KYUNGPOOK Math. J. 55(2015), 817-826
http://dx.doi.org/10.5666/KMJ.2015.55.4.817
pISSN 1225-6951 eISSN 0454-8124

Some Analogues of a Result of Vasconcelos

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Abstract. Let R be a commutative ring with total quotient ring K. Each monomorphic R-module endomorphism of a cyclic R-module is an isomorphism if and only if R has Krull dimension 0. Each monomorphic R-module endomorphism of R is an isomorphism if and only if $R = K$. We say that R has property $(\star)$ if for each nonzero element $a \in R$, each monomorphic R-module endomorphism of R/Ra is an isomorphism. If R has property $(\star)$, then each nonzero principal prime ideal of R is a maximal ideal, but the converse is false, even for integral domains of Krull dimension 2. An integral domain R has property $(\star)$ if and only if R has no R-sequence of length 2; the " if " assertion fails in general for non-domain rings R. Each treed domain has property $(\star)$, but the converse is false.

1. Introduction

All rings considered in this note are commutative; all rings and modules are unital. Our starting point is the following result of Vasconcelos [14, Theorem]: a ring R has (Krull) dimension 0 if and only if each monomorphic R-module endomorphism of a finitely generated R-module is surjective (that is, is an isomorphism). While

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* Corresponding Author.
Received June 21, 2013; accepted September 12, 2013.
2010 Mathematics Subject Classification: Primary 13A15, 13C99; Secondary 13G05, 13F05, 20M14, 13B30.
Key words and phrases: Commutative ring, cyclic module, monomorphism, Krull dimension, monoid ring, integral domain, pullback, treed domain, pseudo-valuation domain, total quotient ring, localization.
The second-named author thanks the University of Tennessee for its warm hospitality during his visit in 2013.

the statement of this result concerns module-theoretic behavior, its proof involves a central ring-theoretic result of Cohen-Seidenberg (cf. [4]), namely, that the partners of any integral ring extension must have the same dimension. In considering whether a result like that of Vasconcelos could be proved without recourse to such ring-theoretic tools, we found an elementary module-theoretic proof of the following analogue (see Proposition 2.1): a ring R has (Krull) dimension 0 if and only if each monomorphic R-module endomorphism of a cyclic R-module is an isomorphism. The rest of this note examines what can be said about a ring R if one imposes such a condition on only some of its cyclic modules.

For instance, Proposition 2.2 shows each monomorphic R-module endomorphism of (the cyclic R-module) R is an isomorphism if and only if R is a total quotient ring. This is perhaps of greater interest because total quotient rings can be of arbitrary dimension (cf. [12]), but one would seek greater generality if R is a (commutative integral) domain, since domains which are total quotient rings must be fields (and hence of no ideal-theoretic interest). For this reason, we focus on rings R that have the following property, which we call property ($\star$): for each nonzero element $a \in R$, each monomorphic R-module endomorphism of (the cyclic R-module) R/Ra is an isomorphism.

It is shown in Proposition 2.4 that rings having property ($\star$) retain some of the low-dimensional flavor of the rings studied by Vasconcelos, in the sense that whenever R has property ($\star$), each nonzero principal prime ideal of R is a maximal ideal. However, Example 2.6 shows that the converse is false, even for two-dimensional domains. A very useful result for domains is given in Proposition 2.5, where it is shown that a domain R has property ($\star$) if and only if there is no R-sequence on R of length 2. Examples of domains of arbitrary dimension that have property ($\star$) are provided by Corollary 2.9: if R is a pseudo-valuation domain (in the sense of [11]), then R has property ($\star$). In addition, Example 2.6 is shown to be best-possible by Corollary 2.8, where it is shown that each one-dimensional domain has property ($\star$). More generally, Corollary 2.7 establishes that each treed domain has property ($\star$). Additional examples of domains having property ($\star$) are given in Theorem 2.10, which provides some pullbacks, not all of which are treed domains, that have property ($\star$). Remark 2.12 (a) shows that the converse of Corollary 2.7 is false, while Remark 2.12 (d) shows that the criterion in Corollary 2.5 involving R-sequences does not characterize property ($\star$) for non-domains. Finally, Remark 2.12 (e) shows that property ($\star$) is somewhat more fragile than zero-dimensionality in regard to its possible stability under globalization.

