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# STRICT TOPOLOGIES AND OPERATORS ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract. Let $X$ be a completely regular Hausdorff space, and $E$ and $F$ be Banach spaces. Let $C_{rc}(X,E)$ be the Banach space of all continuous functions $f : X \to E$ such that $f(X)$ is a relatively compact set in $E$. We establish an integral representation theorem for bounded linear operators $T : C_{rc}(X,E) \to F$. We characterize continuous operators from $C_{rc}(X,E)$, provided with the strict topologies $\beta_z(X,E)$ $(z = \sigma, \tau)$ to $F$, in terms of their representing operator-valued measures.

## 1. Introduction and terminology

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be real Banach spaces, and let $E'$ and $F'$ denote the Banach duals of $E$ and $F$, respectively. By $B_{F'}$ and $B_E$ we denote the closed unit ball in $F'$ and $E$, respectively. By $\mathcal{L}(E,F)$ we denote the Banach space of all bounded linear operators $U : E \to F$, provided with the uniform norm $\|\cdot\|$. Given a locally convex space $(L,\xi)$ by $(L,\xi)'$ we will denote its topological dual. We denote by $\sigma(L,K)$ the weak topology on $L$ with respect to a dual pair $\langle L, K\rangle$.

Assume that $X$ is a completely regular Hausdorff space. Let $C_{rc}(X,E)$ (resp. $C_b(X,E)$) stand for the Banach space of all continuous functions $f : X \to E$ such that $f(X)$ is a relatively compact set in $E$ (resp. bounded continuous functions $f : X \to E$) provided with the uniform norm $\|\cdot\|$. By $C_{rc}(X,E)'$ and $C_{rc}(X,E)''$ we denote the Banach dual and the Banach bidual of $C_{rc}(X,E)$, respectively. Let

$$B_{C_{rc}} = \{f \in C_{rc}(X,E) : \|f\| \le 1\}.$$

We write $C_b(X)$ instead of $C_{rc}(X,\mathbb{R})$. For $f \in C_{rc}(X,E)$ let

$$\widetilde{f}(t) = \|f(t)\|_E \quad \text{for} \quad t \in X.$$

Let $\mathcal{B}$ (resp. $\mathcal{B}a$) be the algebra (resp. $\sigma$-algebra) of Baire sets in $X$, which is the algebra (resp. $\sigma$-algebra) generated by the class $\mathcal{Z}$ of all zero sets of

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functions of $C_b(X)$. Let $M(X)$ stand for the space of all Baire measures on $\mathcal{B}$. Then $M(X)$ with the norm $\|\nu\| = |\nu|(X)$ (= the total variation of $\nu$), is a Dedekind complete Banach lattice (see [21]).

Due to the Alexandrov representation theorem (see [21, Theorem 5.1]) $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \nu \mapsto \varphi_\nu \in C_b(X)'$, where

$$\varphi_\nu(u) = \int_X u d\nu \quad \text{for} \quad u \in C_b(X),$$

and $\|\varphi_\nu\| = \|\nu\|$.

By $M(X, E')$ we denote the set of all finitely additive measures $\mu : \mathcal{B} \to E'$ with the following properties:

(i) For each $x \in E$, the function $\mu_x : \mathcal{B} \to \mathbb{R}$ defined by $\mu_x(A) = \mu(A)(x)$, belongs to $M(X)$,

(ii) $|\mu|(X) < \infty$, where $|\mu|(A)$ stands for the variation of $\mu$ on $A \in \mathcal{B}$.

In view of [11, Theorem 2.5] $C_{rc}(X, E)'$ can be identified with $M(X, E')$ through the linear mapping $M(X, E') \ni \mu \mapsto \Phi_\mu \in C_{rc}(X, E)'$, where

$$\Phi_\mu(f) = \int_X f d\mu \quad \text{for} \quad f \in C_{rc}(X, E),$$

and $\|\Phi_\mu\| = |\mu|(X)$.

In the topological measure theory the so-called strict topologies on $C_b(X)$ and $C_{rc}(X, E)$ are of importance (see [12], [13], [14], [15], [21] for definitions and more details). In this paper we will consider the strict topologies $\beta_z(X, E)$ on $C_{rc}(X, E)$ and $\beta_z(X)$ on $C_b(X)$, where $z = \sigma, \tau$.

Let $M_\sigma(X)$ and $M_\tau(X)$ denote the subspaces of $M(X)$ of all $\sigma$-additive and $\tau$-additive Baire measures, respectively. Then $M_\tau(X) \subset M_\sigma(X)$. It is known that (see [21, §6]):

$$(C_b(X), \beta_z(X))' = \{\varphi_\nu : \nu \in M_z(X)\} = L_z(C_b(X)) \tag{1.1}$$

for $z = \sigma, \tau$, where $L_\sigma(C_b(X))$ and $L_\tau(C_b(X))$ are spaces of all $\sigma$-additive and $\tau$-additive functionals on $C_b(X)$.

For $z = \sigma, \tau$ let

$$M_z(X, E') := \{\mu \in M(X, E') : \mu_x \in M_z(X) \quad \text{for each} \quad x \in E\}.$$

Then for $z = \sigma, \tau$ we have

$$(C_{rc}(X, E), \beta_z(X, E))' = \{\Phi_\mu : \mu \in M_z(X, E')\} \tag{1.2}$$

(see [12, Theorems 4.6 and 4.7]).

The theory of linear operators from $C_{rc}(X, E)$ and $C_b(X, E)$ to a locally convex Hausdorff space $F$ (in particular, a Banach space) has been developed by Katsaras and Liu [15], Aguayo and Sanchez [2], Aguayo and Nova-Yanèz [3] and Khurana [17]. Locally solid topologies on the space $C_b(X, E)$ have been studied in [16], [18], [19]. It is known that the natural strict topologies $\beta_z(X, E)$ on $C_b(X, E)$, where $z = \sigma, \infty, p, g, \tau, t$ are locally solid.

In Section 2 we study locally solid topologies on $C_{rc}(X,E)$. Section 3 is devoted to the study of linear functionals on $C_{rc}(X,E)$. In Section 4 we state an integral representation of bounded linear operators $T: C_{rc}(X,E) \to F$. In Section 5 we characterize continuous operators from $C_{rc}(X,E)$, equipped with the strict topologies $\beta_z(X,E)$, $z = \sigma, \tau$ to $F$, in terms of the corresponding operator measures.

