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SOME FIXED POINT THEOREMS IN GENERALIZED DARBO FIXED POINT THEOREM AND THE EXISTENCE OF SOLUTIONS FOR SYSTEM OF INTEGRAL EQUATIONS

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Abstract. In this paper we introduce the notion of the generalized Darbo fixed point theorem and prove some fixed and coupled fixed point theorems in Banach space via the measure of non-compactness, which generalize the result of Aghajani et al. [6]. Our results generalize, extend, and unify several well-known comparable results in the literature. One of the applications of our main result is to prove the existence of solutions for the system of integral equations.

1. Introduction

The integral equation creates a very important and significant part of the mathematical analysis and has various applications into real world problems. On the other hand, Measures of noncompactness are very useful tools in the wide area of functional analysis such as the metric fixed point theory and the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integro-differential equations, optimal control theory, etc., see [1, 2, 3, 4, 7, 13, 14, 15, 16, 17]. In our investigations, we apply the method associated with the technique of measures of noncompactness in order to generalize the Darbo fixed point theorem [10] and to extend some recent results of Aghajani et al. [6], and also we are going to study the existence of solutions for the following system of integral equations

$$(1.1)\quad \begin{cases} x(t,s) = a(t,s) + f(t,s,x(t,s),y(t,s)) \\ \qquad\qquad + g(t,s,x(t,s),y(t,s)) \int_0^{\alpha_1(t)} \int_0^{\alpha_2(s)} k(t,s,u,v,x(u,v),y(u,v))dudv \\ y(t,s) = a(t,s) + f(t,s,y(t,s),x(t,s)) \\ \qquad\qquad + g(t,s,y(t,s),x(t,s)) \int_0^{\alpha_1(t)} \int_0^{\alpha_2(s)} k(t,s,u,v,y(u,v),x(u,v))dudv, \end{cases}$$

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for $t, s \in \mathbb{R}_+$, $x, y \in E = BC(\mathbb{R}_+ \times \mathbb{R}_+)$. We show that Eq. (1.1) has solutions that belong to $E \times E$ for $E = BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this work. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. We use the following definition of the measure of noncompactness given in [10].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1^0) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$,
(2^0) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
(3^0) $\mu(\bar{X}) = \mu(X)$,
(4^0) $\mu(ConvX) = \mu(X)$,
(5^0) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$,
(6^0) If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n (n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ defined in axiom (1^0) is called the kernel of the measure of noncompactness μ.

One of the properties of the measure of noncompactness is $X_\infty \in \ker\mu$. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, 3, \ldots$, we infer that $\mu(X_\infty) = 0$. Further facts concerning measures of noncompactness and their properties may be found in [9, 10].

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

Theorem 2.2 (Schauder [4]). *Let C be a closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

In the following we state a fixed-point theorem of Darbo type proved by Banaś and Goebel [10].

Theorem 2.3. *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $F : C \to C$ be a continuous mapping. Assume that there exist a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset of C. Then F has a fixed-point in the set C.*

The following theorem was proved in [6].

Theorem 2.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous mapping such that*

$$\mu(FX) \leq \varphi(\mu(X)) \tag{2.1}$$

for any nonempty subset X of Ω where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Then F has at least one fixed point in the set Ω.

The following concept of $O(f;.)$ and its examples was given by Altun and Turkoglu [8].

Let $F([0,\infty)$ be class of all function $f : [0,\infty) \to [0,\infty]$ and let Θ be class of all operators

$$O(\bullet;.) : F([0,\infty)) \longrightarrow F([0,\infty)),\ f \to O(f;.)$$

satisfying the following conditions:

(i) $O(f;t) > 0$ for $t > 0$ and $O(f;0) = 0$,
(ii) $O(f;t) \leq O(f;s)$ for $t \leq s$,
(iii) $\lim_{n\to\infty} O(f;t_n) = O(f;\lim_{n\to\infty} t_n)$,
(iv) $O(f;\max\{t,s\}) = \max\{O(f;t), O(f;s)\}$ for some $f \in F([0,\infty)$.

Example 2.5. If $f : [0,\infty) \to [0,\infty)$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0,\infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then the operator defined by

$$O(f;t) = \int_0^t f(s)ds$$

satisfies the above conditions.

Example 2.6. If $f : [0,\infty) \to [0,\infty)$ is a non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by

$$O(f;t) = \frac{f(t)}{1 + f(t)}$$

satisfies the above conditions.

Example 2.7. If $f : [0,\infty) \to [0,\infty)$ is a non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by

$$O(f;t) = \frac{f(t)}{1 + \mathrm{Ln}(1 + f(t))}$$

satisfies the conditions (i)-(iv).

3. Fixed point theorem

This section is devoted to prove a few generalizations of Darbo fixed point theorem (cf. Theorem 2.3), and as a consequence establish an existence result of coupled fixed point for a class of condensing operators in Banach spaces, which will be used in next section.

