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ON WEAKLY 2-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with $1 \neq 0$. In this paper, we introduce the concept of weakly 2-absorbing primary ideal which is a generalization of weakly 2-absorbing ideal. A proper ideal I of R is called a *weakly 2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. A number of results concerning weakly 2-absorbing primary ideals and examples of weakly 2-absorbing primary ideals are given.

1. Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is said to be proper if $I \neq R$. Let I be a proper ideal of R. Then $\sqrt{I} = \{r \in R : r^k \in I \text{ for some } k \in \mathbb{N}\}$ denotes the radical ideal of R and $Z_I(R) = \{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}$. Note that $\sqrt{0}$ is the set (ideal) of all nilpotent elements of R. The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [5] and studied in [3], [12], and [8]. Various generalizations of prime ideals are also studied in [1] and [9].

Recall that a proper ideal I of R is called a 2-*absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Recently (see [7]), the concept of 2-absorbing ideal is extended to the context of 2-absorbing primary ideal which is a generalization of primary ideal. Recall from [7] that a proper ideal of R is said to be a 2-*absorbing primary ideal* of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recall from [2] ([4]) that a proper ideal I of R is called a *weakly prime ideal* (*weakly primary ideal*) if whenever $0 \neq ab \in I$, then $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$). The concept of weakly prime ideal was extended to the context of weakly 2-absorbing ideal. Recall from [6] that a proper ideal I of R is said to be a *weakly* 2-*absorbing*

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ideal of R if whenever $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this paper, we extend the concept of weakly 2-absorbing ideal to the context of weakly 2-absorbing primary ideal. A proper ideal I of R is said to be a *weakly 2-absorbing primary ideal* of R if whenever $a, b, c \in R$ with $0 \neq abc \in I$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Note that every 2-absorbing primary ideal is clearly a weakly 2-absorbing primary ideal. However, the converse is not true. For example, 0 is always a weakly 2-absorbing primary ideal of R, but it is not always a 2-absorbing primary ideal.

Among many results in this paper, it is shown (Example 2.6) that the radical of a weakly 2-absorbing primary ideal of a ring R need not be a weakly 2-absorbing ideal of R. It is shown (Theorem 2.7) that if I is a proper ideal of R such that $\sqrt{I}$ is a weakly prime ideal of R, then I is a weakly 2-absorbing primary ideal of R. It is shown (Theorem 2.10) that if I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary, then $I^3 = 0$. It is shown (Example 2.11) that if $I^3 = 0$ for some proper ideal I of R, then I need not be a weakly 2-absorbing primary ideal of R. It is shown (Theorem 2.14) that if $\sqrt{0}$ is prime and I is a proper ideal of R, then I is a weakly 2-absorbing primary ideal of R if and only if I is a 2-absorbing primary ideal. If $R = R_1 \times \cdots \times R_n$, then a complete characterization of the nonzero weakly 2-absorbing primary ideals of R is determined (Theorem 2.21–Theorem 2.24). It is shown (Theorem 2.25) that every proper ideal of $R = R_1 \times R_2 \times R_3$ is a weakly 2-absorbing primary ideal of R if and only if R_1, R_2, and R_3 are fields. It is shown (Theorem 2.26) that if every proper ideal of R is weakly 2-absorbing primary, then R has at most three incomparable (under inclusion) prime ideals (and hence at most three distinct maximal ideals). It is shown (Theorem 2.30) that if I is a weakly 2-absorbing primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. In the last section, we give alternative proofs to some results in [2].

2. Weakly 2-absorbing primary ideals

Definition 2.1. A proper ideal I of R is called a *weakly 2-absorbing primary* ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Definition 2.2. Let I be a weakly 2-absorbing primary ideal of R. We say (a, b, c) is a *triple-zero* of I if $abc = 0$, $ab \notin I$, $bc \notin \sqrt{I}$, and $ac \notin \sqrt{I}$.

Note that if I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary ideal, then there exists a triple-zero (a, b, c) of I for some $a, b, c \in R$.

We start with the following result. We omit the proof since it is clear by definitions.

Theorem 2.3. *Let I be a proper ideal of R. Then*

1. *If I is a weakly prime ideal, then I is a weakly 2-absorbing primary ideal.*
2. *If I is a weakly 2-absorbing ideal, then I is a weakly 2-absorbing primary ideal.*
3. *If I is a weakly primary ideal, then I is a weakly 2-absorbing primary ideal.*
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5. *If I is a 2-absorbing primary ideal, then I is a weakly 2-absorbing primary ideal.*

Recall that a ring R is called *quasilocal* if it has exactly one maximal ideal. The proof of the following result is clear, and hence we omit the proof.