## 2. Locally solid topologies on $C_{rc}(X,E)$

Following [16, Section 8] we can introduce the concepts of solidness and locally solid topologies on $C_{rc}(X,E)$.

**Definition 2.1.** (i) A subset $H$ of $C_{rc}(X,E)$ is said to be *solid* whenever $\widetilde{f_1} \le \widetilde{f_2}$, $f_1 \in C_{rc}(X,E)$, $f_2 \in H$ imply $f_1 \in H$.

(ii) A linear Hausdorff topology $\tau$ on $C_{rc}(X,E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets.

The following lemma will be of importance for the study of locally solid topologies on $C_{rc}(X,E)$.

**Lemma 2.1.** *Assume that $f \in C_{rc}(X,E)$ and $\widetilde{f} \le \sum_{i=1}^m u_i$, where $u_i \in C_b(X)^+$, $i = 1, \ldots, m$. Then there exist $f_i \in C_{rc}(X,E)$ such that $f = \sum_{i=1}^m f_i$ and $\widetilde{f_i} \le u_i$, $i = 1, \ldots, m$.*

*Proof.* Assume that $f_i(t) = u_i(t)\big(\sum_{j=1}^m u_j(t)\big)^{-1} f(t)$ if $\sum_{j=1}^m u_j(t) > 0$ and $f_i(t) = 0$ if $\sum_{j=1}^m u_j(t) = 0$, $i = 1, 2, \ldots, m$. Note that $f_i$ are continuous and $f = \sum_{i=1}^m f_i$ and $\widetilde{f_i} \le u_i$, $i = 1, 2, \ldots, m$. To show that $f_i \in C_{rc}(X,E)$, we prove that $\{f_i(t) : t \in X\}$ is a relatively sequentially compact set in $E$. Indeed, let $(t_n)$ be a sequence in $X$. Then there exists a subsequence $(t_{k_n})$ of $(t_n)$ such that $f(t_{k_n}) \to x$ for some $x \in E$ and $u_i(t_{k_n}) \to a_i$, where $a_i \ge 0$ for $i = 1, 2, \ldots, m$.

Assume first that $\sum_{j=1}^m a_j > 0$. Then $f_i(t_{k_n}) \to a_i\big(\sum_{j=1}^m a_j\big)^{-1} x \in E$.

Now assume that $\sum_{j=1}^m a_j = 0$, i.e., $u_i(t_{k_n}) \to 0$ for $i = 1, 2, \ldots, m$. We have $\widetilde{f_i} \le \widetilde{f}$ and $\widetilde{f}(t_{k_n}) \to 0$. Hence $\widetilde{f_i}(t_{k_n}) \to 0$, i.e., $f_i \in C_{rc}(X,E)$ for $i = 1, 2, \ldots, m$. □

Using Lemma 2.1 and arguing as in the proofs of [20, Theorems 1.2 and 2.1] we obtain the following results.

**Proposition 2.2.** *The convex hull* co$H$ *of a solid subset $H$ of $C_{rc}(X,E)$ is solid.*

**Proposition 2.3.** *Let $\tau$ be a locally solid topology on $C_{rc}(X,E)$. Then the $\tau$-closure of a solid subset $H$ of $C_{rc}(X,E)$ is solid.*

**Definition 2.2.** A linear topology $\tau$ on $C_{rc}(X,E)$ that at the some time is locally solid and locally convex will be called a *locally convex-solid topology.*

In view of Propositions 2.2 and 2.3 we see that for a locally convex-solid topology on $C_{rc}(X,E)$ the collection of all $\tau$-closed and solid $\tau$-neighborhoods of 0 forms a local base at 0 for $\tau$.

**Definition 2.3.** A seminorm $\rho$ on $C_{rc}(X,E)$ is said to be *solid* whenever $\rho(f_1) \le \rho(f_2)$ if $f_1, f_2 \in C_{rc}(X,E)$ and $\widetilde{f_1} \le \widetilde{f_2}$.

Arguing as in the proof of [20, Theorem 2.2] we get:

**Proposition 2.4.** *For a locally convex topology $\tau$ on $C_{rc}(X,E)$ the following statements are equivalent*:

(i) *$\tau$ is generated by the family of solid seminorms.*
(ii) *$\tau$ is a locally convex-solid topology.*

Now we establish a mutual relationship between locally convex-solid topologies on $C_{rc}(X,E)$ and the vector lattice $C_b(X)$.

Given a Riesz seminorm $p$ on $C_b(X)$ let us set

$$p^{\vee}(f) := p(\widetilde{f}) \quad \text{for all} \quad f \in C_{rc}(X,E).$$

Clearly $p^{\vee}$ is a solid seminorm on $C_{rc}(X,E)$.

Let $x_0 \in S_E = \{x \in E : \|x\|_E = 1\}$. Given a solid seminorm $\rho$ on $C_{rc}(X,E)$, let

$$\rho^{\wedge}(u) := \rho(u \otimes x_0) \quad \text{for} \quad u \in C_b(X).$$

It is seen that $\rho^{\wedge}$ is well defined because $\rho(u \otimes x_0)$ does not depend on the choice of $x_0 \in S_E$, due to solidness of $\rho$. Clearly $\rho^{\wedge}$ is a Riesz seminorm on $C_b(X)$.

One can easily show the following results (see [20, Lemma 3.1]).

**Proposition 2.5.** (i) *If $\rho$ is a solid seminorm on $C_{rc}(X,E)$, then $(\rho^{\wedge})^{\vee}(f) = \rho(f)$ for all $f \in C_{rc}(X,E)$.*

(ii) *If $p$ is a Riesz seminorm on $C_b(X)$, then $(p^{\vee})^{\wedge}(u) = p(u)$ for all $u \in C_b(X)$.*

Let $\tau$ be a locally convex-solid topology on $C_{rc}(X,E)$. Then in view of Theorem 2.4 $\tau$ is generated by some family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_{rc}(X,E)$. By $\tau^{\wedge}$ we will denote the locally convex-solid topology on $C_b(X)$ generated by the family $\{\rho_\alpha^{\wedge} : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$. One can check that $\tau^{\wedge}$ does not depend on the choice of a family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_{rc}(X,E)$ generating $\tau$.