As usual, if A is a ring, it will be convenient to denote the (Krull) dimension of A by $\dim(A)$; the set of units of A by $U(A)$; the set of prime ideals of A by $\mathrm{Spec}(A)$; the radical of an ideal I of A by $\sqrt{I}$; the set of zero-divisors of A by $Z(A)$; and the total quotient ring of A by $\mathrm{tq}(A) := A_{A \setminus Z(A)}$. Also, if E is an A-module and $P \in \mathrm{Spec}(A)$, we let E_P denote the localization of E at $A \setminus P$. Parts (d) and (e) of Remark 2.12 use the idealization construction of Nagata; a useful reference for basic facts about idealization is [12]. Any unexplained material is standard, as in the textbooks listed in the bibliography.

2. Results

We begin in Proposition 2.1 by showing how a focus on cyclic modules can lead to a simple analogue of the motivating result of Vasconcelos.

Proposition 2.1. *Let R be a ring. Then the following conditions are equivalent:*

(1) *If E is a cyclic R-module and $f \in \mathrm{Hom}_R(E, E)$ is an injection, then f is an isomorphism;*

(2) $\dim(R) = 0$.

Proof. (1) $\Rightarrow$ (2): The contrapositive of this implication can be proved as in the proof of [14, Theorem]. For the sake of completeness, we next provide the details. If (2) fails, pick distinct prime ideals $P \subset Q$ of R, pick an element $b \in Q \setminus P$, put $E := R/P$, let f be the element of $\mathrm{Hom}_R(E, E)$ defined by multiplication by b, and note that f is an injection but not a surjection, so that (1) fails.

(2) $\Rightarrow$ (1): Suppose that $\dim(R) = 0$. Let E be a cyclic R-module and suppose that $f \in \mathrm{Hom}_R(E, E)$ is an injection. We will show that f is surjective. By hypothesis, there exists an ideal I of R such that $E \cong R/I$. Of course, f is surjective (respectively, injective) if and only if the induced map $f_M \in \mathrm{Hom}_{R_M}(E_M, E_M)$ is surjective (respectively, injective) for each $M \in \mathrm{Spec}(R)$; and for any such M, we have $E_M \cong R_M/IR_M$, which is a cyclic R_M-module. Thus, without loss of generality, we may take (R, M) quasi-local and $E = R/I$. Also without loss of generality, $I \neq R$ (since the only endomorphism of the module 0 is surjective), and so $I \subseteq M$. In fact, $\sqrt{I} = M$ by [9, Corollary 2.10], since the zero-dimensionality of R implies that M is the unique prime ideal of R. Next, note that $f(1+I) = b+I$ for some $b \in R$. (If $I \neq 0$, then b is not uniquely determined, but it suffices to pick any such b.) Then f is simply multiplication by b; that is, if $r \in R$, then $f(r+I) = br+I$ $(= b \cdot (r+I))$. If $b \in U(R)$, then $f(b^{-1}+I) = bb^{-1}+I = 1+I$, and so f is surjective since f is R-linear. Hence, without loss of generality, $b \in R \setminus U(R) = M$. It suffices to obtain a contradiction in this case. As above, we have that $\sqrt{Rb} = M$, and so $b \in \sqrt{I}$. Hence, there is a smallest positive integer n such that $b^n \in I$. Consider $\rho := b^{n-1} \in R$. Then $\rho + I \neq 0 \in E$, although $f(\rho + I) = b\rho + I = b^n + I = 0 \in E$, which contradicts the assumption that f is a monomorphism. The proof is complete. □

We next consider rings that satisfy various weaker versions of condition (1) of Proposition 2.1. By focusing on just the cyclic R-module R, Proposition 2.2 gives a new characterization of total quotient rings. As is well known (cf. [12, Theorems 26.2 and 26.4]), such rings can be of arbitrary Krull dimension.

Proposition 2.2. *Let R be a ring. Then the following conditions are equivalent:*

(1) *If $f \in \mathrm{Hom}_R(R, R)$ is an injection, then f is an isomorphism;*

(2) $R = \mathrm{tq}(R)$.

