) = 0$ gives that $b = 0$ by the σ-primeness of M. □

Theorem 3.2. *Let $0 \neq d$ be a derivation of M and let I be a non-zero σ-ideal of M. If m in $Sa_\sigma(M)$ satisfies $[d(a), m]_\alpha = 0$, for all $a \in I$ and $\alpha \in \Gamma$, then $m \in Z(M)$. Furthermore, if $d(I) \subseteq Z(M)$, then M is commutative.*

Proof. For all $a, b \in I$ and $\beta \in \Gamma$, we have $[d(ab), m]_\alpha = 0$. It yields that

$$\begin{aligned} 0 &= [d(a)\beta b + a\beta d(b), m]_\alpha \\ &= [d(a)\beta b, m]_\alpha + [a\beta d(b), m]_\alpha \\ &= d(a)\beta[b,m]_\alpha + [d(a),m]_\alpha\beta b + a\beta[d(b),m]_\alpha + [a,m]_\alpha\beta d(b) \\ &= d(a)\beta[b,m]_\alpha + [a,m]_\alpha\beta d(b), \end{aligned}$$

by using $[d(a), m]_\alpha = 0 = [d(b), m]_\alpha$. Hence, we obtain

$$d(a)\beta[b,m]_\alpha + [a,m]_\alpha\beta d(b) = 0, \tag{3.1}$$

for all $a, b \in I$, $\alpha, \beta \in \Gamma$. Replacing b by $b\gamma m$ in (3.1), we obtain

$$d(a)\beta[b\gamma m, m]_\alpha\beta + [a,m]_\alpha\beta d(b\gamma m) = 0.$$

This gives that

$$(d(a)\beta b\gamma[m,m]_\alpha + d(a)\beta[b,m]_\alpha\beta m + [a,m]_\alpha\beta(d(b)\gamma m + b\gamma d(m)) = 0.$$

Therefore,

$$(d(a)\beta[b,m]_\alpha + [a,m]_\alpha\beta d(b))\gamma m + [a,m]_\alpha\beta b\gamma d(m) = 0.$$

Using (refe2), we obtain that $[a,m]_\alpha\beta b\gamma d(m) = 0$. This yields $[a,m]_\alpha\beta I\gamma d(m) = 0$. The fact that I is a σ-ideal together with m in $Sa_\sigma(M)$, give $\sigma([a,m]_\alpha)\beta I\gamma d(m) = [a,m]_\alpha\beta I\gamma d(m) = 0$. By Lemma 3.1, we obtain that $[a,m]_\alpha = 0$ or $d(m) = 0$. If

$d(m) \neq 0$, then $[a,m]_\alpha = 0$, for all $a \in I$ and $\alpha \in \Gamma$. Let $t \in M$. Now, we have $[t\beta a, m]_\alpha = t\beta[a,m]_\alpha + [t,m]_\alpha\beta a$. Since $[a,m]_\alpha = 0$, we have $[t,m]_\alpha\beta a = 0$. Let $0 \neq a_0 \in I$. Then,

$$[t,m]_\alpha\beta M\gamma a_0 = [t,m]_\alpha\beta M\gamma\sigma(a_0) = 0.$$

Since M is σ-prime, $[t,m]_\alpha = 0$, which proves that m in $Z(M)$. Now, if $d(m) = 0$, then $d([a,m]_\alpha) = [d(a),m]_\alpha + [a,d(m)]_\alpha$ yields that $0 = d([a,m]_\alpha) = [d(a),m]_\alpha$ and consequently

$$d([I,M]_\Gamma) = 0. \tag{3.2}$$

Replace b by $b\gamma c$ in (3.1), where $c \in I$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} 0 &= d(a)\beta[b\gamma c, m]_\alpha + [a,m]_\alpha\beta d(b\gamma c) \\ &= d(a)\beta b\gamma[c,m]_\alpha + d(a)\beta[b,m]_\alpha c + [a,m]_\alpha\beta(d(b)\gamma c + b\gamma d(c)) \\ &= d(a)\beta b\gamma[c,m]_\alpha + [a,m]_\alpha b\gamma d(c) + (d(a)\beta[b,m]_\alpha + [a,m]_\alpha\beta d(b))\gamma d(c). \end{aligned}$$

By using (3.1), we have

$$d(a)\beta b\gamma[c,m]_\alpha + [a,m]_\alpha\beta b\gamma d(c) = 0. \tag{3.3}$$

Now, putting $[c,m]_\alpha$ for c in (3.3), we obtain

$$d(a)\beta b\gamma[[c,m]_\alpha, m]_\alpha + [a,m]_\alpha\beta b\gamma d([c,m]_\alpha) = 0.$$