Next, let $\xi$ be a locally convex-solid topology on $C_b(X)$. Then $\xi$ is generated by some family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$ (see [1, Theorem 6.3]). By $\xi^{\vee}$ we will denote the locally convex-solid topology on $C_{rc}(X,E)$ generated by the family $\{p_\alpha^{\vee} : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_{rc}(X,E)$. One can verify that $\xi^{\vee}$ does not depend on the choice of a family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$ that generates $\xi$.

In view of Proposition 2.5 we can easily get:

**Theorem 2.6.** (i) *For a locally convex-solid topology* $\tau$ *on* $C_{rc}(X,E)$ *we have*: $(\tau^\wedge)^\vee = \tau$.

(ii) *For a locally convex-solid topology* $\xi$ *on* $C_b(X)$ *we have*: $(\xi^\vee)^\wedge = \xi$.

Now we recall definitions of strict topologies $\beta_z(X,E)$ on $C_{rc}(X,E)$ for $z = \sigma, \tau$ (see [12], [13] and [14] for more details). Let $\beta X$ stand for the Stone-Čech compactification of $X$. For a compact subset $Q$ of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v}|_Q \equiv 0\}$, where $\overline{v}$ denotes the unique extension of $v \in C_b(X)$ on $\beta X$. For each $v \in C_Q(X)$ let $\rho_v(f) := \sup_{t\in X} |v(t)|\widetilde{f}(t)$ for $f \in C_{rc}(X,E)$, and let $\beta_Q(X,E)$ be the locally convex-solid topology on $C_{rc}(X,E)$ defined by $\{\rho_v : v \in C_Q(X)\}$.

Now let $\mathcal{C}$ be some family of compact subsets of $\beta X \setminus X$. The *strict topology* $\beta_{\mathcal{C}}(X,E)$ on $C_{rc}(X,E)$ determined by $\mathcal{C}$ is the greatest lower bound (in the class of locally convex Hausdorff topologies) of the topologies $\beta_Q(X,E)$, as $Q$ runs over $\mathcal{C}$.

**Proposition 2.7.** *The strict topology* $\beta_{\mathcal{C}}(X,E)$ *on* $C_{rc}(X,E)$ *is locally convex-solid.*

*Proof.* Since $\beta_{\mathcal{C}}(X,E)$ is an inductive limit topology on $C_{rc}(X,E)$, it has a local base at 0 consisting of all sets of the form:

$$\mathrm{eco}\Big(\bigcup\big\{W_{v_Q} : Q \in \mathcal{C} \ \text{ and } \ v_Q \in C_Q(X)\big\}\Big),$$

where for $v_Q \in C_Q(X)$, $W_{v_Q} = \{f \in C_{rc}(X,E) : \varrho_{v_Q}(f) \le 1\}$ (see [5, Chapter 2.1.4]) (here eco$W$ denotes the balanced convex hull of a set $W$ in $C_{rc}(X,E)$). We shall show that a set $V = \mathrm{eco}W$, where $W = \bigcup\{W_{v_Q} : Q \in \mathcal{C}$ and $v_Q \in C_Q(X)\}$ is solid.

Indeed, let $f \in C_{rc}(X,E)$, $g \in V$ and $\widetilde{f} \le \widetilde{g}$. Then $g = \sum_{i=1}^n \lambda_i g_i$, where $\sum_{i=1}^n |\lambda_i| \le 1$ and $g_i \in W$ for $i = 1,2,\ldots,n$. Hence for each $i = 1,2,\ldots,n$, there exist $Q_i \in \mathcal{C}$ and $v_{Q_i} \in C_{Q_i}(X)$ such that $\varrho_{v_{Q_i}}(g_i) \le 1$. By Lemma 2.1 there exist $f_1,\ldots,f_n \in C_{rc}(X,E)$ such that $f = \sum_{i=1}^n f_i$ and $\widetilde{f_i} \le \widetilde{\lambda g_i}$ for $i = 1,2,\ldots,n$. Let $h_i = \frac{1}{\lambda_i} f_i$ for $i = 1,2,\ldots,n$. Since $\varrho_{v_{Q_i}}$ is a solid seminorm on $C_{rc}(X,E)$ and $\widetilde{h}_i \le \widetilde{g}_i$ for $i = 1,2,\ldots,n$, we have $\varrho_{v_{Q_i}}(h_i) \le \varrho_{v_{Q_i}}(g_i) \le 1$. Hence $h_i \in W_{v_{Q_i}}$, so $h_i \in W$. Then $f = \sum_{i=1}^n \lambda_i h_i \in V$, as desired. □

Let $\mathcal{C}_\sigma$ (resp. $\mathcal{C}_\tau$) be the family of all zero subsets (resp. compact subsets) of $\beta X \setminus X$, and let $\beta_z(X,E) = \beta_{\mathcal{C}_z}(X,E)$, where $z = \sigma, \tau$.

Arguing as in the proof of [20, Theorem 4.2] we get:

**Theorem 2.8.** *For* $z = \sigma, \tau$, *we have*:

$$\beta_z(X,E) = \beta_z(X)^\vee, \quad \beta_z(X,E)^\wedge = \beta_z(X).$$

Now we define two classes of locally convex-solid topologies on $C_{rc}(X,E)$.

**Definition 2.4.** A locally convex-solid topology $\tau$ on $C_{rc}(X,E)$ is said to be:

(i) *$\sigma$-Dini* if $f_n \to 0$ for $\tau$ whenever $(f_n)$ is a sequence in $C_{rc}(X,E)$ such that $\widetilde{f_n}(t) \downarrow 0$ for $t \in X$.
(ii) *Dini* if $f_\alpha \to 0$ for $\tau$ whenever $(f_\alpha)$ is a net in $C_{rc}(X,E)$ such that $\widetilde{f_\alpha}(t) \downarrow 0$ for $t \in X$.

It is known that $\beta_\sigma(X)$ is the finest $\sigma$-Dini topology on $C_b(X)$ and $\beta_\tau(X)$ is the finest Dini topology on $C_b(X)$ (see [21, Corollaries 11.16 and 11.28]).