Proof. The result holds for the zero ring (as its total quotient ring must sensibly be defined to also be the zero ring). Thus, without loss of generality, $R \neq 0$. If $b \in R$, let $\tilde{b} \in \mathrm{Hom}_R(R, R)$ denote multiplication by b, namely, the map given by $r \mapsto br$. As b varies over the elements of R, $\tilde{b}$ varies over the elements of $\mathrm{Hom}_R(R, R)$. Moreover (since $R \neq 0$), $f = \tilde{b}$ is a monomorphism if and only if $b \notin Z(R)$. On the other hand, $f = \tilde{b}$ is surjective if and only if $b \in U(R)$ (in which case, $f = \tilde{b}$ is clearly an isomorphism). Thus, (1) holds if and only if $R \setminus Z(R) = U(R)$. Since it is clear that (2) holds if and only if $R \setminus Z(R) = U(R)$, the proof is complete. □

Seeking a condition that is, in a sense, midway between condition (1) of Proposition 2.1 and condition (1) of Proposition 2.2, we next introduce the main object of study in this note. We will say that a ring R has property $(\star)$ if, for each nonzero element $a \in R$, each monomorphic R-module endomorphism of (the cyclic R-module) R/Ra is an isomorphism. Our first examples of such rings are given in Corollary 2.3, which is an immediate consequence of Proposition 2.1 (or of Vasconcelos' motivating result [14, Theorem]).

Corollary 2.3. *Each ring of Krull dimension* 0 *has property* $(\star)$.

We next show that any ring R that has property $(\star)$ must exhibit certain behavior that was established in [6, Corollary 2.4] for any going-down domain, namely, that each nonzero principal prime ideal of R is a maximal ideal.

Proposition 2.4. *Let R be a ring that has property $(\star)$. Then each nonzero principal prime ideal of R is a maximal ideal.*

Proof. It suffices to rework the proof of the implication (1) $\Rightarrow$ (2) in Proposition 2.1 (taking P in that proof to be the given nonzero principal prime ideal and Q to be, if possible, a strictly larger prime ideal of R). □

Example 2.6 will show that the converse of Proposition 2.4 is false. First, it is convenient to note that for domains, property $(\star)$ admits the following reformulation involving R-sequences (also known as regular sequences).

Proposition 2.5. *Let R be a domain. Then R has property $(\star)$ if and only if there is no R-sequence* (*on* R) *of the form a, b.*

Proof. For all $a, b \in R$, let $\widetilde{(a,b)}$ denote the R-module homomorphism $\widetilde{(a,b)} : R/Ra \to R/Ra$ given by $r + Ra \mapsto br + Ra$, for each $r \in R$. Note that R fails to have property $(\star)$ if and only if there exist elements a, b in R such that $a \neq 0$ and $\widetilde{(a,b)}$ is injective but not surjective. On the other hand (cf. [13, page 84]), if a and b are elements of R, then a, b is an R-sequence (on R) if and only if $Ra + Rb \neq R$, $a \neq 0$ and $\widetilde{(a,b)}$ is injective.

Suppose first that the "if" assertion fails. Hence, R has no R-sequences consisting of two elements, but there exist elements a, b in R such that $a \neq 0$ and $\widetilde{(a,b)}$ is injective but not surjective. Consequently $Ra + Rb = R$, and so $R/Ra = R(b + Ra) = \widetilde{(a,b)}(R/Ra)$, whence $\widetilde{(a,b)}$ is surjective, the desired contradiction.

Next, suppose that the "only if" assertion fails. Then R has an R-sequence a, b, but R has property $(\star)$. The R-sequence condition gives us that $Ra + Rb \neq R$,

$a \neq 0$ and $\widetilde{(a,b)}$ is injective. But then property $(\star)$ ensures that $\widetilde{(a,b)}$ is surjective. Thus $R/Ra = \widetilde{(a,b)}(R/Ra) = (Ra + Rb)/Ra$, and so a standard homomorphism theorem yields that $R = Ra + Rb$, the desired contradiction, which completes the proof. □

Recall from [3] that a domain is said to be an *antimatter domain* if it has no irreducible elements. Note that any antimatter domain D vacuously satisfies the conclusion of Proposition 2.4; Example 2.6 will show that such D need not satisfy the hypothesis of Proposition 2.4. Also, it will be convenient to let $\mathbb{Q}^+$ denote the monoid consisting of the non-negative rational numbers under addition.

