

## TOTAL IDENTITY-SUMMAND GRAPH OF A COMMUTATIVE SEMIRING WITH RESPECT TO A CO-IDEAL

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ABSTRACT. Let  $R$  be a semiring,  $I$  a strong co-ideal of  $R$  and  $S(I)$  the set of all elements of  $R$  which are not prime to  $I$ . In this paper we investigate some interesting properties of  $S(I)$  and introduce the total identity-summand graph of a semiring  $R$  with respect to a co-ideal  $I$ . It is the graph with all elements of  $R$  as vertices and for distinct  $x, y \in R$ , the vertices  $x$  and  $y$  are adjacent if and only if  $xy \in S(I)$ .

### 1. Introduction

The concept of a zero-divisor graph of a commutative ring was introduced by Beck in [5]. In his work all elements of the ring were vertices of the graph. In [3], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors of a ring. In [2], the authors defined the total graph of a ring  $R$  to be the (undirected) graph  $T(\Gamma(R))$  with all elements of  $R$  as vertices, and two distinct vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in Z(R)$ , where  $Z(R)$  is the zero divisors of  $R$ . Also they studied the subgraph  $T_0(\Gamma(R))$  of  $T(\Gamma(R))$  with vertices  $R \setminus \{0\}$ .

Recently, the study of graphs of rings was extended to include semirings as in [7, 8, 9]. Semirings have proven to be useful in theoretical computer science, in particular for studying automata and formal languages. Moreover, co-ideals of semirings play an important role in the structure theory and useful for many purposes. They have properties that are more suited than the properties of ideals, to the study of the graphs of semirings, such as Proposition 2.1(ii) and Theorem 3.7. In [11, 12], the authors introduced the identity-summand graph and identity-summand graph with respect to co-ideal  $I$  of a semiring  $R$ . The identity-summand graph with respect to co-ideal  $I$  denoted by  $\Gamma_I(R)$  is a graph

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with vertices as elements  $S_I(R) = \{x \in R \setminus I : x + y \in I \text{ for some } y \in R \setminus I\}$ , where two distinct vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in I$  [12].

We say that  $r \in R$  is an identity-summand element of  $R$ , if there exists  $1 \neq a \in R$  such that  $r + a = 1$ . The notation  $S(R)$  is used to refer to the set of elements of  $R$  that are identity-summand (we use  $S^*(R)$  to denote the set of non-identity identity-summands of  $R$ ). A semiring  $R$  is called an  $I$ -semiring if  $r + 1 = 1$  for all  $r \in R$  [10]. In [13], the authors introduced the total graph of a commutative semiring with respect to identity-summand elements. Let  $R$  be an  $I$ -semiring. The total graph of  $R$ , denoted by  $T(\Gamma(R))$ , is the graph with all elements of  $R$  as vertices, and for distinct  $x, y \in R$ , the vertices  $x$  and  $y$  are adjacent if and only if  $xy \in S(R)$ .  $S(\Gamma(R))$  (resp.  $S^*(\Gamma(R))$ ) denotes the subgraph of  $T(\Gamma(R))$  with vertex set  $S(R)$  (resp.  $S^*(R)$ ). In this paper the authors extend the results obtained in [13] for a subtractive co-ideal  $I$  of  $R$  and it is shown that these results do not hold for a non-subtractive co-ideal.

Let  $I$  be a co-ideal of  $R$ , we say that  $a \in R$  is prime to  $I$ , if  $r + a \in I$  (where  $r \in R$ ) implies that  $r \in I$  and we define the set of elements of  $R$  which are not prime to  $I$  by  $S(I)$ . The set  $S(I)$  is not necessarily a co-ideal of  $R$ . In this paper we prove some important properties of  $S(I)$  which are so useful in solving the problems of the total graph with respect to a co-ideal of  $R$ .

In Section 3, we show that if  $S(I)$  is finite, then  $S(I)$  is not a co-ideal of  $R$ . In Lemma 3.1(ii), we show that  $S(I)$  is a union of subtractive prime co-ideals of  $R$  and in Theorem 3.7, we show that  $S(I)$  is a union of minimal prime co-ideals of  $I$ . Also, it is shown that if  $\min(I)$  is finite, then the set of subtractive prime co-ideals which construct  $S(I)$  in Lemma 3.1 is equal to the set of minimal prime co-ideals of  $R$ . But if  $\min(I)$  is infinite, this equality is not true, as the Example 3.12 shows.

In Section 4, we introduce and study the total graph of  $R$  with respect to co-ideal  $I$ . At the beginning of the section, it is shown that  $\bar{S}(\Gamma_I(R))$  (the subgraph of  $T(\Gamma_I(R))$  with vertex set  $R \setminus S(I)$ ) is totally disconnected, which implies  $T(\Gamma_I(R))$  is always disconnected. So we investigate only the subgraph  $S(\Gamma_I(R))$  of  $T(\Gamma_I(R))$  (the induced subgraph of  $T(\Gamma_I(R))$  with vertex set  $S(I)$ ). We show that  $S(\Gamma_I(R))$  is connected,  $\text{diam}(S(\Gamma_I(R))) \in \{1, 2\}$  and  $\text{gr}(S(\Gamma_I(R))) \in \{3, \infty\}$ . Moreover, we investigate  $\kappa(S(\Gamma_I(R)))$  and the cut points of  $S(\Gamma_I(R))$  and we consider when  $S(\Gamma_I(R))$  has a cut point. At the end of the section, it is proved that if  $S(\Gamma_I(R)) \cong S(\Gamma_J(T))$ , then  $\Gamma_I(R) \cong \Gamma_J(T)$ , where  $R$  and  $S$  are semirings and  $I, J$  are co-ideals of  $R$  and  $S$ , respectively.

In Section 5, we describe the relation between  $S(\Gamma_I(R))$  and  $S(\Gamma(R/I))$  for a  $Q$ -strong co-ideal of  $R$ . It is shown that  $\text{diam}(S(\Gamma_I(R))) = 1$  if and only if  $\text{diam}(S(\Gamma(R/I))) = 1$  and  $\text{diam}(S(\Gamma_I(R))) = 2$  if and only if  $\text{diam}(S(\Gamma(R/I))) = 2$ .

## 2. Preliminaries

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph  $\Gamma$  by  $E(\Gamma)$  and  $V(\Gamma)$  we denote the set of all edges and vertices, respectively. A graph  $G$  is called connected if for any vertices  $x$  and  $y$  of  $G$  there is a path between  $x$  and  $y$ . Otherwise,  $G$  is called disconnected. The distance between two distinct vertices  $a$  and  $b$ , denoted by  $d(a, b)$ , is the length of the shortest path connecting them (if such a path does not exist, then  $d(a, b) = \infty$ , also  $d(a, a) = 0$ ). The diameter of graph  $\Gamma$ , denoted by  $\text{diam}(\Gamma)$ , is equal to  $\sup\{d(a, b) : a, b \in V(\Gamma)\}$ . A graph is complete if it is connected with diameter less than or equal to one. A clique of a graph is its complete subgraph and the number of vertices in the largest clique of graph  $G$ , denoted by  $w(G)$ , is called the clique number of  $G$ .

A commutative semiring  $R$  is defined as an algebraic system  $(R, +, \cdot)$  such that  $(R, +)$  and  $(R, \cdot)$  are commutative semigroups, connected by  $a(b + c) = ab + ac$  for all  $a, b, c \in R$ , and there exist  $0, 1 \in R$  such that  $r + 0 = r$  and  $r0 = 0r = 0$  and  $r1 = 1r = r$  for each  $r \in R$ . In this paper all semirings considered will be assumed to be commutative semirings with a non-zero identity.

**Definition.** Let  $R$  be a semiring.

(1) A non-empty subset  $I$  of  $R$  is called a *co-ideal*, if it is closed under multiplication and satisfies the condition  $r + a \in I$  for all  $a \in I$  and  $r \in R$  (so  $0 \in I$  if and only if  $I = R$ ). A co-ideal  $I$  of  $R$  is called strong co-ideal provided that  $1 \in I$  [10, 15, 17].

(2) If  $I$  is a co-ideal of  $R$ , then the *co-rad*( $I$ ) of  $I$ , is the set of all  $x \in R$  for which  $nx \in I$  for some positive integer  $n$ . This is a co-ideal of  $R$  contains  $I$  [10].

(3) A co-ideal  $I$  of  $R$  is called *subtractive* if  $x, xy \in I$ , then  $y \in I$  (so every subtractive co-ideal is a strong co-ideal) [10].

(4) A proper co-ideal  $P$  of  $R$  is called *prime* if  $x + y \in P$ , then  $x \in P$  or  $y \in P$ . The set of all prime (resp. minimal prime) co-ideals of  $R$  is denoted by  $\text{co-Spec}(R)$  (resp.  $\text{min}(R)$ ) [10].

(5) A semiring  $R$  is called *co-semidomain*, if  $a + b = 1$  ( $a, b \in R$ ), then either  $a = 1$  or  $b = 1$  [10].

(6) We say that a subset  $T \subseteq R$  is additively closed if  $0 \in T$  and  $a + b \in T$  for all  $a, b \in T$ .

(7) An ideal  $I$  of  $R$  is called *k-ideal* if  $x, x + y \in I$ , then  $y \in I$  for all  $x, y \in R$  [14].