**Corollary 2.9.** (i) *$\beta_\sigma(X,E)$ is the finest $\sigma$-Dini topology on $C_{rc}(X,E)$.*
(ii) *$\beta_\tau(X,E)$ is the finest Dini topology on $C_{rc}(X,E)$.*

*Proof.* (i) Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of Riesz seminorms on $C_b(X)$ that generates $\beta_\sigma(X)$. Assume that $(f_n)$ is a sequence in $C_{rc}(X,E)$ such that $\widetilde{f_n}(t) \downarrow 0$ for all $x \in X$. Then $p_\alpha^\vee(f_n) = p_\alpha(\widetilde{f_n}) \to 0$ and this means that $f_n \to 0$ for $\beta_\sigma(X)^\vee$. In view of Theorem 2.8 we conclude that $f_n \to 0$ for $\beta_\sigma(X,E)$.

Now assume that $\tau$ is a $\sigma$-Dini topology on $C_{rc}(X,E)$. Then $\tau^\wedge \subset \beta_\sigma(X)$, and hence $\tau = (\tau^\wedge)^\vee \subset \beta_\sigma(X)^\vee = \beta_\sigma(X,E)$ (see Theorems 2.6 and 2.8).

(ii) It is similar to (i). □

## 3. Linear functionals on $C_{rc}(X,E)$

Assume that $\mu \in M(X,E')$. For $u \in C_b(X)^+$ let us put

$$|\Phi_\mu|(u) := \sup\{|\Phi_\mu(h)| : h \in C_{rc}(X,E),\ \ \widetilde{h} \le u\}.$$

**Proposition 3.1.** *For $\mu \in M(X,E')$, the functional $|\Phi_\mu| : C_b(X)^+ \to \mathbb{R}^+$ is additive.*

*Proof.* Let $u_1, u_2 \in C_b(X)^+$. First we shall show that

$$|\Phi_\mu|(u_1+u_2) \le |\Phi_\mu|(u_1) + |\Phi_\mu|(u_2).$$

Indeed, let $\varepsilon > 0$. Then there exists $h_0 \in C_{rc}(X,E)$ such that $\widetilde{h_0} \le u_1 + u_2$ and $|\Phi_\mu|(u_1+u_2) \le |\Phi_\mu(h_0)| + \varepsilon$. Then by Lemma 2.1 there exist $h_1, h_2 \in C_{rc}(X,E)$ such that $h_0 = h_1 + h_2$ and $\widetilde{h_i} \le u_i$, $i = 1,2$. Hence

$$\begin{aligned}|\Phi_\mu|(u_1+u_2) \le |\Phi_\mu(h_1+h_2)| + \varepsilon\ &\le |\Phi_\mu(h_1)| + |\Phi_\mu(h_2)| + \varepsilon \\ &\le |\Phi_\mu|(u_1) + |\Phi_\mu|(u_2) + \varepsilon.\end{aligned}$$

Now we shall show that

$$|\Phi_\mu|(u_1) + |\Phi_\mu|(u_2) \le |\Phi_\mu|(u_1+u_2).$$

Indeed, let $\varepsilon > 0$ be given. Then there exist $h_1, h_2 \in C_{rc}(X,E)$ such that $\widetilde{h_i} \le u_i$ and $|\Phi_\mu|(u_i) \le |\Phi_\mu(h_i)| + \frac{\varepsilon}{2}$, $i = 1,2$. Let $g_i = \mathrm{sign}\Phi_\mu(h_i)h_i$ for $i = 1,2$. Then $\widetilde{g_i} \le \widetilde{h_i}$ and $\widetilde{g_1+g_2} \le u_1 + u_2$, and hence

$$\begin{aligned}|\Phi_\mu|(u_1) + |\Phi_\mu|(u_2)\ &\le \Phi_\mu(g_1) + \Phi_\mu(g_2) + \varepsilon = \Phi_\mu(g_1+g_2) + \varepsilon \\ &\le |\Phi_\mu|(u_1+u_2) + \varepsilon.\end{aligned}$$

□

In view of [1, Theorem 1.7] we obtain that $|\Phi_\mu| : C_b(X)^+ \to \mathbb{R}^+$ has a unique linear extension (denoted by $|\Phi_\mu|$ again)

$$|\Phi_\mu| : C_b(X) \to \mathbb{R}$$

defined by

$$|\Phi_\mu|(u) := |\Phi_\mu|(u^+) - |\Phi_\mu|(u^-) \quad \text{for all} \quad u \in C_b(X).$$

Hence for $u \in C_b(X)$,

$$\big||\Phi_\mu|(u)\big| \leq |\Phi_\mu|(u).$$

**Corollary 3.2.** *Let $\mu \in M_\sigma(X, E')$. Then for $u \in C_b(X)$ we have*

$$|\Phi_\mu|(u) = \int_X ud|\mu| = \varphi_{|\mu|}(u).$$

*Proof.* In view of [16, Theorem 2.1] for $u \in C_b(X)^+$ we have,

$$\int_X ud|\mu| = \sup\left\{\left|\int_X gd\mu\right| : g \in C_b(X) \otimes E,\ \widetilde{g} \leq u\right\}$$

and

$$|\Phi_\mu|(u) = \sup\left\{\left|\int_X hd\mu\right| : h \in C_{rc}(X, E),\ \widetilde{h} \leq u\right\}.$$

Since $C_b(X) \otimes E \subset C_{rc}(X, E)$, we get $\int_X ud|\mu| \leq |\Phi_\mu|(u)$.

Now let $h \in C_{rc}(X, E)$ and $\widetilde{h} \leq u$. Then

$$\left|\int_X hd\mu\right| \leq \int_X \widetilde{h}d|\mu| \leq \int_X ud|\mu|,$$

and hence $|\Phi_\mu|(u) \leq \int_X ud|\mu|$, as desired. □

**Corollary 3.3.** *For $\mu \in M(X, E')$ the following statements are equivalent*:

(i) $\mu \in M_\sigma(X, E')$.

(ii) $\Phi_\mu \in (C_{rc}(X, E), \beta_\sigma(X, E))'$.