Theorem 3.1. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T : C \to C$ be a continuous operator such that*

$$O(f;\mu(T(X))) + \varphi(\mu(TX)) \leq \psi[O(f;\mu(X))) + \varphi(\mu(X))], \tag{3.1}$$

for any subset X of C, $O(\bullet;.) \in \Theta$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, where μ is an arbitrary measure of noncompactness, $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing functions such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \geq 0$. Then T has at least one fixed point in C.

Proof. Let $C_0 = C$, we construct a sequence $\{C_n\}$ such that $C_{n+1} = Conv(TC_n)$ for $n \geq 0$. $TC_0 = TC \subseteq C = C_0, C_1 = Conv(TC_0) \subseteq C = C_0$, therefore by continuing this process we have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots .$$

If there exists a natural number N such that $\mu(C_N) = 0$, then C_N is compact. In this case Theorem 2.2 implies that T has a fixed point. So we assume that $\mu(C_n) \neq 0$ for $n = 0, 1, 2, \ldots$. Also by (3.1) we have

$$\begin{aligned} O(f;\mu(C_{n+1})) + \varphi(\mu(C_{n+1})) &= O(f;\mu(Conv(TC_n))) + \varphi(\mu(Conv(TC_n))) \\ &= O(f;\mu(TC_n)) + \varphi(\mu(TC_n)) \\ &\leq \psi[O(f;\mu(C_n)) + \varphi(\mu(C_n))] \\ &\leq \psi^2[O(f;\mu(C_{n-1})) + \varphi(\mu(C_{n-1}))] \\ &\;\;\vdots \\ &\leq \psi^n[O(f;\mu(C_0)) + \varphi(\mu(C_0))] \\ &= \psi^n[O(f;\mu(C)) + \varphi(\mu(C))]. \end{aligned} \tag{3.2}$$

Taking the limit of (3.2), as $n \to \infty$, we have

$$\lim_{n\to\infty}[O(f;\mu(C_{n+1})) + \varphi(\mu(C_{n+1}))] = 0,$$

therefore,

$$\lim_{n\to\infty} O(f;\mu(C_{n+1})) = O(f;\lim_{n\to\infty}\mu(C_{n+1})) = 0,$$

which, from (i), implies that

$$\lim_{n\to\infty}\mu(C_{n+1})) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, \ldots$, then from (6^0), $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is a nonempty convex closed set, invariant under T and belongs to $\ker\mu$. Therefore Theorem 2.2 completes the proof. □

An immediate consequence of Theorem 3.1 is the following.

Theorem 3.2. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $T : C \to C$ are continuous functions. Suppose that there exists a constant $0 < \lambda < 1$, such that for all $X \subseteq C$,*

$$O(f; \mu(T(X))) + \varphi(\mu(TX)) \leq \lambda[O(f; \mu(X))) + \varphi(\mu(X))],$$

where μ is an arbitrary measure of noncompactness and $O(\bullet; .) \in \Theta$. Then T has at least one fixed point in C.

Remark 3.3. It is clear that Theorem 3.1 is a generalization of Theorem 2.3.

Remark 3.4. It is clear that Theorem 3.1 is a generalization of Theorem 2.4 in fact letting $f = I$ the identity mapping on $[0, \infty)$ (which we denote by $I_{[0,\infty)}$), $\varphi \equiv 0$ and $O(f; , t) = t$ in (3.1) (it is obvious that $O(f; t) \in \Theta$) one has

$$\mu(T(X)) = O(f; \mu(T(X))) \leq \psi(O(f; \mu(X))) = \psi(\mu(X)).$$

Corollary 3.5. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T : C \to C$ be a continuous operator such that*

$$O(f; \mu(T(X))) \leq \psi[O(f; \mu(X)))],$$

for any subset X of C and $O(\bullet; .) \in \Theta$, where μ is an arbitrary measure of noncompactness, $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing functions such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \geq 0$. Then T has at least one fixed point in C.

The following corollary gives us a fixed point theorem with a contractive condition of integral type.

Corollary 3.6. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T : C \to C$ be a continuous operator such that for any $X \subseteq C$ one has*

$$\int_0^{\mu(T(X))} f(s)\ ds \leq \psi(\int_0^{\mu(X)} f(s)\ ds),$$

where μ is an arbitrary measure of noncompactness and $f : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, non-negative and such that for each $\epsilon > 0$, $\int_0^\epsilon f(s)\ ds > 0$ and $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing functions such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \geq 0$. Then T has at least one fixed point in C.

Corollary 3.7. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E, $k \in (0, 1)$ and $T : C \to C$ be a continuous operator such that for any $X \subseteq C$ one has*

$$\int_0^{\mu(T(X))} f(s)\ ds \leq k \int_0^{\mu(X)} f(s)\ ds,$$

where μ is an arbitrary measure of noncompactness and $f : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) *on each compact subset of* $[0,\infty)$*, non-negative and such that for each* $\epsilon > 0$*,* $\int_0^\epsilon f(s)\, ds > 0$*. Then* T *has at least one fixed point in* C*.*

Definition 3.8 ([11])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Theorem 3.9 ([10])**.** *Suppose* $\mu_1, \mu_2, \ldots, \mu_n$ *be the measures in* $E_1, E_2, \ldots, E_n$ *respectively. Moreover assume that the function* $F : [0,\infty)^n \to [0,\infty)$ *is convex and* $F(x_1, x_2, \ldots, x_n) = 0$ *if and only if* $x_i = 0$ *for* $i = 1, 2, \ldots, n$*. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \ldots \times E_n$ *where* X_i *denotes the natural projection of* X *into* E_i *for* $i = 1, 2, \ldots, n$*.*

Now, as results from Theorem 3.9, we present the following examples.