Example 2.6. Let K be a perfect field of positive characteristic, let the monoid S be the cardinal sum of two copies of $\mathbb{Q}^+$, and let R denote the monoid ring $K[X; S]$. Then R is a domain of Krull dimension 2 and each nonzero principal prime ideal of R is a maximal ideal, but the ring R does not have property $(\star)$.

Proof. By [1, Theorem 5 (1)], R is an antimatter domain (and a GCD-domain). By combining [10, Theorems 21.4 and 17.1] with [9, Theorem 30.5], we see that the dimension of this domain is the torsion-free rank of S, namely, 2. Since R is an antimatter domain, it has no nonzero principal prime ideals (and so, vacuously, any such ideal is maximal). It remains only to show that R does not have property $(\star)$. By Proposition 2.5, it therefore suffices to produce an R-sequence a, b (on R). The reader can easily check that $a := X^{(0,1)}, b := X^{(1,0)}$ form the desired R-sequence.□

Recall that a domain R is said to be *treed* if Spec(R), as a poset under inclusion, forms a tree; that is, if no prime ideal of R can contain incomparable prime ideals of R. Corollary 2.7 will lead to many examples of rings that have property $(\star)$ by showing that any treed domain has property $(\star)$. In this vein, and with an eye to the comment that preceded Proposition 2.4, we recall that any going-down domain is a treed domain [5, Theorem 2.2] (and the converse is false).

Corollary 2.7. *If R is a treed domain, then R has property* $(\star)$.

Proof. By [7, Theorem 2.2], there is no R-sequence (on R) of the form a, b. Therefore, an application of Proposition 2.5 completes the proof. □

By Corollary 2.7 and the comment preceding it, each going-down domain has property $(\star)$. The applications in the next two corollaries could be inferred from this fact about going-down domains, but we prefer to present those corollaries as consequences of Corollary 2.7, as treed domains are perhaps more widely known than going-down domains.

Corollary 2.8. *If R is a domain of Krull dimension at most* 1*, then R has property* $(\star)$.

Proof. Since R is a treed domain, the assertion follows from Corollary 2.7. For an alternate proof, combine Proposition 2.5 with the well-known fact (cf. [13, Exercise

22, page 104]) that for any non-negative integer n, an n-dimensional ring cannot contain an R-sequence of length $n+1$. □

In view of Corollary 2.8 and Example 2.6, the following result is of some interest, as pseudo-valuation domains can have arbitrary Krull dimension. (To see this, either use [6, Lemma 4.5 (v), Proposition 4.9 (i)] or consider a valuation domain whose value group has rank equal to the desired dimension.)

Corollary 2.9. *Each pseudo-valuation domain has property* ($\star$).

Proof. Each pseudo-valuation domain is a treed domain [11, Corollary 1.3]. Therefore, an application of Corollary 2.7 completes the proof. □

The conclusion in Corollary 2.9 applies as well, and more generally, to the class of locally pseudo-valuation domains (the so-called LPVDs). Since the property of being treed is a local property of domains, this raises the question whether property ($\star$) itself is a local property. We will address some related issues in Remark 2.12. First, we give a family of examples of rings having property ($\star$) in Theorem 2.10. As explained in Remark 2.12(c), some of those rings are not treed.

Theorem 2.10. *Let* (T, M) *be a quasi-local domain with property* ($\star$), *let* D *be a subring of* T/M, *let* $\pi : T \to T/M$ *denote the canonical projection map, and let* R *be the pullback* $R := D \times_{T/M} T$ $(= \pi^{-1}(D))$. *Then* R *has property* ($\star$) *if and only if* D *has property* ($\star$).

Proof. We have that $R/M \cong D$, and so $M \in \mathrm{Spec}(R)$. Also, by the order-theoretic impact of the topological description of $\mathrm{Spec}(R)$ (with the Zariski topology) in [8, Theorem 1.4], each prime ideal of R is comparable with M under inclusion, and the set of prime ideals of R that contain (resp., are contained in) M is isomorphic as a poset to $\mathrm{Spec}(D)$ (resp., $\mathrm{Spec}(T)$).