(iii) $|\Phi_\mu| \in (C_b(X), \beta_\sigma(X))'$.

(iv) $\Phi_\mu(u_n g) \to 0$ *uniformly for* $g \in B_{C_{rc}}$ *whenever* $(u_n)$ *is a sequence in* $C_b(X)$ *such that* $u_n(t) \downarrow 0$ *for* $t \in X$.

*Proof.* (i)$\Longleftrightarrow$(ii) It follows from (1.2).

(i)$\Longrightarrow$(iii) Assume that $\mu \in M_\sigma(X, E')$. Then $|\mu| \in M_\sigma(X)$ (see [10, Proposition 3.9]) and by Corollary 3.2 and (1.1), we get $|\Phi_\mu| \in (C_b(X), \beta_\sigma(X))'$.

(iii)$\Longrightarrow$(iv) Assume that (iii) holds. Let $(u_n)$ be a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$. Then for $g \in B_{C_{rc}}$, by Corollary 3.2 we get

$$|\Phi_\mu(u_n g)| \leq \int_X u_n d|\mu| = |\Phi_\mu|(u_n).$$

Since $|\Phi_\mu| \in L_\sigma(C_b(X))$ (see (1.1)), we get $\Phi_\mu(u_n g) \to 0$, as desired.

(iv)$\Longrightarrow$(i) Assume that (iv) holds. Let $(u_n)$ be a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$. Then for $x \in E$ we get

$$\Phi_\mu(u_n \otimes x) = \int_X (u_n \otimes x) d\mu = \int_X u_n d\mu_x \to 0.$$

Hence by (1.1) $\mu_x \in M_\sigma(X)$, i.e., $\mu \in M_\sigma(X, E')$. □

Arguing similarly as in the proof of Corollary 3.3 and using [10, Proposition 3.9] we get:

**Corollary 3.4.** *For $\mu \in M(X, E')$ the following statements are equivalent*:

(i) $\mu \in M_\tau(X, E')$.
(ii) $\Phi_\mu \in (C_{rc}(X, E), \beta_\tau(X, E))'$.
(iii) $|\Phi_\mu| \in (C_b(X), \beta_\tau(X))'$.
(iv) $\Phi_\mu(u_\alpha g) \to 0$ *uniformly for* $g \in B_{C_{rc}}$ *whenever* $(u_\alpha)$ *is a sequence in* $C_b(X)$ *such that* $u_\alpha(t) \downarrow 0$ *for* $t \in X$.

**Proposition 3.5.** *Let $\mathcal{M}$ be a subset of $M_\sigma(X, E')$. Then the following statements are equivalent*:

(i) $\{\Phi_\mu : \mu \in \mathcal{M}\}$ *is* $\beta_\sigma(X, E)$*-equicontinuous.*
(ii) $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ *is uniformly* $\sigma$*-additive, i.e.,* $\sup\{|\Phi_\mu|(u_n) : \mu \in \mathcal{M}\} \to 0$ *whenever* $(u_n)$ *is a sequence in* $C_b(X)$ *such that* $u_n(t) \downarrow 0$ *for* $t \in X$.
(iii) $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ *is* $\beta_\sigma(X)$*-equicontinuous.*

*Proof.* (i)$\Longrightarrow$(ii) Assume that $\{\Phi_\mu : \mu \in \mathcal{M}\}$ is $\beta_\sigma(X, E)$-equicontinuous. To show that $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ is uniformly $\sigma$-additive, let $(u_n)$ be a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$. Let $\varepsilon > 0$ be given. Then there exists a convex and solid neighborhood $V$ of 0 for $\beta_\sigma(X, E)$ such that $\sup_{\mu \in \mathcal{M}} |\Phi_\mu(f)| \leq \varepsilon$ for all $f \in V$ (see Proposition 2.7). Since $\beta_\sigma(X, E)$ is a $\sigma$-Dini topology (see Corollary 2.9), there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \otimes x_0 \in V$ ($x_0 \in S_E$) for $n \geq n_\varepsilon$. Hence $|\Phi_\mu(u_n \otimes x_0)| \leq \varepsilon$ for $n \geq n_\varepsilon$. Let $n_0 \geq n_\varepsilon$ be fixed and let $h \in C_{rc}(X)$ with $\widetilde{h} \leq u_{n_0}$. Then $h \in V$ because $V$ is solid, and hence $\sup_{\mu \in \mathcal{M}} |\Phi_\mu(h)| \leq \varepsilon$. It follows that $\sup_\mu |\Phi_\mu|(u_n) \leq \varepsilon$ for $n \geq n_\varepsilon$, as desired.

(ii)$\Longleftrightarrow$(iii) See [21, Theorem 11.14].

(iii)$\Longrightarrow$(i) Assume that $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ is $\beta_\sigma(X)$-equicontinuous. Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be the family of solid seminorms that generates $\beta_\sigma(X, E)$ (see Propositions 2.4 and 2.7). Given $\varepsilon > 0$ there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ and $\eta > 0$ such that $\sup\{|\Phi_\mu|(u) : \mu \in \mathcal{M}\} \leq \varepsilon$ for $u \in C_b(X)$ with $\max_{1 \leq i \leq n} \hat{\rho}_{\alpha_i}(u) \leq \eta$. Let $f \in C_{rc}(X, E)$ with $\max_{1 \leq i \leq n} \rho_{\alpha_i}(f) \leq \eta$. Since $\hat{\rho}_{\alpha_i}(\widetilde{f}) = \rho_{\alpha_i}(\widetilde{f} \otimes x_0) = \rho_{\alpha_i}(f)$ ($i = 1, 2, \ldots, n$), we obtain that $\sup\{|\Phi_\mu|(\widetilde{f}) : \mu \in \mathcal{M}\} \leq \varepsilon$. But $|\Phi_\mu(f)| \leq |\Phi_\mu|(\widetilde{f})$, so $\sup\{|\Phi_\mu(f)| : \mu \in \mathcal{M}\} \leq \varepsilon$, and this means that the family $\{\Phi_\mu : \mu \in \mathcal{M}\}$ is $\beta_\sigma(X, E)$-equicontinuous. □

Using [21, Theorem 11.24] we can prove an analogous result for $\beta_\tau(X, E)$ with a similar proof.