Example 3.10 ([10])**.** Let μ be a measure of noncompactness, considering $F(x, y) = \max\{x, y\}$ for any $x, y \in [0,\infty)$, then all the conditions of Theorem 3.9 are satisfied. Therefore, $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.

Example 3.11 ([10])**.** Let μ be a measure of noncompactness. We define $F(x, y) = x + y$ for any $x, y \in [0,\infty)$. Then F has all the properties mentioned in Theorem 3.9. Hence $\widetilde{\mu}(X) = \mu(X_1)+\mu(X_2)$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.

Theorem 3.12. *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and* $T : C \times C \to C$ *be a continuous function such that*

$$\begin{aligned} &O(f; \mu(T(X_1 \times X_2))) + \varphi(\mu(T(X_1 \times X_2))) \\ &\leq \frac{1}{2}\psi[O(f; \mu(X_1) + \mu(X_2)) + \varphi(\mu(X_1) + \mu(X_2))], \end{aligned} \tag{3.3}$$

for any subset X_1, X_2 *of* C*, where* μ *is an arbitrary measure of noncompactness and* $\varphi : [0,\infty) \to [0,\infty)$ *is a nondecreasing, continuous and* $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ *for all* $t, s \geq 0$ *and* $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing function such that* $\lim_{n\to\infty} \psi^n(t) = 0$ *for each* $t \geq 0$*. Also* $O(\bullet; .) \in \Theta$ *and* $O(f; t + s) \leq O(f; t) + O(f; s)$ *for all* $t, s \geq 0$*. Then* T *has at least a coupled fixed point.*

Proof. First note that, from Example 3.11, $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ for any bounded subset $X \subseteq E \times E$ defines a measure of noncompactness on $E \times E$ where X_1 and X_2 denote the natural projections of X. We define a mapping $\widetilde{T} : C \times C \to C \times C$ by

$$\widetilde{T}(x, y) = (T(x, y), T(y, x)).$$

It is obvious that $\widetilde{T}$ is continuous. Now we claim that $\widetilde{T}$ satisfies all the conditions of Theorem 3.1. To prove this, let $X \subseteq C \times C$ be any nonempty subset. Then by (2^0), (3.3) and (ii) we obtain

$$O(f; \widetilde{\mu}(\widetilde{T}(X))) + \varphi(\widetilde{\mu}(\widetilde{T}(X)))$$

$$
\begin{aligned}
&\le O(f;\widetilde{\mu}(T(X_1\times X_2)\times T(X_2\times X_1)))+\varphi(\widetilde{\mu}(T(X_1\times X_2))\times T(X_2\times X_1)))\\
&= O(f;\mu(T(X_1\times X_2))+\mu(T(X_2\times X_1)))\\
&\quad +\varphi(\mu(T(X_1\times X_2))+\mu(T(X_2\times X_1)))\\
&= O(f;\mu(T(X_1\times X_2)))+O(f;\mu(T(X_2\times X_1)))\\
&\quad +\varphi(\mu(T(X_1\times X_2)))+\varphi(\mu(T(X_2\times X_1)))\\
&\le \frac{1}{2}\psi[O(f;\mu(X_1)+\mu(X_2))+\varphi(\mu(X_1)+\mu(X_2))]\\
&\quad +\frac{1}{2}\psi[O(f;\mu(X_2)+\mu(X_1))+\varphi(\mu(X_2)+\mu(X_1))]\\
&= \psi[O(f;\mu(X_1)+\mu(X_2))+\varphi(\mu(X_1)+\mu(X_2))]\\
&= \varphi[O(f;\widetilde{\mu}(X))+\varphi(\widetilde{\mu}(X))].
\end{aligned}
$$

Hence, from Theorem 3.1, $\widetilde{T}$ has at least one fixed point in $C\times C$. Now the conclusion of theorem follows from the fact that every fixed point of $\widetilde{T}$ is a coupled fixed point of T. □

As applications for Theorem 3.12, one can get the following Corollaries 3.13, 3.14 and 3.15.

Corollary 3.13. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T: C\times \to C$ be a continuous function such that*

$$O(f;\mu(T(X_1\times X_2)))\le \frac{1}{2}\psi[O(f;\mu(X_1)+\mu(X_2))],$$

for any subset X_1, X_2 of C, where μ is an arbitrary measure of noncompactness and $\psi:[0,\infty)\to[0,\infty)$ is a nondecreasing function such that $\lim_{n\to\infty}\psi^n(t)=0$ for each $t\ge 0$. Also $O(\bullet;.)\in\Theta$ and $O(f;t+s)\le O(f;t)+O(f;s)$ for all $t,s\ge 0$. Then T has at least a coupled fixed point.