Theorem 2.4. *Let R be a quasilocal ring with maximal ideal $\sqrt{0}$. Then every proper ideal of R is a weakly 2-absorbing primary ideal of R.*

Theorem 2.5. *Let I be a proper ideal of R. Then $\sqrt{I}$ is a weakly 2-absorbing ideal of R if and only if $\sqrt{I}$ is a weakly 2-absorbing primary ideal of R.*

Proof. Since $\sqrt{\sqrt{I}} = \sqrt{I}$, the proof is completed. □

If I is a 2-absorbing primary ideal of R, then $\sqrt{I}$ is a 2-absorbing ideal of R by [7, Theorem 2.2]. However, if I is a weakly 2-absorbing primary ideal, then $\sqrt{I}$ need not be a weakly 2-absorbing ideal of R. We have the following example.

Example 2.6. Let $A = \mathbb{Z}_2[X, Y, W]$ and $I = X^2Y^2W^2A$ be an ideal of A. Let $R = A/I$. Then I/I is the zero ideal of R, and hence 0 is a weakly 2-absorbing primary ideal of R. We show that $\sqrt{0}$ (in R)$= XYWA/I$ is not a weakly 2-absorbing ideal of R. For in the ring R, we have $0 \neq XYW + I \in \sqrt{0}$, but $XY + I \notin \sqrt{0}$, $XW + I \notin \sqrt{0}$, and $YW + I \notin \sqrt{0}$. Thus $\sqrt{0}$ (in R) is not a weakly 2-absorbing ideal of R.

Let I be a proper ideal of R. Since $\sqrt{I} = \sqrt{\sqrt{I}}$, it is clear that $\sqrt{I}$ is a weakly prime ideal of R if and only if $\sqrt{I}$ is a weakly primary ideal of R. Hence we have the following result.

Theorem 2.7. *Let I be a proper ideal of R such that $\sqrt{I}$ is a weakly prime (weakly primary) ideal of R. Then I is a weakly 2-absorbing primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Suppose that $ab \notin \sqrt{I}$. Since $\sqrt{I}$ is a weakly prime ideal of R, we have $c \in \sqrt{I}$, and thus $ac \in \sqrt{I}$. Suppose that $ab \in \sqrt{I}$. Since $0 \neq abc \in I$ and $ab \in \sqrt{I}$, we have $0 \neq ab \in \sqrt{I}$. Since $\sqrt{I}$ is a weakly prime ideal of R and $0 \neq ab \in \sqrt{I}$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Thus $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Thus I is a weakly 2-absorbing primary ideal of R. □

Theorem 2.8. *Let I be a weakly primary ideal of R that is not primary and J be an ideal of R such that $J \subseteq I$. Then J is a weakly 2-absorbing primary ideal of R. In particular, if L is an ideal of R, then $A = I \cap L$ and $B = IL$ are weakly 2-absorbing primary ideals of R.*

Proof. Since I is a weakly primary ideal of R that is not primary, $\sqrt{I} = \sqrt{0}$ by [4, Theorem 2.2]. Hence $\sqrt{J} = \sqrt{I} = \sqrt{0}$. Let $0 \neq abc \in J$ for some $a, b, c \in R$ and suppose that $ab \notin J$. Since $J \subseteq I$, we have $0 \neq abc \in I$. We consider two cases. **Case one**: Suppose that $ab \notin I$. Since I is weakly primary and $ab \notin I$, we have $c \in \sqrt{J} = \sqrt{I} = \sqrt{0}$. Thus $ac \in \sqrt{J}$. **Case two**: Suppose that $ab \in I$. Since $0 \neq abc \in I$, we have $0 \neq ab \in I$. Since I is a weakly primary ideal of R, we have $a \in I \subseteq \sqrt{0}$ or $b \in \sqrt{0}$. Thus $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. Thus J is a weakly 2-absorbing primary ideal of R. The proof of the "in particular statement" is clear since $A, B \subseteq I$. □

Theorem 2.9. *Let I be a weakly 2-absorbing primary ideal of R and suppose that (a, b, c) is a triple-zero of I for some $a, b, c \in R$. Then*

(1) $abI = bcI = acI = 0$.
(2) $aI^2 = bI^2 = cI^2 = 0$.

Proof. (1) Suppose that $abI \neq 0$. Then there exists $i \in I$ such that $abi \neq 0$. Hence $ab(c+i) \neq 0$. Since $ab \notin I$ and I is weakly 2-absorbing primary, we have $a(c+i) \in \sqrt{I}$ or $b(c+i) \in \sqrt{I}$. So $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, a contradiction. Thus $abI = 0$. Similarly it can be easily verified that $bcI = acI = 0$.