**Proposition 3.6.** *Let $\mathcal{M}$ be a subset of $M_\tau(X, E')$. Then the following statements are equivalent*:

(i) $\{\Phi_\mu : \mu \in \mathcal{M}\}$ *is* $\beta_\tau(X, E)$*-equicontinuous.*
(ii) $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ *is uniformly* $\tau$*-additive, i.e.,* $\sup\{|\Phi_\mu|(u_\alpha) : \mu \in \mathcal{M}\} \to 0$ *whenever* $(u_\alpha)$ *is a net in* $C_b(X)$ *such that* $u_\alpha(t) \downarrow 0$ *for* $t \in X$.
(iii) $\{|\Phi_\mu| : \mu \in \mathcal{M}\}$ *is* $\beta_\tau(X)$*-equicontinuous.*

## 4. Integral representation of bounded linear operators on $C_{rc}(X, E)$

By $B(\mathcal{B}, E)$ we denote the Banach space of all totally $\mathcal{B}$-measurable functions $g : X \to E$ (the uniform limits of sequences of $E$-valued $\mathcal{B}$-simple functions), provided with the uniform norm $\|\cdot\|$ (see [7], [8]).

It is known that $C_{rc}(X, E) \subset B(\mathcal{B}, E)$ (see [16]), and one can embed $B(\mathcal{B}, E)$ into $C_{rc}(X, E)''$ by the mapping $\pi : B(\mathcal{B}, E) \to C_{rc}(X, E)''$, where for $g \in B(\mathcal{B}, E)$,

$$\pi(g)(\Phi_\mu) = \int_X g d\mu \quad \text{for} \quad \mu \in M(X, E').$$

Let $i_F : F \to F''$ stand for the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ denote the left inverse of $i_F$, that is, $j_F \circ i_F = id_F$. Note that $j_F$ is $(\sigma(i_F(F), F'), \sigma(F, F'))$-continuous.

Now assume that $T : C_{rc}(X, E) \to F$ is a bounded linear operator. Let $T' : F' \to C_{rc}(X, E)'$ and $T'' : C_{rc}(X, E)'' \to F''$ stand for the conjugate and biconjugate operators of $T$, respectively. Let

$$\hat{T} := T'' \circ \pi : B(\mathcal{B}, E) \to F''.$$

Then $\hat{T}$ is a bounded operator. For $A \in \mathcal{B}$ let us put

$$m(A)(x) := \hat{T}(\mathbb{1}_A \otimes x) \quad \text{for} \quad x \in E.$$

Then $m : \mathcal{B} \to \mathcal{L}(E, F'')$ will be called a *representing measure* of $T$.

We define the semivariation $\widetilde{m}(A)$ of $m$ on $A \in \mathcal{B}$ by

$$\widetilde{m}(A) = \sup \|\sum m(A_i)(x_i)\|_{F''},$$

where the supremum is taken over all finite $\mathcal{B}$-partitions $(A_i)$ of $A$ and $x_i \in B_E$ for each $i$. For $y' \in F'$ let $m_{y'} : \mathcal{B} \to E'$ be vector measures defined by $m_{y'}(A)(x) := m(A)(x)(y')$ for $A \in \mathcal{B}, x \in E$. Let $|m_{y'}|(A)$ stand for the variation of $m_{y'}$ on $A$. Then for $A \in \mathcal{B}$ (see [7, §4, Proposition 5]),

$$\widetilde{m}(A) = \sup\{|m_{y'}|(A) : y' \in B_{F'}\}.$$

Since $\hat{T} : B(\mathcal{B}, E) \to F''$ is bounded, we have

$$\hat{T}(g) = \int_X g dm \quad \text{for} \quad g \in B(\mathcal{B}, E),$$

$\|\hat{T}\| = \widetilde{m}(X)$, and for each $y' \in F'$ we have,

$$\hat{T}(g)(y') = \int_X g dm_{y'} \quad \text{for} \quad g \in B(\mathcal{B}, E),$$

(see [7, §6], [8, §1, G-H]). Moreover, from the general properties of the operator $\hat{T}$ it follows immediately that

$$\hat{T}(C_{rc}(X,E)) \subset i_F(F). \tag{4.1}$$

For each $x \in E$ we can define a vector measure $m_x : \mathcal{B} \to F''$ by

$$m_x(A)(y') := m(A)(x)(y') \quad \text{for} \quad A \in \mathcal{B},\ y' \in F'.$$

For $x \in E$ and $y' \in F'$ let

$$m_{x,y'}(A) := m(A)(x)(y') \quad \text{for} \quad A \in \mathcal{B}.$$

An integral representation of weakly compact operators $T : C_{rc}(X,E) \to F$ was established by Katsaras and Liu (see [15, Theorem 3]). Now we state a general Riesz representation theorem for bounded linear operators $T : C_{rc}(X,E) \to F$.

**Theorem 4.1.** *Let $T : C_{rc}(X,E) \to F$ be a bounded linear operator, and $m : \mathcal{B} \to \mathcal{L}(E,F'')$ its representing measure. Then the following statements hold*:

(i) *$m_{y'} \in M(X,E')$ for each $y' \in F'$.*
(ii) *The mapping $F' \ni y' \mapsto m_{y'} \in M(X,E')$ is $(\sigma(F',F), \sigma(M(X,E'), C_{rc}(X,E)))$-continuous.*
(iii) *For each $y' \in F'$, $y'(T(f)) = \int_X f dm_{y'}$ for $f \in C_{rc}(X,E)$.*
(iv) *$T(f) = j_F(\int_X f dm)$ for $f \in C_{rc}(X,E)$.*
(v) *$\|T\| = \widetilde{m}(X)$.*

*Conversely, let $m : \mathcal{B} \to \mathcal{L}(E,F'')$ be a vector measure satisfying* (i) *and* (ii). *Then there exists a unique bounded linear operator $T : C_{rc}(X,E) \to F$ such that* (iii) *holds and $m(A)(x) = (T'' \circ \pi)(\mathbb{1}_A \otimes x)$ for all $A \in \mathcal{B}$, $x \in E$. In consequence, the vector measure $m : \mathcal{B} \to \mathcal{L}(E,F'')$ satisfying* (i), (ii) *and* (iii) *is uniquely determined by a bounded linear operator $T : C_{rc}(X,E) \to F$.*