Corollary 3.14. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T: C\times \to C$ be a continuous function such that*

$$\mu(T(X_1\times X_2))\le \frac{1}{2}\varphi(\mu(X_1)+\mu(X_2)),$$

for any subset X_1, X_2 of C, where μ is an arbitrary measure of noncompactness and $\varphi:[0,\infty)\to[0,\infty)$ is a nondecreasing functions such that $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t\ge 0$. Then T has at least a coupled fixed point.

Corollary 3.15. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T: C\times \to C$ be a continuous function. Assume that there exists a constant $k\in[0,1)$ such that*

$$\mu(T(X_1\times X_2))\le \frac{k}{2}(\mu(X_1)+\mu(X_2)),$$

for any subset X_1, X_2 of C, where μ is an arbitrary measure of noncompactness. Then T has at least a coupled fixed point.

4. Applications

In what follows we will work in the classical Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ equipped with the standard norm

$$||x|| = \sup\{|x(t,s)| : t, s \geq 0\}.$$

Now, we present the definition of a special measure of noncompactness in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ which will be used in the sequel, a measure that was introduced and studied in [10].

To do this, let X be fix a nonempty and bounded subset of $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and fix a positive number T. For $x \in X$ and $\epsilon > 0$, denote by $\omega^T(x,\epsilon)$ the modulus of the continuity of function x on the interval $[0,T]$, i.e.,

$$\omega^T(x,\epsilon) = \sup\{|x(t,s) - x(u,v)| : t,s,u,v \in [0,T], |t-u| \leq \epsilon, |s-v| \leq \epsilon\}.$$

Further, let us put

$$\omega^T(X,\epsilon) = \sup\{\omega^T(x,\epsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X,\epsilon)$$

and

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Moreover, for two fixed numbers $t, s \in \mathbb{R}_+$ let us the define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+ \times \mathbb{R}_+)}$ by the following formula

$$\mu(X) = \omega_0(X) + \alpha(X),$$

where $\alpha(X) = \limsup_{t,s\to\infty} \mathrm{diam} X(t,s)$, $X(t,s) = \{x(t,s) : x \in X\}$ and $\mathrm{diam} X(t,s) = \sup\{|x(t,s) - y(t,s)| : x, y \in X\}$. Similar to [10] (cf. also [9]), it can be shown that the function μ is the measure of noncompactness in the space E.

As an application of our results we are going to study the existence of solutions for the system of integral equations (1.1). Consider the following assumptions

- (A_1) $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, nondecreasing and $\lim_{t\to\infty} \alpha_i(t) = \infty$, $i = 1, 2$.
- (A_2) The function $a : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded.
- (A_3) $k : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant M such that

(4.1)
$$M = \sup\{\int_0^{\alpha_1(t)} \int_0^{\alpha_2(s)} |k(t,s,u,v,x(u,v),y(u,v))| du dv : t, s \in \mathbb{R}_+, x, y \in E)\}.$$

Moreover,

$$\lim_{t,s\to\infty} \left| \int_0^{\alpha_1(t)} \int_0^{\alpha_2(s)} [k(t,s,u,v,x_2(u,v),y_2(u,v)) - k(t,s,u,v,x_1(u,v),y_1(u,v))]dudv \right| = 0 \tag{4.2}$$

uniformly respect to $x_1, y_1, x_2, y_2 \in E$.

(A_4) The functions $f, g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists an upper semicontinuous and nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Also there exist two bounded functions $a_1, a_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with bound

$$K = \max\{ \sup_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+} a_1(t,s), \sup_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+} a_2(t,s)\}$$

and a positive constant D such that

$$|f(t,s,x_2,y_2) - f(t,s,x_1,y_1)| \leq \frac{a_1(t,s)\varphi(|x_2 - x_1| + |y_2 - y_1|)}{D + \varphi(|x_2 - x_1| + |y_2 - y_1|)},$$

and

$$|g(t,s,x_2,y_2) - g(t,s,x_1,y_1)| \leq \frac{a_2(t,s)\varphi(|x_2 - x_1| + |y_2 - y_1|)}{D + \varphi(|x_2 - x_1| + |y_2 - y_1|)},$$

for all $t, s \in \mathbb{R}_+$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Additionally we assume that φ is superadditive, i.e., $\varphi(t) + \varphi(s) \leq \varphi(t+s)$ for all $t, s \in \mathbb{R}_+$. Moreover, we assume that $2K(1 + M) \leq D$.