A strong co-ideal  $I$  of a semiring  $R$  is called a *partitioning strong co-ideal* (=  $Q$ -strong co-ideal) if there exists a subset  $Q$  of  $R$  such that  $R = \cup\{qI : q \in Q\}$ , where  $qI = \{qt : t \in I\}$  and if  $q_1, q_2 \in Q$ , then  $(q_1I) \cap (q_2I) \neq \emptyset$  if and only if  $q_1 = q_2$  [10]. Let  $I$  be a  $Q$ -strong co-ideal of a semiring  $R$  and let  $R/I = \{qI : q \in Q\}$ . Then  $R/I$  forms a semiring under the binary operations  $\oplus$  and  $\odot$  defined as follows:  $(q_1I) \oplus (q_2I) = q_3I$ , where  $q_3$  is the unique element

in  $Q$  such that  $(q_1I + q_2I) \subseteq q_3I$ , and  $(q_1I) \odot (q_2I) = q_3I$ , where  $q_3$  is the unique element in  $Q$  such that  $(q_1q_2)I \subseteq q_3I$  [10]. If  $q_e$  is the unique element in  $Q$  such that  $1 \in q_eI$ , then  $q_eI = I$  is the identity of  $R/I$ . Note that every  $Q$ -strong co-ideal is subtractive [10]. Throughout this paper we shall assume unless otherwise stated, that  $q_eI$  is the identity element of  $R/I$ .

**Proposition 2.1.** (i) [11, Proposition 2.5] *Let  $R$  be a commutative  $I$ -semiring. Then the following statements hold:*

- (1) *If  $a + a = 1$  for some  $a \in R$ , then  $a = 1$ ;*
- (2) *If  $J$  is a co-ideal, then  $J$  is a strong co-ideal of  $R$ . Moreover, if  $xy \in J$ , then  $x, y \in J$  for every  $x, y \in R$ . In particular,  $J$  is subtractive;*
- (3) *The set  $(1 : x) = \{r \in R : r + x = 1\}$  is a strong co-ideal of  $R$  for every  $x \in S(R)$ .*

(ii) [12, Proposition 2.1] *Let  $I$  be a subtractive co-ideal of a semiring  $R$ . Then the following hold:*

- (1) *If  $xy \in I$ , then  $x, y \in I$  for all  $x, y \in R$ ;*
- (2)  *$I = \text{co-rad}(I)$ ;*
- (3)  *$(I : a) = \{r \in R : r + a \in I\}$  is a subtractive co-ideal of  $R$  for all  $a \in R$ ;*
- (4) *If  $I$  is a  $Q$ -strong co-ideal of  $R$  and  $q_eI$  is the identity element in  $R/I$ , then  $q_eI \oplus qI = q_eI$  and  $qI \oplus qI = qI$  for all  $qI \in R/I$ .*

(iii) [12, Theorem 4.6] *Let  $I$  be a subtractive co-ideal of a semiring  $R$ .*

- (1) *If  $\{P_\alpha\}_{\alpha \in \Lambda}$  is the set of all prime strong co-ideals of  $R$  containing  $I$ , then  $I = \bigcap_{\alpha \in \Lambda} P_\alpha$ .*
- (2) *If  $P_1, \dots, P_n$  are the only distinct minimal prime strong co-ideals of  $R$  containing  $I$ , then  $\bigcap_{i=1}^n P_i = I$  and  $I \neq \bigcap_{1 \leq i \leq n, i \neq j} P_i$  for each  $1 \leq j \leq n$ .*

**Proposition 2.2.** (i) [12, Theorem 2.8] *If  $I$  is a subtractive co-ideal of  $R$  with  $|S(\Gamma_I(R))| \geq 3$ , then  $\Gamma_I(R)$  is not a complete graph. In particular,  $\text{diam}(\Gamma_I(R)) = 2$  or  $3$ .*

(ii) [13, Theorem 4.4] *Let  $R$  be an  $I$ -semiring which is not co-semidomain. Then  $S^*(\Gamma(R))$  is connected if and only if  $|\min(R)| \neq 2$ . Moreover if  $S^*(\Gamma(R))$  is connected, then  $\text{diam}(S^*(\Gamma(R))) \in \{1, 2\}$ .*

(iii) [12, Theorem 4.7] *If  $I$  is a subtractive co-ideal of  $R$  which is not prime, then  $w(\Gamma_I(R)) = |\min(I)|$ .*

### 3. Some properties of $S(I)$

In this section we introduce  $S(I)$ , the set of elements of  $R$  which are not prime to  $I$ . We give an interesting lemma which will be useful in next sections. Also, we investigate one of the most important properties of  $S(I)$ , with respect to minimal prime co-ideals of  $I$ . We begin with the following definition.

**Definition.** Let  $R$  be a semiring and  $I$  be a strong co-ideal of  $R$ . We define  $S(I)$  by  $S(I) = \{r \in R : \exists x \in R \setminus I \text{ such that } r + x \in I\}$  and  $\bar{S} = R \setminus S(I)$ .

Compare the following lemma with [13, Lemma 3.6 and Lemma 2.4].

**Lemma 3.1.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . The following statements hold:*

- (i) *If  $I \neq S(I)$ , then  $|S(I)| \neq 1, 2$ . Moreover if  $S(I)$  is finite, then  $S(I)$  is not a co-ideal of  $R$ .*
- (ii)  *$S(I)$  is a union of subtractive prime co-ideals of  $R$  containing  $I$ . Moreover, if  $xy \in S(I)$ , then  $x, y \in S(I)$  for each  $x, y \in R$ .*
- (iii) *If  $S(I)$  is a co-ideal of  $R$ , then  $S(I)$  is a prime co-ideal of  $R$ .*
- (iv)  *$\bar{S}(I)$  is an ideal of  $R$ .*
- (v)  *$S(I)$  is a co-ideal of  $R$  if and only if  $\bar{S}(I)$  is a prime ideal of  $R$ .*
- (vi)  *$Q_i = R \setminus P_i$  is a prime  $k$ -ideal of  $R$  for each  $P_i \in \min(I)$ .*
- (vii)  *$\bar{S}(I) = \bigcap_{i \in \Lambda} Q_i$ .*

*Proof.* (ii) Set

$$\Sigma = \{J : J \text{ is subtractive and each element of } J \text{ is not prime to } I \text{ and } I \subseteq J\}.$$

Since  $I \in \Sigma$ ,  $\Sigma \neq \emptyset$ . By Zorn's lemma,  $\Sigma$  has a maximal element. Let  $P$  be a maximal element of  $\Sigma$ . We show that  $P$  is a prime co-ideal of  $R$ . Let  $x + y \in P$  and  $x, y \notin P$ . Since  $P \subset (P : x)$  and  $P$  is maximal in  $\Sigma$ ,  $(P : x) \notin \Sigma$ . So there exists  $z \in (P : x)$  such that  $z$  is prime to  $I$ . We show that  $(P : z) \in \Sigma$ . Let  $w \in (P : z) \setminus I$ . Since  $z$  is prime to  $I$ ,  $w + z \notin I$ . Because  $w + z \in P$ ,  $w + z + u \in I$  for some  $u \notin I$ , which implies  $w + u \in I$  because  $z$  is prime to  $I$ . Thus  $w$  is not prime to  $I$  because  $u \notin I$ . Hence  $(P : z) \in \Sigma$ , a contradiction, because  $P \subset (P : z)$  and  $P$  is maximal in  $\Sigma$ . Thus  $P$  is a prime co-ideal of  $R$ . Since  $S(I)$  is a union of maximal elements of  $\Sigma$ ,  $S(I)$  is a union of prime co-ideals of  $R$ . Now, let  $xy \in S(I)$ . So  $xy \in P_i$  for some  $P_i \in \Sigma$ . By Proposition 2.1(ii),  $x, y \in P_i \subseteq S(I)$ .

Other parts are similar to the proof of [13, Lemma 3.6 and Lemma 2.4].  $\square$

In the next example we show that the condition “ $I$  is subtractive” is not superfluous in Lemma 3.1(ii).

**Example 3.2.** Assume that  $R = \{0, 1, 2, 3, 4, 5\}$ . Define

$$a + b = \begin{cases} 5 & \text{if } a \neq 0, b \neq 0, a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0 \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 3 & \text{if } a = b = 2, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then  $(R, +, *)$  is easily checked to be a commutative semiring. An inspection will show that  $I = \{1, 4, 5\}$  is a co-ideal of  $R$  which is not subtractive (because

$2 * 4 = 5 \in I$  and  $2 \notin I$ ). It can be easily seen that  $S(I) = \{1, 2, 3, 4, 5\}$ . As we see  $S(I)$  is finite but  $S(I)$  is a co-ideal of  $R$ .

We next give several lemmas in order to gain Theorem 3.7.