We will prove the “if” assertion first. So, suppose that D has property ($\star$). By Proposition 2.5, it suffices to derive a contradiction from the assumption that a, b is an R-sequence on R. Our analysis will break naturally into two cases that are determined by where a and b reside relative to M. In any event, note that $0 \neq a, b \in R \setminus U(R)$; and that b, a is also an R-sequence on R, by [7, Lemma 2.1 (b)].

Case 1: either $a \notin M$ or $b \notin M$. By the preceding observation, we may assume, without loss of generality, that $a \notin M$. Now, by a standard homomorphism theorem, $D/Da \cong R/(Ra + M)$ as R-modules. It will simplify the notation to point out that $Ra + M = Ra$; that is, $M \subseteq Ra$. (In detail, note that $a \in T \setminus M = U(T)$, so that, working in T, we have $M = (a^{-1}M)a \subseteq (TM)a = Ma \subseteq Ra$.) Let f denote multiplication by b on D/Da. We claim that f is a monomorphism. Suppose, on the contrary, that there exists $t \in R \setminus Ra$ such that $bt \in Ra$. Since a, b is an R-sequence on R, we conclude that $t \in Ra$, the desired contradiction. This proves the above claim; that is, f is a monomorphism.

Since D has property ($\star$) and f is a monomorphism, f must be surjective.

Therefore, there exists $u \in R$ such that $bu - 1 \in Ra + M = Ra$. Thus, $1 \in Ra + Rb$, contradicting that a, b is an R-sequence on R.

Case 2: both a and b are elements of M. Since a, b is an R-sequence on R, we have that $b \notin Ra$. We claim that $b \notin Ta$. To see this, suppose, on the contrary, that $b = qa$ for some $q \in T \setminus R$. Then $q \in U(T)$ (since $T \setminus U(T) = M \subseteq R$). Observe that $b \in M \setminus Ra$, and so $b + Ra \neq 0 \in R/Ra$. However, $b \cdot (b + Ra) = b^2 + Ra = (q^2 a)a + Ra = 0 \in R/Ra$, since $q^2 a \in Ta \subseteq M \subseteq R$. This contradicts that a, b is an R-sequence on R, thus proving the above claim; that is, $b \notin Ta$.

Note that multiplication by b on T/Ta is not surjective, since the image of this endomorphism is contained in M/Ta. Consequently, since T has property $(\star)$, this endomorphism is not injective. Hence, there exists $c \in T \setminus Ta$ such that $bc \in Ta$. Write $bc = ta$, with $t \in T$. If $c \in U(T)$, then $b^2 = (tc^{-1}b)a \in Ra$ (since $tc^{-1}b \in Tb \subseteq M \subseteq R$), so that multiplication by b annihilates the nonzero element $b + Ra \in R/Ra$, which contradicts that a, b is an R-sequence on R. Therefore, $c \in T \setminus U(T) = M \subseteq R$. As $bc = ta$ and $c \notin Ra$, the fact that a, b is an R-sequence on R ensures that $t \notin R$. In particular, $t \notin M$, and so $t^{-1} \in T$. Then $d := ct^{-1}$ satisfies $bd = a$ and $d \in MT = M \subseteq R$. As a, b is an R-sequence on R, it follows that $d \in Ra$. Thus, $c = td \in T(Ra) = Ta$, the desired contradiction. This completes the proof of the "if" assertion.

Suppose the "only if" assertion fails. Then R has property $(\star)$ but D does not have property $(\star)$. Use $\pi|_R$ to identify $D = R/M$. By Proposition 2.5, there exist $a, b \in R$, such that $\overline{a} := a + M, \overline{b} := b + M$ is a regular sequence on D (that is, an "R"-sequence of D-modules). Necessarily, neither a nor b is an element of M. As above, a standard homomorphism theorem gives that $D/D\overline{a} \cong R/(Ra + M)$. By Proposition 2.5, it cannot be the case that a, b is an R-sequence on R. It also cannot be the case that $Ra + Rb = R$ (since $D\overline{a} + D\overline{b} \neq D$). Hence, there exists $s \in R \setminus Ra$ such that $bs \in Ra$. Then, *a fortiori*, $bs \in Ra + M$, whence $\overline{s} := s + M$ satisfies $\overline{b} \cdot \overline{s} \in D\overline{a}$. Since $\overline{a}, \overline{b}$ is a regular sequence on D, it follows that $\overline{s} \in D\overline{a}$; that is, $s \in Ra + M$. However, $M = (Ma^{-1})a \subseteq (MT)a = Ma \subseteq Ra$, and so $s \in Ra$, the desired contradiction, which completes the proof. □