(A_5) The functions $H_1, H_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $H_1(t,s) = |f(t,s,0,0)|$ and $H_2(t,s) = |g(t,s,0,0)|$ are bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ with

$$H_0 = \max\{ \sup_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+} H_1(t,s), \sup_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+} H_2(t,s)\}.$$

Theorem 4.1. *If the assumptions (A_1)-(A_5) are satisfied, then the system of equation (1.1) has at least one solution $(x,y) \in E \times E$.*

Proof. Define the operator $T : E \times E \to E$ associated with the integral equation (1.1) by

$$T(x,y)(t,s) = a(t,s) + f(t,s,x(t,s),y(t,s)) + g(t,s,x(t,s),y(t,s))[F(x,y)(t,s)], \tag{4.3}$$

where,

$$F(x,y)(t,s) = \int_0^{\alpha_1(t)} \int_0^{\alpha_2(s)} k(t,s,u,v,x(u,v),y(u,v))dudv. \tag{4.4}$$

Solving Eq. (1.1) is equivalent to finding a coupled fixed point of the operator T defined on the space $E \times E$. For better readability, we break the proof into a sequence of cases.

Case 1: T transforms the space $E \times E$ into E.

By considering conditions of theorem we infer that $T(x,y)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+$. Now we prove that $T(x,y)\in E$ for any $(x,y)\in E\times E$. For arbitrarily fixed $(t,s)\in\mathbb{R}_+\times\mathbb{R}_+$ we have

$$\begin{aligned} |(T(x,y))(t,s)| &\leq |a(t,s)| + |f(t,s,x(t,s),y(t,s))| \\ &\quad + |g(t,s,x(t,s),y(t,s))||F(x,y)(t,s)| \\ &\leq |a(t,s)| + \frac{K\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_0 \\ &\quad + [\frac{K\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_0]M. \end{aligned} \tag{4.5}$$

Indeed,

$$\begin{aligned} |f(t,s,x(t,s),y(t,s))| &\leq |f(t,s,x(t,s),y(t,s)) - f(t,s,0,0)| + |f(t,s,0,0)| \\ &\leq \frac{a_1(t,s)\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_1(t,s) \\ &\leq \frac{K\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_0, \\ |g(t,s,x(t,s),y(t,s))| &\leq |g(t,s,x(t,s),y(t,s)) - g(t,s,0,0)| + |g(t,s,0,0)| \\ &\leq \frac{a_2(t,s)\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_2(t,s) \\ &\leq \frac{K\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_0, \\ |(F(x,y)(t,s)| &= |\int_0^{\alpha_1(t)}\int_0^{\alpha_2(s)} k(t,s,u,v,x(u,v),y(u,v))dudv| \\ &\leq \int_0^{\alpha_1(t)}\int_0^{\alpha_2(s)} |k(t,s,u,v,x(u,v),y(u,v))|dudv \leq M. \end{aligned}$$

By assumption (A_4), we get

$$\begin{aligned} ||T(x,y)|| &\leq ||a|| + (\frac{K\varphi(||x||+||y||)}{D+\varphi(||x||+||y||)} + H_0)(1+M) \\ &\leq ||a|| + (K+H_0)(1+M). \end{aligned} \tag{4.6}$$

Thus T maps the space $E\times E$ into E. More precisely, from (4.6) we obtain that $T(\overline{B}_r\times\overline{B}_r)\subseteq\overline{B}_r$, where $r=||a||+(K+H_0)(1+M)$.

Case 2: we show that map $T:\overline{B}_r\times\overline{B}_r\to\overline{B}_r$ is continuous.
To do this, let us fix arbitrarily $\epsilon>0$ and take $(x,y),(z,w)\in\overline{B}_r\times\overline{B}_r$ such that $||(x,y)-(z,w)||\leq\epsilon$. Then

$$\begin{aligned} &|(T(x,y)(t,s)) - (T(z,w)(t,s))| \\ &= |f(t,s,x(t,s),y(t,s)) + g(t,s,x(t,s),y(t,s))[F(x,y)(t,s)] \\ &\quad - f(t,s,z(t,s),w(t,s)) - g(t,s,z(t,s),w(t,s))[F(z,w)(t,s)]| \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&\le |f(t,s,x(t,s),y(t,s)) - f(t,s,z(t,s),w(t,s))| \\
&\quad + |g(t,s,x(t,s),y(t,s))||(F(x,y))(t,s) - (F(z,w))(t,s)| \\
&\quad + |g(t,s,x(t,s),y(t,s) - g(ts,z(t,s),w(t,s))||(F(z,w))(t,s)| \\
&\le \frac{a_1(t,s)\varphi(|x(t,s)-z(t,s)| + |y(t,s)-w(t,s)|)}{D+\varphi(|x(t,s)-z(t,s)| + |y(t,s)-w(t,s)|)} \\
&\quad + [\frac{K\varphi(|x(t,s)|+|y(t,s)|)}{D+\varphi(|x(t,s)|+|y(t,s)|)} + H_0]|(F(x,y))(t,s) - (F(z,w))(t,s)| \\
&\quad + [\frac{a_2(t,s)\varphi(|x(t,s)-z(t,s)| + |y(t,s)-w(t,s)|)}{D+\varphi(|x(t,s)-z(t,s)| + |y(t,s)-w(t,s)|)}]M \\
&\le \frac{K(1+M)\varphi(||x-z||+||y-w||)}{D+\varphi(||x-z||+||y-w||)} \\
&\quad + [\frac{K\varphi(||x||+||y||)}{D+\varphi(||x||+||y||)} + H_0]|(F(x,y))(t,s) - (F(z,w))(t,s)|.
\end{aligned}
$$