*Proof.* Let $y' \in F'$. Since $y' \circ T \in C_{rc}(X,E)'$, there exists a unique $\mu_{y'\circ T} \in M(X,E')$ such that

$$(y' \circ T)(f) = \int_X f d\mu_{y'\circ T} \quad \text{for} \quad f \in C_{rc}(X,E).$$

For $A \in \mathcal{B}$ and $x \in E$ we have

$$\begin{aligned} m_{y'}(A)(x) &= m(A)(x)(y') = \hat{T}(\mathbb{1}_A \otimes x)(y') \\ &= T''(\pi(\mathbb{1}_A \otimes x))(y') = \pi(\mathbb{1}_A \otimes x)(T'(y')) \\ &= \pi(\mathbb{1}_A \otimes x)(y' \circ T) = \int_X (\mathbb{1}_A \otimes x) d\mu_{y'\circ T} = \mu_{y'\circ T}(A)(x). \end{aligned}$$

It follows that $m_{y'} = \mu_{y'\circ T} \in M(X,E')$ and

$$(y' \circ T)(f) = \int_X f dm_{y'} \quad \text{for} \quad f \in C_{rc}(X,E).$$

This means that (i) and (iii) hold. Since the mapping $T' : F' \to C_{rc}(X,E)'$ is $(\sigma(F',F), \sigma(C_{rc}(X,E)', C_{rc}(X,E)))$-continuous, the mapping $F' \ni y' \mapsto m_{y'} \in M(X,E')$ is $(\sigma(F',F), \sigma(M(X,E'), C_{rc}(X,E)))$-continuous, i.e., (ii) holds.

Note that using (4.1) we have $T(f) = j_F(\hat{T}(f)) = j_F(\int_X f dm)$ for $f \in C_{rc}(X,E)$, i.e., (iv) holds. Using (iii) we get

$$\begin{aligned} \|T\| = \|T'\| &= \sup\{\|T'(y')\| : y' \in B_{F'}\} \\ &= \sup\{\|y' \circ T\| : y' \in B_{F'}\} \\ &= \sup\{|m_{y'}|(X) : y' \in B_{F'}\} \end{aligned}$$

i.e., (v) holds.

Conversely, let $m : \mathcal{B} \to \mathcal{L}(E, F'')$ be a vector measure satisfying (i) and (ii). Then $m_{y'} \in M(X,E')$ and the mapping $F' \ni y' \mapsto m_{y'} \in M(X,E')$ is $(\sigma(F',F), \sigma(M(X,E'), C_{rc}(X,E)))$-continuous.

For $f \in C_{rc}(X,E)$ define a linear mapping $\Psi_f : F' \to \mathbb{R}$ by

$$\Psi_f(y') = \int_X f dm_{y'} \quad \text{for all} \ \ y' \in F'.$$

Then by (ii) $\Psi_f$ is a $\sigma(F',F)$-continuous linear functional, so there exists a unique $y_f \in F$ such that $\Psi_f = i_F(y_f)$, i.e., $\Psi_f(y') = y'(y_f)$ for each $y' \in F'$. For each $f \in C_{rc}(X,E)$ let us put

$$T(f) = y_f.$$

Then $T : C_{rc}(X,E) \to F$ is a linear mapping and for each $y' \in F'$ we have

$$\begin{aligned} \sup\{|y'(T(f))| : \|f\| \le 1\} &= \sup\{\{|y'(y_f)| : \|f\| \le 1\} \\ &= \sup\left\{\Big|\int_X f dm_{y'}\Big| : \|f\| \le 1\right\} \\ &\le \sup\left\{\int_X \|f\| d|m_{y'}| : \|f\| \le 1\right\} \\ &\le |m_{y'}|(X) < \infty. \end{aligned}$$

This means that $\{T(f) : \|f\| \le 1\}$ is $\sigma(F,F')$-bounded, so $\sup\{\|T(f)\|_F : \|f\| \le 1\} < \infty$, i.e., $T$ is bounded. Moreover, for each $y' \in F'$ we have

$$y'(T(f)) = y'(y_f) = \Psi_f(y') = \int_X f dm_{y'} \quad \text{for} \ \ f \in C_{rc}(X,E),$$

i.e., $T$ satisfies (iii).

Assume that $S : C_{rc}(X,E) \to F$ is another bounded linear operator such that for each $y' \in F'$,

$$y'(S(f)) = \int_X f dm_{y'} \quad \text{for} \ \ f \in C_{rc}(X,E).$$

Then $y'(S(f)) = y'(T(f))$ for all $f \in C_{rc}(X,E)$, so $S = T$.

Let $m_o(A)(x) = (T'' \circ \pi)(\mathbb{1}_A \otimes x)$ for all $A \in \mathcal{B}$, $x \in E$. Then by the first part of the proof, for each $y' \in F'$, we get

$$y'(T(f)) = \int_X f d(m_o)_{y'} \quad \text{for} \quad f \in C_{rc}(X, E).$$

Hence $(m_o)_{y'} = m_{y'} \in M(X, E')$. It follows that

$$m(A)(x) = m_o(A)(x) = (T'' \circ \pi)(\mathbb{1}_A \otimes x).$$

Thus the proof is complete. □

**Definition 4.1.** A measure $m : \mathcal{B} \to \mathcal{L}(E, F'')$ is said to be a *representing measure* if it satisfies conditions (i) and (ii) of Theorem 4.1.

## 5. Continuous operators on $C_{rc}(X, E)$ with strict topologies

First we distinguish two classes of operators on $C_{rc}(X, E)$.

**Definition 5.1.** A bounded linear operator $T : C_{rc}(X, E) \to F$ is said to be:

(i) *$\sigma$-additive*, if $\|T(u_n g)\|_F \to 0$ uniformly for $g \in B_{C_{rc}}$ whenever $(u_n)$ is a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$.

(ii) *$\tau$-additive*, if $\|T(u_\alpha g)\|_F \to 0$ uniformly for $g \in B_{C_{rc}}$ whenever $(u_\alpha)$ is a net in $C_b(X)$ such that $u_\alpha(t) \downarrow 0$ for $t \in X$.