**Lemma 3.3.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . If  $T$  is an additively closed subset of  $R$  such that  $T \cap I = \emptyset$ , then  $\sum = \{S : T \subseteq S, S \text{ is an additively closed subset of } R \text{ and } S \cap I = \emptyset\}$  has a maximal element.*

*Proof.* It is straightforward by Zorn's lemma.  $\square$

**Lemma 3.4.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . If  $T$  is an additively closed subset of  $R$  such that  $I \cap T = \emptyset$ , then there exists a subtractive prime co-ideal  $Q$  containing  $I$  which is maximal with respect to the property of not meeting  $T$ .*

*Proof.* Let  $\sum = \{J : I \subseteq J, J \text{ is a subtractive co-ideal of } R \text{ and } J \cap T = \emptyset\}$ . Since  $I \in \sum$ ,  $\sum \neq \emptyset$ . By Zorn's lemma  $\sum$  has a maximal element  $Q$ . We show that  $Q$  is a prime co-ideal of  $R$ . Let  $a + b \in Q$  and  $a \notin Q$ . So  $Q \subset (Q : b)$ . Since  $Q$  is maximal in  $\sum$ , there exists  $s \in T$  such that  $s \in (Q : b) \cap T$ . We show that  $Q = (Q : as)$ . If not, then  $Q \subset (Q : as)$  implies that  $(Q : as) \cap T \neq \emptyset$ . Let  $r \in (Q : as) \cap T$ . Then  $as \in (Q : r)$ . By Proposition 2.1(ii),  $s \in (Q : r)$ . So  $s + r \in Q \cap T$ , a contradiction. Thus  $Q = (Q : as)$ . We claim that  $(Q : as) = (Q : a) \cap (Q : s)$ . Since  $Q = (Q : as)$ ,  $(Q : as) \subseteq (Q : a) \cap (Q : s)$ . Now, let  $r \in (Q : a) \cap (Q : s)$ . Thus  $a, s \in (Q : r)$ . Hence  $as \in (Q : r)$  by Proposition 2.1(ii), which gives  $r \in (Q : as)$ , as needed. Thus  $(Q : as) = (Q : a) \cap (Q : s)$ . As  $b \in (Q : a) \cap (Q : s)$ ,  $b \in Q$ . Therefore  $Q$  is a prime co-ideal of  $R$ .  $\square$

Let  $S$  be a subset of  $R$ . We denote the set of elements of  $R \setminus S$  by  $S^c$ .

**Lemma 3.5.** *Let  $R$  be a semiring,  $I$  be a subtractive co-ideal of  $R$  and  $P$  be a prime co-ideal of  $R$ . Then  $P \in \min(I)$  if and only if  $P^c$  is an additively closed subset of  $R$  which is maximal with respect to the property of not meeting  $I$ . Moreover, every minimal prime co-ideal of  $I$  is subtractive.*

*Proof.* Let  $P$  be a prime co-ideal of  $R$  which  $P^c$  is maximal with respect to the property of not meeting  $I$ . We show that  $P \in \min(I)$ . Let  $Q \subseteq P$ , where  $Q$  is a prime co-ideal of  $R$  containing  $I$ . The definition of a prime co-ideal implies that  $Q^c$  is an additively closed subset of  $R$  and  $P^c \subseteq Q^c$ . Since  $P^c$  is maximal with respect to the property of not meeting  $I$ ,  $I \cap Q^c \neq \emptyset$ . Let  $x \in I \cap Q^c$ , then  $x \in I$  and  $x \notin Q$ , a contradiction. So  $P \in \min(I)$ .

Conversely, let  $P \in \min(I)$ , so  $I \cap P^c = \emptyset$ . We claim  $P^c$  is maximal with respect to the property of not meeting  $I$ : by Lemma 3.3, there exists a maximal additively closed subset  $M$  of  $R$  with respect to the property of not meeting  $I$  such that  $P^c \subseteq M$ . By the proof of Lemma 3.4, there exists a subtractive prime co-ideal  $Q$  containing  $I$ , which is maximal with respect to the property of not meeting  $M$ . Hence  $Q \cap M = \emptyset$  and  $Q \subseteq M^c$ , so  $Q \subseteq M^c \subseteq P$ . Since  $Q$  is prime and  $P \in \min(I)$ ,  $Q = M^c = P$ . Hence  $P^c = M$  is maximal with

respect to the property of not meeting  $I$ . For the moreover statement, since  $Q$  is subtractive,  $P$  is subtractive too.  $\square$

**Proposition 3.6.** *Let  $R$  be a semiring,  $I$  be a subtractive co-ideal of  $R$  and  $P$  a prime co-ideal of  $R$ . Then  $P \in \min(I)$  if and only if for each  $x \in P$  there exists  $y \notin P$  and a positive integer  $i$  such that  $y + ix \in I$ .*

*Proof.* Assume the condition holds, we show  $P \in \min(I)$ . Let  $Q$  be a prime co-ideal of  $R$  containing  $I$ , which  $Q \subseteq P$ . Choose  $x \in P \setminus Q$ . By assumption, there exists  $y \notin P$  and a positive integer  $i$  such that  $y + ix \in I$ . Since  $I \subseteq Q$  and  $Q$  is prime,  $ix \in Q$ , which implies  $x \in Q$ , a contradiction. Thus  $P \in \min(I)$ .

Conversely, let  $P \in \min(I)$ . Let  $x \in P$  and  $T = \{y + ix : y \in P^c, i \in \mathbb{N} \cup \{0\}\}$  (Note that  $0x = 0$ ). Then  $T$  is an additively closed subset of  $R$  which properly contains  $P^c$ . By Lemma 3.5,  $P^c$  is maximal with respect to property not meeting  $I$ . Thus  $I \cap T \neq \emptyset$ . Hence there exists a positive integer  $i$  and  $y \notin P$  such that  $y + ix \in I$ .  $\square$

Now we are in a position to prove our main theorem in this section.

**Theorem 3.7.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then  $S(I) = \cup_{P_\alpha \in \min(I)} P_\alpha$ .*

*Proof.* Let  $P_\alpha$  be a minimal prime co-ideal of  $I$  and  $x \in P_\alpha \setminus I$ . By Proposition 3.6, there exists  $y \notin P_\alpha$  such that  $y + ix \in I$  for some integer  $i \neq 0$ . Since  $I$  is subtractive and  $ix = (1 + 1 + \dots + 1)x$ ,  $x \in (I : y)$  by Proposition 2.1(ii). So  $x + y \in I$ , which gives  $x \in S(I)$ . Thus  $\cup_{P_\alpha \in \min(I)} P_\alpha \subseteq S(I)$ . Now, let  $x \in S(I) \setminus I$ , so there exists  $y \in R \setminus I$  such that  $x + y \in I$ . Since  $y \notin I$ , there exists  $P_\alpha \in \min(I)$  such that  $y \notin P_\alpha$ , because  $\cap_{P_\alpha \in \min(I)} P_\alpha = I$ , by Proposition 2.1(iii). Since  $x + y \in I \subseteq P_\alpha$  and  $y \notin P_\alpha$ ,  $x \in P_\alpha$ . Thus  $S(I) = \cup_{P_\alpha \in \min(I)} P_\alpha$ .  $\square$

In the next example we show that the condition “ $I$  is subtractive” can not be omitted in Theorem 3.7.

**Example 3.8.** Let  $R$  be the semiring of Example 3.2 and  $I = \{1, 4, 5\}$ . So  $P = \{1, 3, 4, 5\}$  is the only minimal prime co-ideal of  $I$ . It can be easily seen that  $S(I) = \{1, 2, 3, 4, 5\}$ . As we see,  $I \neq \cap_{P \in \min(I)} P$  and  $S(I) \neq \cup_{P \in \min(I)} P$ .

In Lemma 3.1 and Theorem 3.7, we see two set of prime co-ideals for  $S(I)$ . In the following we answer to this question that are two set of prime co-ideals for  $S(I)$  equal?

The following lemma is useful in the proof of next corollary.

**Lemma 3.9.** *Let  $P_1, P_2, \dots, P_n$  be subtractive prime co-ideals of a semiring  $R$ . If  $I$  is a strong co-ideal of  $R$  such that  $I \subseteq \cup_{i=1}^n P_i$ , then  $I \subseteq P_r$  for some  $1 \leq r \leq n$ .*

*Remark 3.10.* Let  $P$  and  $I$  be strong co-ideals of a semiring  $R$  with  $P$  prime and  $I \subseteq P$ . Then the non-empty set  $\Delta = \{Q \in \text{Spec}(R) : I \subseteq Q \subseteq P\}$  has

a minimal element  $P_1$  with respect to inclusion (by partially ordering  $\Delta$  by reverse inclusion and using Zorn's Lemma), so  $P_1$  is an element of  $\min(I)$ , the set of minimal prime strong co-ideals of  $R$  containing  $I$ . Thus if  $P$  is a prime strong co-ideal of the commutative semiring  $R$  and  $P$  contains the strong co-ideal  $I$  of  $R$ , then there exists a minimal prime strong co-ideal  $Q$  of  $R$  with  $I \subseteq Q \subseteq P$ .