Furthermore, with due attention to the condition (A_2) there exists $T > 0$ such that for $t > T$ we have

(4.8)

$$
\begin{aligned}
&|(F(x,y))(t,s) - (F(z,w))(t,s)| \\
&= |\int_0^{\alpha_1(t)}\int_0^{\alpha_2(s)}[k(t,s,u,v,x(u,v),y(u,v)) - k(t,s,u,v,z(u,v),w(u,v))]dudv| \\
&< \epsilon.
\end{aligned}
$$

Suppose that $t, s > T$. It follows (4.7) and (4.8) that

$$|T(x,y)(t,s) - T(z,w)(t,s)| < \epsilon. \tag{4.9}$$

If $t, s \in [0,T]$, then we obtain

$$|(F(x,y))(t,s) - (F(z,w))(t,s)| \le \alpha_T^2\omega_1(k,\epsilon), \tag{4.10}$$

where we denoted

$$\alpha_T = \sup\{\alpha_i(t) : t \in [0,T], i = 1,2\},$$

and

$$
\begin{aligned}
\omega_1(k,\epsilon) = \sup\{&|k(t,s,u,v,x,y) - k(t,s,u,v,z,w)| : t,s \in [0,T], u,v \in [0,\alpha_T], \\
&x,y,z,w \in [-r,r], ||(x,y)-(z,w)|| \le \epsilon\}.
\end{aligned}
$$

By using the continuity of k on $[0,T]\times[0,T]\times[0,\alpha_T]\times[0,\alpha_T]\times[-r,r]\times[-r,r]$, we have $\omega_1(k,\epsilon) \to 0$ as $\epsilon \to 0$. Now, linking the inequalities (4.7) and (4.10) we deduce that

$$|T(x,y)(t,s) - T(z,w)(t,s)| \le \epsilon + [K + H_0]\alpha_T^2\omega_1(k,\epsilon). \tag{4.11}$$

The above established facts we conclude that T is continuous on $\overline{B}_r \times \overline{B}_r$.

Case 3: In the sequel, we show that for any nonempty set $X_1, X_2 \subseteq \overline{B}_r$,

$$\mu(T(X_1 \times X_2)) \leq \frac{1}{2}\varphi(\mu(X_1) + \mu(X_2)).$$

Indeed, by virtue of assumptions (A_1)-(A_5), we conclude that for any (x, y), $(z, w) \in X_1 \times X_2$ and $t, s \in \mathbb{R}_+$,

$$\begin{aligned}
&|(T(x,y))(t,s) - (T(z,w))(t,s)| \\
&\leq \frac{K(1+M)\varphi(|x(t,s) - z(t,s)| + |y(t,s) - w(t,s)|)}{D + \varphi(|x(t,s) - z(t,s)| + |y(t,s) - w(t,s)|)} \\
&\quad + [\frac{K\varphi(|x(t,s)| + |y(t,s)|}{D + \varphi(|x(t,s)| + |y(t,s)|)} + H_0]\beta(t,s) \\
&\leq \frac{1}{2}\varphi(|x(t,s) - z(t,s)| + |y(t,s) - w(t,s)|) \\
&\quad + [\frac{K\varphi(|x(t,s)| + |y(t,s)|}{D + \varphi(|x(t,s)| + |y(t,s)|)} + H_0]\beta(t,s),
\end{aligned}$$

where

$$\begin{aligned}
\beta(t,s) = \sup\{|\int_0^{\alpha_1(t)}\int_0^{\alpha_2(s)} &[k(t,s,u,v,x(u,v),y(u,v)) \\
&- k(t,s,u,v,z(u,v),w(u,v))]dudv| : x, y \in E\}.
\end{aligned}$$

This estimate allows us to derive the following one

(4.12)

$$\begin{aligned}
&\operatorname{diam}(T(X_1 \times X_2))(t,s) \\
&\leq \frac{1}{2}\varphi(\operatorname{diam}X_1(t,s) + \operatorname{diam}X_2(t,s)) + [\frac{K\varphi(|x(t,s)| + |y(t,s)|}{D + \varphi(|x(t,s)| + |y(t,s)|)} + H_0]\beta(t,s).
\end{aligned}$$

Consequently, in view of the upper semicontinuity of the function φ and from (4.12) and assumption (4.2) that

$$\begin{aligned}
\limsup_{t,s\to\infty} &\operatorname{diam}(T(X_1 \times X_2))(t,s) \\
&\leq \frac{1}{2}\varphi(\limsup_{t,s\to\infty}\operatorname{diam}X_1(t,s) + \limsup_{t,s\to\infty}\operatorname{diam}X_2(t,s)).
\end{aligned} \tag{4.13}$$