Corollary 2.11. *Let $R \subseteq T$ be domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ *as sets. If T has property $(\star)$, then R has property $(\star)$.*

Proof. As any field has property $(\star)$, we may assume that R and T are not fields. Also without loss of generality, $R \neq T$. Then, by [2, Proposition 3.3 and Theorem 3.25], R is a quasi-local ring, say with maximal ideal M, such that if K denotes the quotient field of R and $(M :_K M) := \{u \in K \mid uM \subseteq M\}$, with $\pi : (M :_K M) \to (M :_K M)/M$ the canonical projection map, then there exists a field $k \subseteq (M :_K M)/M$ such that T equals the pullback $\pi^{-1}(k)$. As T is quasi-local and k has property $(\star)$, the assertion now follows from Theorem 2.10. For an alternate proof, one could combine [7, Proposition 2.5] with the criterion for property $(\star)$ in Proposition 2.5. □

Since any pseudo-valuation domain can be obtained as the pullback of its canonically associated valuation domain and a subfield of the residue field of that valuation domain (cf. [2, Proposition 2.6]), Corollary 2.11 reduces the assertion in Corollary 2.9 to the case of a valuation domain. An additional application of Theorem 2.10 will be given in Remark 2.12 (c).

Remark 2.12. (a) The converse of Corollary 2.7 is false, even for two-dimensional Noetherian domains. Indeed, let R be any two-dimensional Noetherian local domain which is not a Cohen-Macaulay ring. An example of such R was given in [7, Remark 2.4], where it was also shown (for any such R) that R has no R-sequence of the form a, b. Hence, by Proposition 2.5, R has property $(\star)$. Of course, R is not treed (cf. [13, Theorem 144]).

(b) Many examples of valuation domains that are also antimatter domains were given in [3]. By Corollary 2.7 (or Corollary 2.9), each of those domains has property $(\star)$. Their behavior is to be contrasted with that of the antimatter domain given in Example 2.6. Noetherian domains form another prominent class of domains that contains some domains having property $(\star)$ and some domains that do not have property $(\star)$. Indeed, recall from (a) that any two-dimensional (Noetherian) local non-Cohen-Macaulay domain has property $(\star)$, while one can show, by combining Proposition 2.4 with [13, Theorems 5 and 142], that any Noetherian unique factorization domain of Krull dimension at least 2 must fail to have property $(\star)$. These facts help to show the diversity of the class of domains that have property $(\star)$. Our point here is that if D is a Noetherian domain (more generally, a domain satisfying the ascending chain condition on principal ideals) but not a field, then D is far from being an antimatter domain, since each nonzero nonunit of D is a product of irreducible elements of D.

(c) Consider the data in Theorem 2.10, with D also assumed to be quasi-local and treed. Once again, we use the order-theoretic impact of the topological description of $\mathrm{Spec}(R)$ (with the Zariski topology) in [8, Theorem 1.4]. This shows that R is quasi-local and also that R is treed if and only if T is treed; in addition, it follows from [8, Proposition 2.1 (5)] that $\dim(R) = \dim(D) + \dim(T)$. Thus, if T is one of the two-dimensional Noetherian local non-Cohen-Macaulay domains considered in (a) and D is a d-dimensional valuation subring of the field T/M, then Theorem 2.10 implies that the pullback $R := D \times_{T/M} T$ is a $(d + 2)$-dimensional non-treed domain having property $(\star)$, where, as usual, we take $\infty + 2 := \infty$.