We characterize $(\beta_z(X, E), \|\cdot\|_F)$-continuous linear operators $T : C_{rc}(X, E) \to F$ for $z = \sigma, \tau$.

**Theorem 5.1.** *Let $T : C_{rc}(X, E) \to F$ be a bounded linear operator, and $m : \mathcal{B} \to \mathcal{L}(E, F'')$ its representing measure. Then the following statements are equivalent*:

(i) *$T$ is $(\beta_\sigma(X, E), \|\cdot\|_F)$-continuous.*

(ii) *$T$ is $\sigma$-additive.*

(iii) *$\widetilde{m}(Z_n)$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathcal{Z}$.*

*Proof.* (i)$\Longrightarrow$(ii) Assume that $T$ is $(\beta_\sigma(X, E), \|\cdot\|_F)$-continuous. Let $(u_n)$ be a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$, and let $\varepsilon > 0$ be given. Then there exists a solid neighborhood $V$ of 0 for $\beta_\sigma(X, E)$ such that $\|T(f)\|_F \le \varepsilon$ for all $f \in V$ (see Proposition 2.7). Choose $n_\varepsilon \in \mathbb{N}$ such that $u_n \otimes x_0 \in V$ for $n \ge n_\varepsilon$, where $x_0 \in S_E$ (see Corollary 2.9). Hence $u_n g \in V$ for all $g \in B_{C_{rc}}$ and $n \ge n_\varepsilon$, and it follows that $\sup_{g \in B_{C_{rc}}} \|T(u_n g)\|_F \le \varepsilon$ for $n \ge n_\varepsilon$.

(ii)$\Longrightarrow$(iii) Assume that $T$ is $\sigma$-additive. Then for $y' \in F'$ we have

$$(y' \circ T)(f) = \int_X f dm_{y'} \quad \text{for} \quad f \in C_{rc}(X, E)$$

and by Corollary 3.3, $m_{y'} \in M_\sigma(X, E')$, and hence $|m_{y'}| \in M_\sigma(X)$. In view of Corollary 3.2 we have

$$|y' \circ T|(u) = \int_X u d|m_{y'}| \quad \text{for} \quad u \in C_b(X).$$

Let $(u_n)$ be a sequence in $C_b(X)$ such that $u_n(t) \downarrow 0$ for $t \in X$. We shall show that $\sup_{y' \in B_{F'}} |y' \circ T|(u_n) \to 0$. For each $y' \in F'$ there exists a sequence $(h_{y',n})$ in $C_{rc}(X, E)$ with $\widetilde{h}_{y',n} \le u_n$ and such that

$$|y' \circ T|(u_n) \le \Big| \int_X h_{y',n} dm_{y'} \Big| + \frac{1}{n}.$$

Let $g_{y',n}(t) = \frac{h_{y',n}(t)}{u_n(t)}$ for $t \in X$. Then $g_{y',n} \in B_{C_{rc}}$ for $n \in \mathbb{N}$. Hence for each $y' \in B_{F'}$, we get

$$\begin{aligned} |y' \circ T|(u_n) &\le \Big| \int_X h_{y',n} dm_{y'} \Big| + \frac{1}{n} = \Big| \int_X u_n g_{y',n} dm_{y'} \Big| + \frac{1}{n} \\ &\le \sup_{g \in B_{C_{rc}}} \Big| \int_X u_n g\, dm_{y'} \Big| + \frac{1}{n} = \sup_{g \in B_{C_{rc}}} |y'(T(u_n g)| + \frac{1}{n} \\ &\le \sup_{g \in B_{C_{rc}}} \|T(u_n g)\|_F + \frac{1}{n}. \end{aligned}$$

Hence $\sup_{y' \in B_{F'}} |y' \circ T|(u_n) \to 0$, as desired. By [21, Theorem 11.14] the family $\{|m_{y'}| : y' \in B_{F'}\}$ in $M_\sigma(X)$ is uniformly $\sigma$-additive, i.e.,

$$\widetilde{m}(Z_n) = \sup\{|m_{y'}|(Z_n) : y' \in B_{F'}\} \to 0 \quad \text{whenever } \; Z_n \downarrow \emptyset,\ (Z_n) \subset \mathcal{Z}.$$

(iii)$\Longrightarrow$(i) Assume that (iii) holds. Then $|m_{y'}| \in M_\sigma(X)$ for each $y' \in F'$. Note that for $A \in \mathcal{B}$, $x \in E$ we have $|m_{x,y'}(A)| \le |m_{y'}|(A) \cdot \|x\|_E$. It follows that $m_{x,y'} \in M_\sigma(X)$ for $x \in E$, i.e., $m_{y'} \in M_\sigma(X, E')$, and hence $y' \circ T \in (C_{rc}(X, E), \beta_\sigma(X, E))'$ (see (1.2)). In view of [21, Theorem 11.14] the family $\{|y' \circ T| : y' \in B_{F'}\}$ is $\beta_\sigma(X)$-equicontinuous, and hence by Proposition 3.5 the family $\{y' \circ T : y' \in B_{F'}\}$ is $\beta_\sigma(X, E)$-equicontinuous. This means that $T$ is $(\beta_\sigma(X, E), \|\cdot\|_F)$-continuous. □

Arguing as in the proof of Theorem 5.1 and using Proposition 3.6 we get:

**Theorem 5.2.** *Let $T : C_{rc}(X, E) \to F$ be a bounded linear operator, and $m : \mathcal{B} \to \mathcal{L}(E, F'')$ its representing measure. Then the following statements are equivalent*:

(i) *$T$ is $(\beta_\tau(X, E), \|\cdot\|_F)$-continuous.*
(ii) *$T$ is $\tau$-additive.*
(iii) *$\widetilde{m}(Z_n) \to 0$ whenever $Z_\alpha \downarrow \emptyset$, $(Z_\alpha) \subset \mathcal{Z}$.*

*Remark.* For weakly compact operators $T : C_{rc}(X, E) \to F$, the equivalences (i)$\Longleftrightarrow$(iii) in Theorems 5.1 and 5.2 were derived in a different way in [15, Theorems 4 and 5].

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