Next, fix arbitrarily $T > 0$ and $\epsilon > 0$. Let us choose $t_1, t_2, s_1, s_2 \in [0, T]$, with $|t_2 - t_1| \leq \epsilon$, $|s_2 - s_1| \leq \epsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$ and $s_1 \leq s_2$. Then, for $(x, y) \in X_1 \times X_2$ we get

$$\begin{aligned}
&|f(t_2, s_2, x(t_2,s_2), y(t_2,s_2)) - f(t_1, s_1, x(t_1,s_1), y(t_1,s_1))| \\
&\leq |f(t_2, s_2, x(t_2,s_2), y(t_2,s_2)) - f(t_2, s_2, x(t_1,s_1), y(t_1,s_1))| \\
&\quad + |f(t_2, s_2, x(t_1,s_1), y(t_1,s_1)) - f(t_1, s_1, x(t_1,s_1), y(t_1,s_1))| \\
&\leq \frac{K\varphi(|x(t_2,s_2) - x(t_1,s_1)| + |y(t_2,s_2) - y(t_1,s_1)|)}{D + \varphi(|x(t_2,s_2) - x(t_1,s_1)| + |y(t_2,s_2) - y(t_1,s_1)|)}
\end{aligned}$$

$$+ |f(t_2, s_2, x(t_1, s_1), y(t_1, s_1)) - f(t_1, s_1, x(t_1, s_1), y(t_1, s_1))|$$
$$\leq \frac{1}{2(1+M)}\varphi(\omega^N(x,\epsilon) + \omega^T(y,\epsilon)) + \omega^T(f,\epsilon),$$

and

$$\begin{aligned}
&|(F(x,y)(t_2,s_2) - (F(x,y)(t_1,s_1)| \\
&\leq \int_0^{\alpha_1(t_2)}\int_0^{\alpha_2(s_2)} |k(t_2,s_2,u,v,x(u,v),y(u,v)) \\
&\qquad\qquad - k(t_1,s_1,u,v,x(u,v),y(u,v))|dudv \\
&\quad + \int_{\alpha_1(t_1)}^{\alpha_1(t_2)}\int_{\alpha_2(s_1)}^{\alpha_2(s_2)} |k(t_1,s_1,u,v,x(u,v),y(u,v))|dudv \\
&\leq \int_0^{\alpha_1(t_2)}\int_0^{\alpha_2(s_2)} \omega^T(k,\epsilon)dudv + \int_{\alpha_1(t_1)}^{\alpha_1(t_2)}\int_{\alpha_2(s_1)}^{\alpha_2(s_2)} K^T dudv \\
&\leq \alpha_T^2\omega^T(k,\epsilon) + \omega^T(\alpha_1,\epsilon)\omega^T(\alpha_2,\epsilon)K^T,
\end{aligned}$$

and

$$\begin{aligned}
&|g(t_2,s_2,x(t_2,s_2),y(t_2,s_2))(F(x,y)(t_2,s_2) - g(t_1,s_1,x(t_1,s_1),y(t_1,s_1))(F(x,y)(t_1,s_1)| \\
&\leq |g(t_2,s_2,x(t_2,s_2),y(t_2,s_2))(F(x,y)(t_2,s_2) - g(t_1,s_1,x(t_1,s_1),y(t_1,s_1))(F(x,y)(t_2,s_2)| \\
&\quad + |g(t_1,s_1,x(t_1,s_1),y(t_1,s_1))(F(x,y)(t_2,s_2) - g(t_1,s_1,x(t_1,s_1),y(t_1,s_1))(F(x,y)(t_1,s_1)| \\
&\leq \frac{K\varphi(|x(t_2,s_2)-x(t_1,s_1)| + |y(t_2,s_2)-y(t_1,s_1)|)}{D+\varphi(|x(t_2,s_2)-x(t_1,s_1)| + |y(t_2,s_2)-y(t_1,s_1)|)}|(F(x,y)(t_2,s_2)| \\
&\quad + [\frac{K\varphi(|x(t_1,s_1)| + |y(t_1,s_1)|)}{D+\varphi(|x(t_1,s_1)| + |y(t_1,s_1)|)} + H_0]|(F(x,y)(t_2,s_2) - (F(x,y)(t_1,s_1)| \\
&\leq \frac{M}{2(1+M)}\varphi(\omega^T(x,\epsilon)+\omega^T(y,\epsilon)) + (K+H_0)[\alpha_T^2\omega^T(k,\epsilon) + \omega^T(\alpha_1,\epsilon)\omega^T(\alpha_2,\epsilon)K^T].
\end{aligned}$$