**Corollary 3.11.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . If  $\min(I)$  is finite, then two sets of prime co-ideals which are defined in Theorem 3.7 and Lemma 3.1 are equal. So  $S(I) = \cup P_i$ , where  $P_i$ 's are subtractive minimal prime co-ideals of  $R$  and are maximal in*

$$\sum = \{J : \text{each element of } J \text{ is not prime to } I \text{ and } I \subseteq J\}.$$

*Proof.* Let  $P_i \in \min(I)$ , we show that  $P_i$  is maximal in  $\sum$ . If  $P_i$  is not maximal in  $\sum$ , there exists a maximal element  $Q$  in  $\sum$  such that  $P_i \subset Q$ . Because  $Q \subseteq S(I)$  and  $S(I) = \cup_{i=1}^n P_i$  by Theorem 3.7,  $Q \subseteq P_j$  for some minimal prime co-ideal  $P_j$  of  $I$  by Lemma 3.9. So  $P_i = Q = P_j$ , a contradiction. So each minimal prime co-ideal of  $I$  is maximal in  $\sum$ . Conversely, let  $Q$  be maximal in  $\sum$ . If  $Q \notin \min(I)$ , there exists  $P_i \in \min(I)$  such that  $P_i \subseteq Q$  by Remark 3.10. But we showed that each minimal prime co-ideal of  $I$  is maximal in  $\sum$ , hence  $P_i = Q$ .  $\square$

The following example shows that the condition  $\min(I)$  is finite can not be omitted in Corollary 3.11.

**Example 3.12.** Let  $R = (\mathbb{Z}^+, \text{gcd}, \text{lcm})$  (take  $\text{gcd}(0, 0) = 0$  and  $\text{lcm}(0, 0) = 0$ ). It is clear that  $I = \{1\}$  and  $S(I) = \mathbb{Z}^+ \setminus \{0\}$  are co-ideals of  $R$ . We show that  $\min(I)$  is infinite. Suppose, on the contrary,  $\min(I)$  is finite. Hence  $S(I) \subseteq P_i$  for some  $P_i \in \min(I)$  by Lemma 3.9, which implies  $S(I) = P_i$ . Thus  $P_i$  is the only minimal prime co-ideal of  $R$  by Theorem 3.7. So  $S(I) = P_i = I$  by Proposition 2.1(iii), a contradiction. Therefore  $\min(I)$  is infinite. It is clear that each co-ideal of  $R$  is contained in the maximal co-ideal  $S(I)$  of  $R$ . Thus  $S(I)$  is the only maximal element of  $\sum$ , which is not a minimal prime co-ideals of  $I$ .

#### 4. Total graph with respect to a co-ideal

In this section we apply the results we have obtained about  $S(I)$  in order to study the total graph with respect to co-ideal  $I$ . We begin with the key definition of this paper.

**Definition.** Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . The total graph of  $R$  with respect to co-ideal  $I$  of  $R$ , denoted by  $T(\Gamma_I(R))$ , is the graph with all elements of  $R$  as vertices, and for distinct  $x, y \in R$ , the vertices  $x$  and  $y$  are adjacent if and only if  $xy \in S(I)$ .  $S(\Gamma_I(R))$  (resp.  $\bar{S}(\Gamma_I(R))$ ) denotes the subgraph of  $T(\Gamma_I(R))$  with vertex set  $S(I)$  (resp.  $\bar{S}(I)$ ).



**Lemma 4.1.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then the following statements are equivalent:*

- (i)  $T(\Gamma_I(R))$  is an empty graph;
- (ii)  $S(I) = \{1\}$ ;
- (iii)  $I = \{1\}$  and  $R$  is an  $I$ -semiring and a co-semidomain.

*Proof.* (i) $\Rightarrow$ (ii) Let  $1 \neq x \in S(I)$ . Then  $x$  and  $1$  are adjacent in  $T(\Gamma_I(R))$ , a contradiction.

(ii) $\Rightarrow$ (iii) Since  $I \subseteq S(I)$ ,  $I = \{1\}$ . Since  $\{1\}$  is a co-ideal of  $R$ ,  $R$  is an  $I$ -semiring. Moreover,  $\{1\} = S(I) = S(\{1\}) = S(R)$ . Hence  $R$  is a co-semidomain.

(iii) $\Rightarrow$ (i) Let  $x$  and  $y$  be adjacent in  $T(\Gamma_I(R))$ . So  $xy \in S(R) = S(I) = \{1\}$ . Hence  $x = x(1 + y) = x + xy = x + 1 = 1$ . By the similar way  $y = 1$ . Thus  $T(\Gamma_I(R))$  is an empty graph.  $\square$

A coclique in a graph  $G$  is a set of pairwise nonadjacent vertices.

**Proposition 4.2.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then*

- (i) Each  $x \in \bar{S}(\Gamma_I(R))$  and  $y \in R$  are not adjacent;
- (ii)  $\bar{S}(\Gamma_I(R))$  is a coclique in  $T(\Gamma_I(R))$ ;
- (iii)  $T(\Gamma_I(R))$  is always disconnected.

*Proof.* The proof is similar to [13, Theorem 3.7].  $\square$

As we see in Proposition 4.2,  $T(\Gamma_I(R))$  is always disconnected. In Theorem 4.4, we show that the subgraph  $S(\Gamma_I(R))$  of  $T(\Gamma_I(R))$  with vertex set  $S(I)$  is connected. So we study only the subgraph  $S(\Gamma_I(R))$  of  $T(\Gamma_I(R))$ .

**Lemma 4.3.** *Let  $R$  be a semiring and  $I$  be a strong co-ideal of  $R$ . The following statements are equivalent:*

- (i)  $S(I)$  is a strong co-ideal;
- (ii)  $S(\Gamma_I(R))$  is a complete graph;
- (iii)  $\text{diam}(S(\Gamma_I(R))) = 1$ .

*Proof.* (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i) We show that  $S(I)$  is a co-ideal of  $R$ . Let  $x \in S(I)$  and  $s \in R$ . So  $x + r \in I$  for some  $r \in R \setminus I$ . Since  $I$  is a co-ideal of  $R$ ,  $x + s + r \in I$ , which gives  $x + s \in S(I)$ . Now, let  $x, y \in S(I)$ . Since  $\text{diam}(S(\Gamma_I(R))) = 1$ ,  $xy \in S(I)$ , which implies  $S(I)$  is a co-ideal of  $R$ .  $\square$

**Theorem 4.4.** *Let  $R$  be a semiring and  $I$  be a strong co-ideal of  $R$ . Then*

- (i)  $S(\Gamma_I(R))$  is a connected graph and  $\text{diam}(S(\Gamma_I(R))) \in \{1, 2\}$ . Moreover,  $\text{diam}(S(\Gamma_I(R))) = 2$  if and only if  $S(I)$  is not a co-ideal;
- (ii)  $\text{gr}(S(\Gamma_I(R))) \in \{3, \infty\}$ .

*Proof.* The proof is similar to [13, Theorems 3.3 and 3.4]. For the moreover statement in (i) use Lemma 4.3.  $\square$

**Theorem 4.5.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$  such that  $I \neq S(I)$ . Then*

- (i)  $|S(I)| = 3$  if and only if  $\text{gr}(S(\Gamma_I(R))) = \infty$ ;
- (ii)  $|S(I)| \geq 4$  if and only if  $\text{gr}(S(\Gamma_I(R))) = 3$ .

*Proof.* (i) Let  $\text{gr}(S(\Gamma_I(R))) = \infty$ . By Lemma 3.1,  $|S(I)| \neq 1, 2$ . Suppose, on the contrary,  $|S(I)| \geq 4$ . Since the induced subgraph of  $S(\Gamma_I(R))$  with vertex set  $I$  is a complete subgraph of  $S(\Gamma_I(R))$  and  $\text{gr}(S(\Gamma_I(R))) = \infty$ , we get  $|I| = 1$  or 2. We consider two cases:

Case 1: Let  $|I| = 1$ . Since  $|S(I)| \geq 4$  and  $S(I) = I \cup S_I(R)$ , we have  $|S_I(R)| \geq 3$ . So  $\text{diam}(\Gamma_I(R)) = 2$  or 3 by Proposition 2.2. Hence there exist  $x, y \in R \setminus I$  such that  $d(x, y) = 2$ . So there exists  $z \in R \setminus I$  such that  $x - z - y$  is a path in  $\Gamma_I(R)$ . Thus  $x, y \in (I : z)$ , which gives  $x, y, xy \in (I : z) \subseteq S(I)$ , because  $(I : z)$  is a co-ideal of  $R$  by Proposition 2.1(ii). So  $1 - x - y - 1$  is a cycle in  $S(\Gamma_I(R))$  and  $\text{gr}(S(\Gamma_I(R))) = 3$ , a contradiction.

Case 2: Let  $|I| = 2$ . Then  $|S_I(R)| \geq 2$ . If  $|S_I(R)| \geq 3$ , then by the similar argument as in case 1,  $\text{gr}(S(\Gamma_I(R))) = 3$  a contradiction. Hence we assume  $|S_I(R)| = 2$ . Let  $1 \neq a \in I$  and  $b, c \in S_I(R)$ . Since  $b$  and  $c$  are the only elements of  $S_I(R)$ ,  $b + c \in I$ . So  $a, b \in (I : c)$ , which gives  $ab \in (I : c) \subseteq S(I)$ . So  $1 - a - b - 1$  is a path in  $S(\Gamma_I(R))$ , which implies  $\text{gr}(S(\Gamma_I(R))) = 3$ , a contradiction.

Therefore  $|S(I)| = 3$ .