Using (3.2), we have $d(a)\beta b\gamma[[c,m]_\alpha,m]_\alpha = 0$ so that $d(a)\beta I\gamma[[c,m]_\alpha,m]_\alpha = 0 = d(a)\beta I\gamma\sigma([[c,m]_\alpha,m]_\alpha)$. By Lemma 3.1, we obtain that $d(a) = 0$ or $[[c,m]_\alpha,m]_\alpha = 0$, for all $a,c \in I$, $m \in M$ and $\alpha \in \Gamma$. If $d(a) = 0$, for all $a \in I$, then for any $t \in M$ we get $d(t\alpha a) = d(t)\alpha a = 0$, for all $a \in I$ and $\alpha \in \Gamma$. Since $m\beta a \in I$, for all $m \in M$ and $\beta \in \Gamma$, we have $d(t)\alpha m\beta a = d(t)\alpha m\beta\sigma(a) = 0$ and as $0 \neq I$, then $d(t) = 0$, for all $t \in M$. Consequently,

$$[[c,m]_\alpha, m]_\alpha = 0. \tag{3.4}$$

Replace c by $c\beta a$ in (3.4) we have

$$\begin{aligned} 0 &= [[c\beta a, m]_\alpha, m]_\alpha = [c\beta[a,m]_\alpha + [c,m]_\alpha\beta a, m]_\alpha \\ &= [c\beta[a,m]_\alpha, m]_\alpha + [c,m]_\alpha a, m]_\alpha \\ &= c\beta[[a,m]_\alpha,m]_\alpha + [c,m]_\alpha\beta[a,m]_\alpha + [[c,m]_\alpha,m]_\alpha\beta a + [c,m]_\alpha\beta[a,m]_\alpha \\ &= [c,m]_\alpha\beta[a,m]_\alpha + [c,m]_\alpha\beta[a,m]_\alpha, \end{aligned}$$

by using (3.4). This yields $2[c,m]_\alpha\beta[a,m]_\alpha = 0$. Hence, by using the 2-torsion freeness of M, we obtain

$$[c,m]_\alpha\beta[a,m]_\alpha = 0. \tag{3.5}$$

Now, replacing c by $y\gamma c$, for all $y \in I$ and $\gamma \in \Gamma$, we have $[y\gamma c, m]_\alpha \beta [a, m]_\alpha = 0$. This follows that

$$\begin{aligned} 0 &= (y\gamma[c, m]_\alpha + [y, m]_\alpha \gamma c)\beta[a, m]_\alpha \\ &= y\gamma[c, m]_\alpha \beta[a, m]_\alpha + [y, m]_\alpha \gamma c\beta[a, m]_\alpha \\ &= [y, m]_\alpha \gamma c\beta[a, m]_\alpha, \end{aligned}$$

by using (3.5). Hence, we have $[y, m]_\alpha \gamma I\beta[a, m]_\alpha = 0$, for all $a \in I$ and $\alpha, \beta, \gamma \in \Gamma$. Therefore,

$$[y, m]_\alpha \gamma I\beta[a, m]_\alpha = 0 = [y, m]_\alpha \gamma I\beta\sigma([a, m]_\alpha),$$

for all $a \in I$ and $\alpha, \beta, \gamma \in \Gamma$. By using Lemma 3.1 we have seen that $[y, m]_\alpha = 0$ or $[a, m]_\alpha = 0$. If $[y, m]_\alpha = 0$, then $m \in Z(M)$. If $[a, m]_\alpha = 0$, for all $a \in I$ and $\alpha \in \Gamma$. For $x \in M$ and $\beta \in \Gamma$, we have $0 = [x\beta a, m]_\alpha = x\beta[a, m]_\alpha + [x, m]_\alpha \beta a = [x, m]_\alpha \beta a = 0$. Hence, $0 = [x, m]_\alpha \beta I = [x, m]_\alpha \beta I\gamma 1 = [x, m]_\alpha \beta I\gamma\sigma(1)$. Once again using Lemma 3.1, we have $[x, m]_\alpha = 0$, which gives that m is in $Z(M)$.

Now, assume that $d(I) \subseteq Z(M)$ and let m in M. We see that we conclude $Sa_\sigma(M) \subseteq Z(M)$ by the first part of the theorem. Consider the fact, we obtain that $m + \sigma(m)$ and $m - \sigma(m)$ are elements of $Sa_\sigma(M)$ and then we conclude that $m + \sigma(m) \in Z(M)$ and $m - \sigma(m) \in Z(M)$ and therefore $2m$ is in $Z(M)$. By the 2-torsion freeness of M, $m \in Z(M)$ proving the commutativity of M. □

Theorem 3.3. *Let $d \neq 0$ be a derivation of M and let $a \in Sa_\sigma(M)$. If $d([m, a]_\alpha) = 0$, for all $m \in M$ and $\alpha \in \Gamma$, then $a \in Z(M)$. In particular, if $d([x, y]_\alpha) = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$, then M is commutative.*

Proof. If $d(a) = 0$, then by hypothesis,

$$0 = d([m, a]_\alpha) = [d(m), a]_\alpha + [m, d(a)]_\alpha == [d(m), a]_\alpha.$$

Hence, $[d(m), a]_\alpha = 0$, for all $m \in M$ and $\alpha \in \Gamma$. By Theorem 3.2, $a \in Z(M)$.