(d) Proposition 2.5 does not extend to arbitrary rings. In fact, the "if" assertion of Proposition 2.5 fails for the following data. Consider the ring $A := \mathbb{Z}_{2\mathbb{Z}}$ and the idealization $R := A(+)A$. Then R is a non-domain (since $Z(R) = 0(+)A \neq 0$); R is a Noetherian local one-dimensional ring with unique maximal ideal $2A(+)A$; and Proposition 2.4 shows that R does not have property $(\star)$, since $0(+)A = A(0, 1)$ is a nonzero principal prime ideal of R which is not a maximal ideal of R. Finally, since $\dim(R) < 2$, it follows from [13, Theorem 132] that R has no R-sequence of the form a, b.

(e) We consider here the behavior of property $(\star)$ under globalization. In this paragraph, we get a positive result for domains. To wit: if R is a domain such that R_M has property $(\star)$ for each maximal ideal M of R, then R has property $(\star)$. For a proof, we may assume, without loss of generality, that R is not a field. Then the assertion follows by combining [7, Lemma 2.1 (a)] with the criterion for property $(\star)$ in Proposition 2.5.

Next, if the ambient ring R is not necessarily a domain, we have the following positive result. If R is a ring whose Jacobson radical $J(R)$ contains $Z(R)$ and if R_M has property $(\star)$ for each maximal ideal M of R, then R has property $(\star)$. For a proof, let $0 \neq a \in R$ and consider an arbitrary monomorphism $f \in \mathrm{Hom}_R(R/Ra, R/Ra)$. Our task is to show that f is an isomorphism or, equivalently, that the induced map $f_M : (R/Ra)_M \to (R/Ra)_M$ is an isomorphism for each maximal ideal M of R. Of course, f_M inherits the property of being a monomorphism from f, for each M. Note also that for each M, we have a canonical isomorphism $(R/Ra)_M \cong R_M/(Ra)_M$ and $(Ra)_M = R_M(a/1)$. Therefore, since each R_M is assumed to have property $(\star)$, we will be done if $(a/1) \neq (0/1) \in R_M$ for each M. This, in turn, follows from the hypothesis that $Z(R) \subseteq J(R)$ (for if $(a/1) \neq (0/1)$ in R_M, then $(0 :_R a) \not\subseteq M$, although $(0 :_R a) \subseteq Z(R) \subseteq J(R) \subseteq M$, the desired contradiction).

We pause to give an example of a quasi-local non-domain R satisfying the hypothesis that $Z(R) \subseteq J(R)$ from the preceding paragraph. Let A and $R = A(+)A$ be as constructed in (d). Then $Z(R) = 0(+)A \subset 2A(+)A = J(R)$. Notice that we showed in (d) that R does not have property $(\star)$.

Finally, despite the positive results in the first two paragraph of this part (e), we show next that property $(\star)$ does not globalize in general. In other words, there exists a ring R such that R_M has property $(\star)$ for each maximal ideal M of R but R does not have property $(\star)$. Necessarily, by the above remarks, any such R must fail to be a domain and must fail to satisfy that $Z(R) \subseteq J(R)$. To construct a satisfactory R, we let (A, N) be a quasi-local treed domain which is not a field, and put $R := A \times A$. The maximal ideals of R are $N \times A$ and $A \times N$; the localizations of R at these maximal ideals are each canonically isomorphic as rings to $A_N \cong A$. In particular, both of these localizations are treed domains and so, by Corollary 2.7, R_M has property $(\star)$ for each maximal ideal M of R. It remains to show that R does not have property $(\star)$. To that end, first use the hypothesis that A is not a field to pick a nonzero nonunit $\rho \in N$, and then let $a := (1, 0)$, $b := (0, \rho) \in R$. It suffices to show that if f denotes multiplication by b on $E := R/Ra$, then f is a monomorphism but not an isomorphism. By means of the canonical isomorphism $E \cong A$, f can be identified with the A-module homomorphism $A \to A$ given by multiplication by ρ. This map is clearly a monomorphism but not surjective.

In closing, we raise the following question about localization which is, in a sense, dual to the question that was studied in Remark 2.12(d). Can one find useful conditions on a ring R that has property $(\star)$ and a multiplicatively closed subset S of R so as to ensure that R_S also has property $(\star)$?

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