Conversely, let  $S(I) = \{1, a, b\}$ . Since  $S(I)$  is not a co-ideal of  $R$  by Lemma 3.1,  $ab \notin S(I)$ . Thus  $a - 1 - b$  is the only path in  $S(\Gamma_I(R))$ . So  $\text{gr}(S(\Gamma_I(R))) = \infty$ .

- (ii) It is clear by (i) and Theorem 4.4 and Lemma 3.1. □

The following example shows that the condition “ $I$  is subtractive” in Theorem 4.5 is not superfluous.

**Example 4.6.** Let  $R = (\{0, 1, 2, 3\}, +, \times)$ , where

$$a + b = \begin{cases} 3 & \text{if } a, b \neq 0, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0 \end{cases}$$

and  $1 \times 1 = 1, 2 \times 1 = 1 \times 2 = 2, 3 \times 1 = 1 \times 3 = 3, 2 \times 2 = 1, 2 \times 3 = 3 \times 2 = 3, 3 \times 3 = 3$ , moreover  $r \times 0 = 0 \times r = 0$  for all  $r \in R$ . It can be easily seen that  $I = \{1, 3\}$  is a co-ideal of  $R$ , which is not subtractive. As we see,  $S(I) = \{1, 2, 3\}$  and  $\text{gr}(S(\Gamma_I(R))) = 3$ .

**Proposition 4.7.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . If  $a \in I$ , then  $a$  is adjacent to every vertex of  $S(\Gamma_I(R))$ . Moreover, the converse is true if  $\min(I)$  is finite.*

*Proof.* Let  $a \in I$  and  $r \in S(I)$ . By Theorem 3.7,  $r \in P_i$  for some  $P_i \in \min(I)$ . Hence  $ar \in P_i \subseteq S(I)$ . Thus  $a$  is adjacent to every vertex of  $S(\Gamma_I(R))$ . Conversely, let  $\min(I)$  be finite and  $a$  be adjacent to every vertex of  $S(\Gamma_I(R))$ .

To prove our claim, we show  $a \in P_i$  for each  $P_i \in \min(I)$ . By Lemma 3.9, for each  $P_i \in \min(I)$ , there exists  $x_i \in P_i$  such that  $x_i \notin \cup_{i \neq j} P_j$ . As  $a$  is adjacent to every other vertex and  $x_i \in S(I)$ ,  $ax_i \in S(I)$ . By Lemma 3.5, each  $P_i \in \min(I)$  is subtractive. So  $ax_i \notin P_j$  for each  $P_i \neq P_j \in \min(I)$  by Proposition 2.1(ii). So  $ax_i \in P_i$ , which implies  $a \in P_i$  by Proposition 2.1(ii). So  $a \in P_i$  for each  $P_i \in \min(I)$ , which implies  $a \in I$ , by Proposition 2.1(iii).  $\square$

The following example shows that the condition “ $\min(I)$  is finite” in Proposition 4.7 is not superfluous.

**Example 4.8.** Let  $R = (\mathbb{Z}^+, \text{gcd}, \text{lcm})$  and  $I = \{1\}$  (take  $\text{gcd}(0, 0) = 0$  and  $\text{lcm}(0, 0) = 0$ ). In Example 3.12, it is shown that  $\min(I)$  is infinite. It can be easily seen that 2 is adjacent to every other vertex in  $S(\Gamma_I(R))$  and  $2 \notin I$ .

A vertex  $x$  of a connected graph  $G$  is a cut-point of  $G$  if there are vertices  $y$  and  $z$  of  $G$  such that  $x$  is in every path from  $y$  to  $z$  (and  $x \neq y, x \neq z$ ). Equivalently, for a connected graph  $G$ ,  $x$  is a cut-point of  $G$  if  $G - \{x\}$  is not connected.

The connectivity of a graph  $G$ , denoted by  $\kappa(G)$ , is defined to be the minimum number of vertices that are necessary to remove from  $G$  in order to produce a disconnected graph.

**Theorem 4.9.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then:*

- (i)  $S(\Gamma_I(R))$  has cut point if and only if  $|\min(I)| = 2$  and  $I = \{1\}$ .
- (ii) If  $V$  is the set of minimum number of vertices that are necessary to remove from  $S(\Gamma_I(R))$  in order to produce a disconnected graph, then  $V \subseteq \cup V_i$ , where  $V_i = P_i \cap (\cup_{j \neq i} P_j)$  and  $P_i, P_j$ 's are minimal prime co-ideal of  $I$ . Moreover,  $\kappa(S(\Gamma_I(R))) \leq \min\{|V_i|\}$ .

*Proof.* (i) Let  $x$  be a cut-point of  $S(\Gamma_I(R))$ . Thus  $S(\Gamma_I(R)) \setminus \{x\}$  is not connected. Hence  $I = \{x\} = \{1\}$ , because if  $I \setminus \{x\} \neq \emptyset$ , then  $S(\Gamma_I(R)) \setminus \{x\}$  is connected by Proposition 4.7, which is a contradiction. So  $I = \{x\} = \{1\}$ . Since  $\{1\}$  is a co-ideal of  $R$ ,  $R$  is an  $I$ -semiring. Because  $S(\Gamma_I(R)) \setminus \{1\} = S(\Gamma(R)) \setminus \{1\} = S^*(\Gamma(R))$  is not connected,  $|\min(R)| = 2$  by Proposition 2.2(ii).

Conversely, let  $I = \{1\}$  and  $|\min(R)| = 2$ . Then  $S(\Gamma_I(R)) = S(\Gamma(R))$ . It is clear that if we remove 1 from the vertex set of  $S(\Gamma_I(R))$ , we gain the  $S^*(\Gamma(R))$ , which is disconnected by Proposition 2.2(ii). So  $S(\Gamma_I(R))$  has cut point 1.

(ii) Let  $V$  be the set of minimum number of vertices that are necessary to remove from  $S(\Gamma_I(R))$  in order to produce a disconnected graph and  $V_i = P_i \cap (\cup_{j \neq i} P_j)$ . We show that  $V \subseteq \cup V_i$ . Suppose, on the contrary, there exists  $x \in V$  such that  $x \notin \cup V_i$ . Let  $V' = S(I) \setminus V$ . By definition of  $V$ , the induced subgraph with vertex set  $V'$  is not connected but the induced subgraph with vertex set  $V' \cup \{x\}$  is connected. Let  $a, b \in V'$  such that, there is no path between them. Since the induced subgraph with vertex set  $V' \cup \{x\}$  is connected, we have the path  $a - x_1 - x_2 - \dots - x_n - x - y_1 - \dots - y_m - b$ . So  $ax_n, xy_1 \in S(I)$ .

Since  $x \notin \cup V_i$ , there are no minimal prime co-ideals  $P_i, P_j \in \min(I)$  such that  $x \in P_i \cap P_j$  (if  $x \in P_i \cap P_j$ , then  $x \in V_i, V_j$ , a contradiction). Hence there is only one minimal prime co-ideal  $P_i$  such that  $x \in P_i$ . So  $P_i$  is the only minimal prime co-ideal of  $I$  such that  $xx_n, xy_1 \in P_i$  by Proposition 2.1(ii). This implies  $x_n, y_1 \in P_i$ . So we have the path  $a - x_1 - \cdots - x_n - y_1 - \cdots - y_m - b$  in the induced subgraph with vertex set  $V'$ , which is a contradiction. Thus  $V \subseteq \cup V_i$ .

For the moreover statement we consider two cases:

Case 1:  $\min(I) = \{P_1, P_2\}$ , then  $P_1 \cap P_2 = I$  by Proposition 2.1(iii). Since every element of  $I$  is adjacent to each element of  $P_1$  and  $P_2$  by Proposition 4.7, we must remove all elements of  $I$  to gain a disconnected graph. So  $\kappa(S(\Gamma_I(R))) \geq |P_1 \cap P_2| = |I|$ . Moreover, it can be easily seen that no elements of  $P_1 \setminus I = P_1 \setminus P_2$  and  $P_2 \setminus I = P_2 \setminus P_1$  are adjacent. So  $\kappa(S(\Gamma_I(R))) = |P_1 \cap P_2| = |I|$ .

Case 2:  $|\min(I)| \geq 3$ . Let  $\min(I) = \{P_i\}_{i \in K}$  and  $V_i = P_i \cap (\cup_{j \neq i, j \in K} P_j)$  for each  $i \in K$ . If for each  $i \in K$ ,  $V_i$  is an infinite set, then there is nothing to prove. Assume that there is an  $i \in K$  such that  $V_i$  is a finite set. Let  $|V_i| = n$ . We show that the induced subgraph with vertex set  $S(I) \setminus V_i = V'_i$  is not connected. It can be easily seen that  $P_i \cap V'_i = P_i \setminus \cup_{i \neq j} P_j$  and  $P_j \cap V'_i = P_j \setminus P_i$ . We divide the proof into two steps:

Step 1: We claim that  $V'_i \cap P_i \neq \emptyset$ . Because if  $V'_i \cap P_i = \emptyset$ , then  $V_i = P_i$ . So  $P_i \subseteq \cup_{i \neq j} P_j$ . Since  $|V_i| = n$  and  $V_i = P_i$ ,  $P_i = \{x_1, x_2, \dots, x_n\}$ . So there exists  $P_i \neq P'_j \in \min(I)$  such that  $x_j \in P'_j$  for each  $x_j \in P_i$ . So  $P_i \subseteq \cup_{j=1}^n P'_j$ , which implies  $P_i \subseteq P'_j$  for some  $1 \leq j \leq n$  by Lemma 3.9, a contradiction. Thus  $V'_i \cap P_i \neq \emptyset$  and  $V_i \neq P_i$ . Also, Since  $P_j$  is a minimal prime co-ideal of  $I$ ,  $P_j \cap V'_i = P_j \setminus P_i \neq \emptyset$ .