If $d(a) \neq 0$, then we have

$$\begin{aligned} 0 = d([a\beta m, a]_\alpha) &= d(a\beta[m, a]_\alpha + [a, a]_\alpha \beta m) \\ &= d(a\beta[m, a]_\alpha) \\ &= d(a)\beta[m, a]_\alpha + a\beta[m, a]_\alpha) \\ &= d(a)\beta[m, a]_\alpha, \end{aligned}$$

since $d([m, a]_\alpha) = 0$, for all $m \in M$ and $\alpha \in \Gamma$. That is

$$d(a)\beta[m, a]_\alpha = 0. \tag{3.6}$$

For any $s \in M$ and $\gamma \in \Gamma$, we have $m\gamma s \in M$. Replacing m by $m\gamma s$ in (3.6), we have

$$\begin{aligned} 0 &= d(a)\beta[ms, a]_\alpha \\ &= d(a)\beta(m\gamma[s, a]_\alpha + [m, a]_\alpha \gamma s) \\ &= d(a)\beta m\gamma[s, a]_\alpha + d(a)\beta[m, a]_\alpha \gamma s \\ &= d(a)\beta m\gamma[s, a]_\alpha, \end{aligned}$$

by using (3.6), so that $d(a)\beta M\gamma[s,a]_\alpha = 0$, for all s in M. Since $a \in Sa_\sigma(M)$, we obtain

$$d(a)\beta M\gamma[s,a]_\alpha = d(a)\beta M\gamma\sigma([s,a]_\alpha) = 0,$$

for all $s \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since M is σ-prime, we get that $[s,a]_\alpha = 0$. This proves $a \in Z(M)$.

Now, suppose that $d([x,y]_\alpha) = 0$, for all $x, y \in M$, $\alpha, \beta, \gamma \in \Gamma$. By using the first part of the theorem, we obtain that $Sa_\sigma(M) \subseteq Z(M)$. For m in M, the fact that $m+\sigma(m)$ and $m-\sigma(m)$ are elements of $Sa_\sigma(M)$. By the 2-torsion freeness of M, we get $m \in Z(M)$ and hence $M \subseteq Z(M)$. This implies that M is commutative. □

References

[1] W. E. Barnes, *On the* Γ*-rings of Nobusawa*, Pacific J. Math., **8**(1966), 411-422.

[2] Y. Ceven, *Jordan left derivations on completely prime* Γ*-ring*, C.U. Fen-Edebiyat Fakultesi Fen Bilimlere Dergisi, **23(2)**(2002), 39-43.

[3] A. K. Halder and A. C. Paul, *Commutativity of two torsion free* σ*-prime with non-zero derivations*, Journal of Physical Sciences, **15**(2011), 27-32.

[4] A. K. Halder and A. C. Paul, *Jordan left derivations on Lie ideals of prime* Γ*-rings*, Panjab Univ. Journal of Math., **44**(2012), 23-29.

[5] I. N. Herstein, *Topic in ring theory*, Univ of Chicago Press, Chicago, 1969.

[6] M. F. Hoque and A. C. Paul, *On centralizers of semiprime gamma rings*, Int. Math. Forum, **6(13)**(2011), 627-638.

[7] K. W. Jun and B. D. Kim, *A note on Jordan left derivations*, Bull. Korean Math. Soc., **33(2)**(1996), 221-228.

[8] M. R. Khan, D. Arora and M. A. Khan, *Remarks on derivations of* σ*-prime rings*, Inter. Journal of Algebra, **4(16)**(2010), 761-767.

[9] N. Nabusawa, *On a generalization of the ring theory*, Osaka J. Math., **1**(1964), 65-75.

[10] L. Oukhtite and S. Salhi, *On commutativity of* σ*-prime rings*, Glas. Mat. Ser. III, **41(1)**(2006), 57-64.

[11] L. Oukhtite and S. Salhi, *On derivations of* σ*-prime rings*, Inter Journal of Algebra, **1(5)**(2007), 241-246.

[12] L. Oukhtite and S. Salhi, σ*-prime rings with a special kind of automorphism*, Int. J. Contemp Math Sci., **2(3)**(2004), 127-133.

[13] A. C. Paul and M. S. Uddin, *Lie and Jordan structure in simple* Γ*-rings*, Journal of Physical Sciences, **11**(2010), 77-86.

[14] A. C. Paul and M. S. Uddin, *Lie structure in simple gamma rings*, International Journal of Pure and Applied Sciences and Technology, **4(2)**(2010), 63-70.

[15] A. C. Paul and M. S. Uddin, *Simple gamma rings with involutions*, IOSR Journal of Mathematics, **4(3)**(2012), 40-48.

[16] M. Sapanci and A. Nakajima, *Jordan derivations on completely prime gamma rings*, Math. Japonica, **46(1)**(1997), 47-51.