Therefore,

(4.14)

$$\begin{aligned}
&|(T(x,y))(t_2,s_2) - (T(x,y))(t_2,s_2)| \\
&\leq |a(t_2,s_2) - a(t_1,s_1)| + |f(t_2,s_2,x(t_2,s_2),y(t_2,s_2)) - f(t_1,s_1,x(t_1,s_1),y(t_1,s_1))| \\
&\quad + |g(t_2,s_2,x(t_2,s_2),y(t_2,s_2))(F(x,y)(t_2,s_2) - g(t_1,s_1,x(t_1,s_1),y(t_1,s_1))(F(x,y)(t_1,s_1)| \\
&\leq \omega^T(a,\epsilon) + \frac{\lambda}{2(1+M)}(\omega^T(x,\epsilon)+\omega^T(y,\epsilon)) + \omega^T(f,\epsilon) \\
&\quad + \frac{M}{2(1+M)}\varphi(\omega^T(x,\epsilon)+\omega^T(y,\epsilon)) + (K+H_0)[\alpha_T^2\omega^T(k,\epsilon) + \omega^T(\alpha_1,\epsilon)\omega^T(\alpha_2,\epsilon)K^T],
\end{aligned}$$

where we define

$$\begin{aligned}
\omega^T(f,\epsilon) = \sup\{&|f(t_2,s_2,x,y) - f(t_1,s_1,x,y)| : t_1,t_2,s_1,s_2 \in [0,T], \\
&|t_2-t_1| \leq \epsilon, |s_2-s_1| \leq \epsilon, x,y \in [-r,r]\},
\end{aligned}$$

$$\omega^T(k,\epsilon) = \sup\{|k(t_2,s_2,u,v,x,y) - k(t_1,s_1,u,v,x,y)| : t_1,t_2,s_1,s_2 \in [0,T], \\ |t_2-t_1| \le \epsilon, |s_2-s_1| \le \epsilon, u,v \in [0,\alpha_T], x,y \in [-r,r]\},$$
$$\omega^T(\alpha_i,\epsilon) = \sup\{|\alpha_i(t)-\alpha_i(s)| : t,s \in [0,T], |t-s| \le \epsilon, i=1,2\},$$
$$\omega^T(x,\epsilon) = \sup\{|x(t_2,s_2) - x(t_1,s_1)| : t_1,t_2,s_1,s_2 \in [0,T], \\ |t_2-t_1| \le \epsilon, |s_2-s_1| \le \epsilon\},$$
$$K^T = \sup\{|k(t,s,u,v,x,y)| : t,s \in [0,T], u,v \in [0,\alpha_T], x,y \in [-r,r]\},$$
$$\omega^T(a,\epsilon) = \sup\{|a(t_2,s_2) - a(t_1,s_1)| : t_1,t_2,s_1,s_2 \in [0,T], \\ |t_2-t_1| \le \epsilon, |s_2-s_1| \le \epsilon\},$$

Since (x,y) is an arbitrary element of $X_1 \times X_2$, the inequality (4.14) implies that

$$\omega^T(T(X_1\times X_2),\epsilon) \le \omega^T(a,\epsilon) + \frac{1}{2}\varphi(\omega^T(X_1,\epsilon)+\omega^T(X_2,\epsilon)) + \omega^T(f,\epsilon) \\ + (K+H_0)[\alpha_T^2\omega^T(k,\epsilon) + \omega^T(\alpha_1,\epsilon)\omega^T(\alpha_2,\epsilon)K^T]. \tag{4.15}$$

In view of the uniform continuity of the functions a, f and k on $[0,T]\times[0,T]$ and $[0,T]\times[0,T]\times[-r,r]$ and $[0,T]\times[0,T]\times[0,\alpha_T]\times[0,\alpha_T]\times[-r,r]\times[-r,r]$ respectively, we have that $\omega^T(a,\epsilon)\to 0, \omega^T(f,\epsilon)\to 0$ and $\omega^T(k,\epsilon)\to 0$. Moreover, it is obvious that the constant K^T is finite and $\omega^T(\alpha_1,\epsilon)\to 0$ and $\omega^T(\alpha_2,\epsilon)\to 0$ as $\epsilon\to 0$. Thus, linking the established facts with the estimate (4.15) we get

$$\omega_o(T(X_1\times X_2)) \le \frac{1}{2}\varphi(\omega_o(X_1)+\omega_o(X_2)). \tag{4.16}$$

Finally, from (4.13), (4.16) and the definition of the measure of noncompactness μ, we obtain

(4.17)

$$\begin{aligned}
&\mu(T(X_1\times X_2)) \\
&= \omega_0(T(X_1\times X_2)) + \limsup_{t,s\to\infty} \operatorname{diam}(T(X_1\times X_2))(t,s) \\
&\le \frac{1}{2}\varphi(\omega_o(X_1)+\omega_o(X_2)) + \frac{1}{2}\varphi(\limsup_{t,s\to\infty}\operatorname{diam}X_1(t,s) + \limsup_{t,s\to\infty}\operatorname{diam}X_2(t,s)) \\
&\le \frac{1}{2}\varphi(\omega_o(X_1) + \limsup_{t,s\to\infty}\operatorname{diam}X_1(t,s) + \omega_o(X_2) + \limsup_{t,s\to\infty}\operatorname{diam}X_2(t,s)) \\
&= \frac{1}{2}\varphi(\mu(X_1)+\mu(X_2)).
\end{aligned}$$

Finally, applying Corollary 3.14, we obtain the desired result. □

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