Step 2: Let  $a \in P_i \cap V'_i$  and  $b \in P_j \cap V'_i$  (Note that  $P_i$  is the only minimal prime co-ideal of  $I$  with  $a \in P_i$ , because  $P_i \cap V'_i = P_i \setminus \cup_{i \neq j} P_j$ ). Now, we claim that there is no path between  $a$  and  $b$ . Suppose, on the contrary, there is a path between  $a$  and  $b$ . By Theorem 4.4,  $a$  and  $b$  are adjacent or  $d(a, b) = 2$ . If  $a$  and  $b$  are adjacent, then  $ab \in S(I) = \cup_{P_j \in \min(I)} P_j$ . Since  $P_i$  is the only minimal prime co-ideal of  $I$  such that  $a \in P_i$ ,  $ab \in P_i$  by Proposition 2.1(ii). Hence  $b \in P_i$ , a contradiction. If  $d(a, b) = 2$ , then there exists  $c \in V'_i$  such that  $a - c - b$  is a path in  $S(\Gamma_I(R))$ . Since  $ac \in S(I) = \cup_{P_j \in \min(I)} P_j$  and  $P_i$  is the only minimal prime co-ideal containing  $a$ ,  $ac \in P_i$ . So  $c \in P_i \cap V'_i$ . By the similar argument  $P_i$  is the only minimal prime co-ideal containing  $c$ , which implies  $b \in P_i$ , a contradiction.

So there is no path between  $a$  and  $b$ , which implies the induced subgraph with vertex set  $V'_i$  is not connected. Hence  $\kappa(S(\Gamma_I(R))) \leq \min\{|V_i|\}$ .  $\square$

**Example 4.10.** Let  $X = \{a, b, c\}$  and  $R = (P(X), \cup, \cap)$  a semiring with  $1_R = X$ , where  $P(X)$  is the set of all subsets of  $R$ . If  $I = \{X\}$ , then  $\min(I) = \{P_1, P_2, P_3\}$ , where  $P_1 = \{\{a\}, \{a, c\}, \{a, b\}, X\}$ ,  $P_2 = \{\{b\}, \{b, c\}, \{b, a\}, X\}$  and  $P_3 = \{\{c\}, \{a, c\}, \{b, c\}, X\}$ . It can be easily seen that  $|P_i \cap (P_j \cup P_k)| = 3$  and  $\kappa(S(\Gamma_I(R))) = 3$ .

**Theorem 4.11.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then  $S(\Gamma_I(R))$  contains  $|\min(I)|$  disjoint complete subgraphs.*

*Proof.* Let  $\min(I) = \{P_i\}_{i \in J}$ . Set  $V_i = P_i \setminus \cup_{i \neq j \in J} P_j$ . Then  $V_i \subseteq S(I)$  and  $V_i \cap V_j = \emptyset$  for each  $i \neq j$ . Assume  $G_i$  to be a subgraph of  $S(\Gamma_I(R))$  with vertex set  $V_i$ . It is clear that  $G_i$  is a complete subgraph of  $S(\Gamma_I(R))$ , because  $P_i \in \min(I)$  is a co-ideal of  $R$  for each  $i \in J$ . We show that  $x$  and  $y$  are not adjacent for each  $x \in G_i, y \in G_k$ . If  $x$  and  $y$  are adjacent, then  $xy \in S(I)$ . So  $xy \in P_t$  for some  $P_t \in \min(I)$ . Thus  $x, y \in P_t$  by Proposition 2.1(ii). Since  $P_i$  is the only co-ideal containing  $x$  and  $P_j$  is the only co-ideal containing  $y$ ,  $P_i = P_j = P_t$ . Thus  $G_i = G_j$ , which implies  $V_i = V_j$ , a contradiction.  $\square$

Two graphs  $G$  and  $G'$  with vertex set  $V$  and  $W$  respectively, are isomorphic if there is a bijection function  $f : V \rightarrow W$  such that for all  $v, w \in V(G)$ :  $\{v, w\} \in E(G) \Leftrightarrow (f(v), f(w)) \in E(G')$ .

**Theorem 4.12.** *Let  $R$  and  $T$  be semirings and  $I, J$  be co-ideals of  $R$  and  $T$  respectively with  $\min(I)$  and  $\min(J)$  are finite. If  $S(\Gamma_I(R)) \cong S(\Gamma_J(T))$ , then  $\Gamma_I(R) \cong \Gamma_J(T)$ .*

*Proof.* Let  $f$  be a bijection (one-to-one correspondence) from  $S(I)$  to  $S(J)$ . Since  $S(\Gamma_I(R)) \cong S(\Gamma_J(T))$ ,  $|S(I)| = |S(J)|$ . So  $|S_I(R) \cup I| = |S_J(T) \cup J|$ . We claim that  $|S_I(R)| = |S_J(T)|$ . For this we show that  $|I| = |J|$ . Since  $S(\Gamma_I(R)) = S(\Gamma_J(T))$ , there is a one-to-one corresponding between the vertices of  $S(\Gamma_I(R))$  and  $S(\Gamma_J(T))$ . Let  $a \in I$ . By Proposition 4.7,  $a$  is adjacent to every vertex of  $S(\Gamma_I(R))$ . We show that  $f(a) \in J$ . Let  $t \in S(J)$ . So there exists  $r \in S(I)$  such that  $f(r) = t$ . Because  $a$  and  $r$  are adjacent in  $S(\Gamma_I(R))$ ,  $f(a)$  and  $f(r)$  are adjacent in  $S(\Gamma_J(T))$ . So  $f(a)$  is adjacent to each element of  $S(\Gamma_J(T))$ . Thus  $f(a) \in J$  by Proposition 4.7. Hence  $|I| \leq |J|$ . By the similar way  $|J| \leq |I|$ . So  $|I| = |J|$ , which implies  $|S_I(R)| = |S_J(T)|$ . Now, let  $a$  and  $b$  are adjacent in  $\Gamma_I(R)$ . So  $a + b \in I$ . We claim that  $f(a)$  and  $f(b)$  are adjacent in  $\Gamma_J(T)$ . For this we show that  $f(a) + f(b) \in J$ . It suffices to show that for each  $t \in S(J)$   $(f(a) + f(b))t \in S(J)$  by Proposition 4.7. Let  $t \in S(J)$  and  $f(s) = t$  for some  $s \in R$ . Since  $a + b \in I$ ,  $(a + b)s \in S(I)$  by Proposition 4.7. So  $as + bs \in S(I) = \cup_{i=1}^n P_i$ . Thus  $as + bs \in P_i$  for some  $P_i \in \min(I)$ . Since  $P_i$  is a prime co-ideal of  $R$ ,  $as \in P_i$  or  $bs \in P_i$ . Let  $as \in P_i \subseteq S(I)$ . Thus  $a$  and  $s$  are adjacent in  $S(\Gamma_I(R))$  which implies  $f(a)$  and  $f(s)$  are adjacent in  $S(\Gamma_J(T))$ . Hence  $f(a)f(s) \in S(J)$ . So  $f(a)f(s) \in Q_j$  for some  $Q_j \in \min(J)$ . Thus  $f(a)f(s) + f(b)f(s) \in Q_j \subseteq S(J)$ , which gives  $(f(a) + f(b))t = (f(a) + f(b))f(s) \in S(J)$ , as needed.  $\square$

### 5. Total graph with respect to a Q-strong co-ideal

In this section we investigate interrelation between  $S(\Gamma_I(R))$  and  $S(\Gamma(R/I))$ .

**Proposition 5.1.** *Let  $I$  be a Q-strong co-ideal of a semiring  $R$  and let  $x, y \in R$  such that  $x \in q_1 I$  and  $y \in q_2 I$  for some  $q_1, q_2 \in Q$ . Then*

(i)  $x$  is adjacent to  $y$  in  $S(\Gamma_I(R))$  if and only if  $q_1I$  is adjacent to  $q_2I$  in  $S(\Gamma(R/I))$ . Moreover,  $x \in S(I)$  if and only if  $q_1I \in S(R/I)$ .

(ii)  $qI \cap S(I) \neq \emptyset$  if and only if all distinct elements of  $qI$  are adjacent in  $S(\Gamma_I(R))$ .

(iii) If  $qI \cap S(I) \neq \emptyset$ , then  $qI \subseteq S(I)$ .

(iv)  $S(\Gamma_I(R))$  contains at least  $|S(I) \cap Q|$  disjoint complete subgraphs.

*Proof.* (i) Let  $x$  be adjacent to  $y$  in  $S(\Gamma_I(R))$ . So  $xy \in S(I)$ , which implies  $xy + r \in I$  for some  $r \in R \setminus I$ . Since  $I$  is a  $Q$ -co-ideal of  $R$ , there exists the unique element  $q_3 \in Q$  such that  $r \in q_3I$ . Because  $r \in q_3I$  and  $r \notin I$ ,  $I \neq q_3I$ . Let  $q_1I \odot q_2I \oplus q_3I = q_4I$ . So  $xy + r \in q_4I \cap I$ , which gives  $q_4I = I$ . Hence  $q_1I \odot q_2I \oplus q_3I = I$  which implies  $q_1I \odot q_2I \in S(R/I)$ . So  $q_1I$  is adjacent to  $q_2I$  in  $S(\Gamma(R/I))$ .

Conversely, let  $q_1I$  be adjacent to  $q_2I$  in  $S(\Gamma(R/I))$ . Hence  $q_1I \odot q_2I \oplus q_3I = I$  for some  $I \neq q_3I \in R/I$ . This implies  $xy + r \in I$  for each  $x \in q_1I$ ,  $y \in q_2I$  and  $r \in q_3I$ . So  $xy \in (I : r) \subseteq S(I)$ . Hence  $x$  and  $y$  are adjacent in  $S(\Gamma_I(R))$ . The moreover statement is clear by similar argument.

(ii) Let  $qI \cap S(I) \neq \emptyset$  and  $x \in qI \cap S(I)$ . Since  $x \in S(I)$ , there exists  $r \in R \setminus I$  such that  $x + r \in I$ . Since  $I$  is a  $Q$ -strong co-ideal of  $R$ , there exists the unique element  $q' \in Q$  such that  $r \in q'I$ . Since  $x + r \in (qI \oplus q'I) \cap I$ ,  $qI \oplus q'I = I$ . Thus  $qI \in (\{I\} : q'I)$ . By Proposition 2.1(ii),  $(\{I\} : q'I)$  is a co-ideal which gives  $qI \odot qI \in (\{I\} : q'I)$ . Thus  $qI \odot qI \oplus q'I = I$ . Hence for each  $a, b \in qI$  there exists  $r \in q'I$ , such that  $ab + r \in I$ . Thus  $a$  and  $b$  are adjacent in  $S(\Gamma_I(R))$  for each  $a, b \in qI$ .

Conversely, let  $x$  and  $y$  be two elements of  $qI$ . By assumption  $x$  and  $y$  are adjacent in  $S(\Gamma_I(R))$ . Hence  $xy \in S(I)$ , which gives  $x, y \in S(I)$  by Lemma 3.1(ii).

(iii) Let  $x \in qI$  and  $y \in qI \cap S(I)$ . By (ii),  $x$  and  $y$  are adjacent. So  $xy \in S(I)$ , which gives  $x \in S(I)$  by Lemma 3.1(ii). So  $qI \subseteq S(I)$ .

(iv) For each  $q \in Q$ , if  $q \in S(I)$ , then the induced subgraph of  $S(\Gamma_I(R))$  with vertex set  $\{qI\} = \{qa : a \in I\}$  is a complete subgraph of  $S(\Gamma_I(R))$  by (ii). So  $S(\Gamma_I(R))$  contains at least  $|S(I) \cap Q|$  complete subgraphs.  $\square$

**Theorem 5.2.** *Let  $R$  be a semiring and  $I$  be a  $Q$ -strong co-ideal of  $R$ .*

(i) *If  $qI \in S(R/I)$ , then  $qI \subseteq S(I)$ ;*

(ii)  *$S(\Gamma_I(R))$  is complete if and only if  $S(\Gamma(R/I))$  is complete;*

(iii)  *$\text{diam}(S(\Gamma_I(R))) = 1$  if and only if  $\text{diam}(S(\Gamma(R/I))) = 1$ ;*

(iv)  *$\text{diam}(S(\Gamma_I(R))) = 2$  if and only if  $\text{diam}(S(\Gamma(R/I))) = 2$ ;*

(v)  *$\text{gr}(S(\Gamma_I(R))) \leq \text{gr}(S(\Gamma(R/I)))$ . Moreover, if  $|\min(I)| \geq 3$ , then*

$$\text{gr}(S(\Gamma_I(R))) = \text{gr}(S(\Gamma(R/I))).$$

*Proof.* (i) If  $S(R/I) = \{I\}$ , then  $R/I$  is a co-semidomain. Hence  $I$  is prime by [10, Theorem 3.8]. This gives  $I = S(I)$ . Suppose that  $S(R/I) \neq \{I\}$ . Let  $a \in I$ ,  $qa \in qI$  and  $qI \in S(R/I)$ . Hence there exists  $I \neq q'I \in S(R/I)$  such that  $qI \oplus q'I = I$ . Thus  $qa + q'b \in I$  for some  $b \in I$ , which implies  $qa \in S(I)$ .

(ii) Let  $S(\Gamma_I(R))$  be complete and  $q_1I, q_2I \in S(R/I)$ . By (i),  $q_1I, q_2I \subseteq S(I)$ , where  $q_iI = \{q_i a : a \in I\}$ . Hence  $q_1, q_2 \in S(I)$ , which gives  $q_1q_2 \in S(I)$ , because  $S(\Gamma_I(R))$  is a complete graph. So there exists  $r \in R \setminus I$ , such that  $q_1q_2 + r \in I$ . Since  $I$  is a  $Q$ -strong co-ideal of  $R$ , there exists a unique element  $q_3 \in Q$  such that  $r \in q_3I$  (hence  $q_3I \neq I$ ). If  $q_1I \odot q_2I \oplus q_3I = qI$ , then  $q_1q_2 + r \in qI \cap I$ , which gives  $qI = I$ . This implies that  $q_1I \odot q_2I \in (\{I\} : q_3I) \subseteq S(R/I)$ . Thus  $S(\Gamma(R/I))$  is a complete graph.

Conversely, let  $S(\Gamma(R/I))$  be a complete graph. Let  $x, y \in S(I)$ . Since  $I$  is a  $Q$ -strong co-ideal, there exist  $q_1, q_2 \in Q$  such that  $x \in q_1I$  and  $y \in q_2I$ . Since  $S(\Gamma(R/I))$  is a complete graph,  $q_1I \odot q_2I \in S(R/I)$ . Let  $q_1I \odot q_2I = q_3I$ , where  $q_3$  is the unique element of  $Q$  such that  $(q_1q_2)I \subseteq q_3I$ . So  $xy \in q_3I$ . Since  $q_3I \subseteq S(I)$  (by (i)),  $xy \in q_3I \subseteq S(I)$ . Thus  $S(I)$  is a complete graph.

(iii) and (iv) are clear by (ii) and Theorem 4.4.

(v) If  $\text{gr}(S(\Gamma(R/I))) = \infty$ , there is nothing to prove. Let  $\text{gr}(S(\Gamma(R/I)))$  be finite. So  $\text{gr}(S(\Gamma(R/I))) = 3$  by Theorem 4.4. Let  $q_1I - q_2I - q_3I - q_1I$  be a path in  $S(\Gamma(R/I))$  for some  $q_1I, q_2I, q_3I \in S(R/I)$ . By (i),  $q_1 - q_2 - q_3 - q_1$  is a path in  $S(\Gamma_I(R))$ . So  $\text{gr}(S(\Gamma_I(R))) = 3$ . Thus  $\text{gr}(S(\Gamma_I(R))) \leq \text{gr}(S(\Gamma(R/I)))$ . For the moreover statement, let  $|\min(I)| \geq 3$  and  $P_1, P_2$  and  $P_3 \in \min(I)$ . By Lemma 3.9,  $P_1 \not\subseteq P_2 \cup P_3$ ,  $P_2 \not\subseteq P_1 \cup P_3$  and  $P_3 \not\subseteq P_2 \cup P_1$ . So  $\{1, a, b, c\} \subseteq S(I)$  for some  $a \in P_1 \setminus P_2 \cup P_3$ ,  $b \in P_2 \setminus P_1 \cup P_3$  and  $c \in P_3 \setminus P_2 \cup P_1$ . So  $\text{gr}(S(\Gamma_I(R))) = 3$  by Theorem 4.5. We show that  $\text{gr}(S(\Gamma(R/I))) = 3$ . Since  $I$  is a  $Q$ -strong co-ideal of  $R$ , there exist  $q_1, q_2, q_3 \in Q$  such that  $a \in q_1I$ ,  $b \in q_2I$  and  $c \in q_3I$ . Hence  $a = q_1a_1$  and  $b = q_1b_1$  for some  $a_1, b_1 \in I$ . We show that  $I \neq q_1I \neq q_2I \neq q_3I$ . Let  $q_1I = q_2I$ . Since  $a \in P_1$  and  $P_1$  is subtractive,  $q_1 \in P_1$  by Proposition 2.1(ii). So  $b = q_1b_1 \in P_1$  because  $q_1, b_1 \in P_1$ , a contradiction. Thus  $I, q_1I, q_2I$  and  $q_3I$  are distinct elements of  $R/I$ . By (i),  $I, q_1I, q_2I$  and  $q_3I \in S(R/I)$ . So  $|S(R/I)| \geq 4$ , which implies  $\text{gr}(S(\Gamma(R/I))) = 3$  by Theorem 4.4. Thus  $\text{gr}(S(\Gamma_I(R))) = \text{gr}(S(\Gamma(R/I)))$ .  $\square$

The following example shows that the condition “ $|\min(I)| \geq 3$ ” can not be omitted in Theorem 5.2.

**Example 5.3.** Let  $X = \{a, b, c\}$  and  $R = (P(X), \cup, \cap)$ , where  $P(X)$  = the set of all subsets of  $X$ . An inspection will show that  $I = \{X, \{a, b\}\}$  is a  $Q$ -co-ideal of  $R$ , where  $Q = \{q_e = X, q_1 = \{b, c\}, q_2 = \{a, c\}, q_3 = \{c\}\}$  such that  $q_1I = \{\{b\}, \{b, c\}\}$  and  $q_2I = \{\{a\}, \{a, c\}\}$  and  $q_3I = \{\{c\}, \emptyset\}$ . It can be easily seen that  $\min(I) = \{P_1, P_2\}$ , where  $P_1 = \{\{a\}, \{a, c\}, \{a, b\}, X\}$  and  $P_2 = \{\{b\}, \{b, c\}, \{a, b\}, X\}$ . We see that  $S(R/I) = \{q_eI, q_1I, q_2I\}$  and  $S(I) = P_1 \cup P_2 = \{X, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b\}\}$ . Hence  $\{a\} - \{a, b\} - X - \{a\}$  is a cycle in  $S(\Gamma_I(R))$  and there is no cycle in  $S(\Gamma(R/I))$ . So  $\text{gr}(S(\Gamma_I(R))) = 3$  and  $\text{gr}(S(\Gamma(R/I))) = \infty$ .

**Theorem 5.4.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ , which is not prime. Then*

(i)  $\omega(\Gamma_I(R)) < \omega(S(\Gamma_I(R)))$  if and only if  $\min(I)$  is finite with  $3 \leq |\min(I)|$  or  $|\min(I)| = 2$  with  $|P_i| \geq 3$ , where  $P_i \in \min(I)$ .

(ii)  $\omega(\Gamma_I(R)) = \omega(S(\Gamma_I(R)))$  if and only if  $|\min(I)|$  is infinite or  $|\min(I)| = 2$  with  $|P_i| = 2$  for each  $P_i \in \min(I)$ .

(iii)  $\omega(S(\Gamma(R/I))) \leq \omega(S(\Gamma_I(R)))$ . Moreover,  $\omega(S(\Gamma_I(R))) = \omega(S(\Gamma(R/I)))$  if and only if  $I = \{1\}$  or  $\omega(S(\Gamma(R/I)))$  is infinite.

*Proof.* (i) Let  $\omega(\Gamma_I(R)) < \omega(S(\Gamma_I(R)))$ . If  $\omega(\Gamma_I(R))$  is infinite, then  $\omega(\Gamma_I(R)) < \omega(S(\Gamma_I(R)))$  implies that  $\omega(S(\Gamma_I(R)))$  is infinite. Hence

$$\omega(\Gamma_I(R)) = \omega(S(\Gamma_I(R))),$$

a contradiction. Therefore  $\omega(\Gamma_I(R))$  is finite

By Proposition 2.2(iii),  $\omega(\Gamma_I(R)) = |\min(I)|$ . Hence  $|\min(I)|$  is finite. We show that  $3 \leq |\min(I)|$  or  $|\min(I)| = 2$  with  $|P_i| \geq 3$ , where  $P_i \in \min(I)$ . Since  $I$  is not a prime co-ideal of  $R$ ,  $|\min(I)| \neq 1$  by Proposition 2.1(iii). Suppose on the contrary,  $\omega(\Gamma_I(R)) = |\min(I)| = 2$  and  $|P_1| = |P_2| = 2$ . Let  $P_1 = \{1, a\}$  and  $P_2 = \{1, b\}$ . By Theorem 3.7,  $S(I) = \{1, a, b\}$  and  $ab \notin S(I)$  by Lemma 3.1. Hence  $a - 1 - b$  is the only path in  $S(\Gamma_I(R))$  and  $\omega(S(\Gamma_I(R))) = 2$ , a contradiction.

Conversely, if  $\min(I) = \{P_1, P_2, \dots, P_n\}$ , where  $n \geq 3$ , then  $\omega(\Gamma_I(R)) = n$  by Proposition 2.2(iii). By Proposition 2.1(iii),  $I = P_1 \cap P_2 \cap \dots \cap P_n$  and  $I \neq \bigcap_{i=1, i \neq j}^n P_i$  for each  $1 \leq j \leq n$ . If  $P_1 \cap P_j \subseteq \bigcup_{i \neq 1, j} P_i$ , then  $P_1 \cap P_j \subseteq P_i$  for some  $1 \leq i \neq j \leq n$  by Lemma 3.9. This implies  $P_1 \subseteq P_i$  or  $P_j \subseteq P_i$  by [10, Lemma 2.7], a contradiction. So  $P_1 \cap P_j \not\subseteq \bigcup_{i \neq 1, j} P_i$ . Let  $a_j \in P_1 \cap P_j \setminus \bigcup_{i \neq 1, j} P_i$ . Then  $\{1, a_1, a_2, \dots, a_n\} \subseteq P_1$ , which gives  $|P_1| \geq n + 1$ . Since the induced subgraph of  $S(\Gamma_I(R))$  with vertex set  $P_1$  is a complete subgraph of  $S(\Gamma_I(R))$ ,  $\omega(S(\Gamma_I(R))) \geq n + 1$ . So  $\omega(\Gamma_I(R)) < \omega(S(\Gamma_I(R)))$ . If  $\min(I) = \{P_1, P_2\}$  with  $|P_i| \geq 3$ , then  $\omega(\Gamma_I(R)) = 2$  by Proposition 2.2(iii). Since the induced subgraph with vertex set  $P_i$  of  $S(\Gamma_I(R))$  is a complete subgraph of  $S(\Gamma_I(R))$  and  $|P_i| \geq 3$ ,  $\omega(S(\Gamma_I(R))) \geq 3$ . So  $\omega(\Gamma_I(R)) < \omega(S(\Gamma_I(R)))$ .

(ii) Let  $\omega(\Gamma_I(R)) = \omega(S(\Gamma_I(R)))$ . So  $|\min(I)|$  is infinite or  $|\min(I)| = 2$  with  $|P_i| = 2$  for each  $P_i \in \min(I)$  by (i).

Conversely, if  $|\min(I)|$  is infinite, then  $\omega(\Gamma_I(R)) = \infty$  by Proposition 2.2(iii). We show that every clique in  $\Gamma_I(R)$  is a clique in  $S(\Gamma_I(R))$ . Let  $T$  be a clique in  $\Gamma_I(R)$  and  $x, y \in T$ . It is clear that  $x, y \in S(I)$  because  $S_I(R) \subseteq S(I)$ . Since  $\omega(\Gamma_I(R)) \geq 3$ , there exists  $z \in T$  such that  $x, y \neq z$ . Since  $T$  is a clique,  $x, y \in (I : z)$  and so  $xy \in (I : z) \subseteq S(I)$ . So  $T$  is a clique in  $S(\Gamma_I(R))$ . Hence  $\omega(S(\Gamma_I(R))) = \infty$ . If  $|\min(I)| = 2$  with  $|P_i| = 2$  for each  $P_i \in \min(I)$ , then  $\omega(\Gamma_I(R)) = 2$  by Proposition 2.2(iii) and  $\omega(S(\Gamma_I(R))) = 2$  by the proof of (i).

(iii) Let  $\{q_e I, q_1 I, \dots, q_n I, \dots\}$  be a clique in  $S(\Gamma(R/I))$ . Then  $\{q_e, q_1, \dots, q_n, \dots\}$  is a clique in  $S(\Gamma_I(R))$  by Proposition 5.1(i). So  $\omega(S(\Gamma(R/I))) \leq \omega(S(\Gamma_I(R)))$ . For the moreover statement, it is clear that if  $I = \{1\}$ , then  $\omega(S(\Gamma(R/I))) = \omega(S(\Gamma_I(R)))$ . If  $\omega(S(\Gamma(R/I)))$  is infinite, then  $\omega(S(\Gamma(R/I))) = \omega(S(\Gamma_I(R)))$ , because  $\omega(S(\Gamma(R/I))) \leq \omega(S(\Gamma_I(R)))$ .



Conversely, let  $\omega(S(\Gamma_I(R))) = \omega(S(\Gamma(R/I)))$ . Let  $\omega(S(\Gamma(R/I)))$  be finite we show that  $I = \{1\}$ . We know that  $q_e \in I$ . We claim that for each  $a \in I$ ,  $a = q_e$ . Suppose, on the contrary, there exists  $a \in I$  such that  $a \neq q_e$ . Since  $I$  is not prime,  $R/I$  is not co-semidomain by [10, Theorem 3.8]. Hence  $|S(R/I)| \geq 2$  which implies  $\omega(S(\Gamma(R/I))) \geq 2$ . Let  $\{q_e I, q_1 I, q_2 I, \dots, q_n I\}$  be a clique in  $\omega(S(\Gamma(R/I)))$ . By Proposition 4.7 and 5.1(i),  $\{q_e, q_1, q_2, \dots, q_n, a\}$  is a clique in  $\omega(S(\Gamma_I(R)))$ , a contradiction. So for each  $a \in I$ ,  $a = q_e$ . So  $1 = q_e$  and  $I = \{1\}$ .  $\square$

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