(2) Suppose that $ai_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Hence from (1) we have $a(b+i_1)(c+i_2) = ai_1i_2 \neq 0$. It implies either $a(b+i_1) \in I$ or $a(c+i_2) \in \sqrt{I}$ or $(b+i_1)(c+i_2) \in \sqrt{I}$. Thus $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, a contradiction. Therefore $aI^2 = 0$. Similarly, one can easily show that $bI^2 = cI^2 = 0$. □

Theorem 2.10. *If I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary, then $I^3 = 0$.*

Proof. Suppose that I is a weakly 2-absorbing primary ideal that is not a 2-absorbing primary ideal of R. Then there exists (a, b, c) a triple-zero of I for some $a, b, c \in R$. Assume that $I^3 \neq 0$. Hence $i_1i_2i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. By Theorem 2.9, we obtain $(a+i_1)(b+i_2)(c+i_3) = i_1i_2i_3 \neq 0$. This implies that $(a+i_1)(b+i_2) \in I$ or $(a+i_1)(c+i_3) \in \sqrt{I}$ or $(b+i_2)(c+i_3) \in \sqrt{I}$. Thus we have $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, a contradiction. Thus $I^3 = 0$. □

Corollary 2.11. *If I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary, then $\sqrt{I} = \sqrt{0}$.*

Recall that a ring R is said to be *reduced* if $\sqrt{0} = 0$.

Corollary 2.12. *Let R be a reduced ring and $I \neq 0$ be a proper ideal of R. Then I is a weakly 2-absorbing primary ideal if and only if I is a 2-absorbing primary ideal of R.*

The following example shows that a proper ideal I of R with the property $I^3 = 0$ need not be a weakly 2-absorbing primary ideal of R. We have the following example.

Example 2.13. Let $R = \mathbb{Z}_{90}$. Then $I = \{0, 30, 60\}$ is an ideal of R and clearly $I^3 = 0$. Since $0 \neq 2 \cdot 3 \cdot 5 = 30 \in I$, $2 \cdot 3 = 6 \notin I$, $2 \cdot 5 = 10 \notin \sqrt{I}$, and $3 \cdot 5 = 15 \notin \sqrt{I}$, we conclude that I is not a weakly 2-absorbing primary ideal of R.

Let I be a proper ideal of R. Since $\sqrt{\sqrt{I}} = \sqrt{I}$, we remind the reader again that $\sqrt{I}$ is a prime ideal of R if and only if $\sqrt{I}$ is a primary ideal of R. We have the following result.

Theorem 2.14. *Suppose that $\sqrt{0}$ is a prime (primary) ideal of R. Let I be a proper ideal of R. Then I is a weakly 2-absorbing primary ideal of R if and only if I is a 2-absorbing primary ideal of R.*

Proof. Suppose that I is a weakly 2-absorbing primary ideal of R. Assume that $abc \in I$ for some $a, b, c \in R$. If $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence assume that $abc = 0$ and $ab \notin I$. Since $abc = 0$ and $\sqrt{0}$ is a prime ideal of R, we conclude that $a \in \sqrt{0}$ or $b \in \sqrt{0}$ or $c \in \sqrt{0}$. Since $\sqrt{0} \subseteq \sqrt{I}$, we conclude that $ac \in \sqrt{0} \subseteq \sqrt{I}$ or $bc \in \sqrt{0} \subseteq \sqrt{I}$. Thus I is a 2-absorbing primary ideal of R. The converse is clear. □

Theorem 2.15. *Suppose that $\{0\}$ has a triple-zero (a, b, c) for some $a, b, c \in R$ such that $ab \notin \sqrt{0}$. Let I be a weakly 2-absorbing primary ideal of R. Then I is not a 2-absorbing primary ideal of R if and only if $I \subseteq \sqrt{0}$.*

Proof. Suppose that I is not a 2-absorbing primary ideal of R. Then $I \subseteq \sqrt{0}$ by Corollary 2.11. Conversely, suppose that $I \subseteq \sqrt{0}$. By hypothesis, we conclude that $ab \notin I$, $ac \notin \sqrt{0}$, and $bc \notin \sqrt{0}$. Thus (a, b, c) is a triple-zero of I. Hence I is not a 2-absorbing primary ideal of R. □

Theorem 2.16. *Let $I_1, I_2, \ldots, I_n$ be weakly 2-absorbing primary ideals of R such that every I_i is not 2-absorbing primary. Then $I = \cap_{i=1}^{n} I_i$ is a weakly 2-absorbing primary ideal of R.*

Proof. Observe that $\sqrt{I_i} = \sqrt{0}$ for each $1 \leq i \leq n$ by Corollary 2.11. Thus $\sqrt{I} = \sqrt{0}$. Suppose that $a, b, c \in R$ with $0 \neq abc \in I$ and $ab \notin I$. Then $ab \notin I_k$ for some $1 \leq k \leq n$. Hence $bc \in \sqrt{I_k} = \sqrt{0} = \sqrt{I}$ or $ac \in \sqrt{I_k} = \sqrt{0} = \sqrt{I}$. Hence I is a weakly 2-absorbing ideal of R. □

Theorem 2.17. *Let $f : R \to R'$ be a homomorphism of commutative rings. Then the following statements hold.*

(1) *If f is a monomorphism and J' is a weakly 2-absorbing primary ideal of R', then $f^{-1}(J')$ is a weakly 2-absorbing primary ideal of R.*
(2) *If f is an epimorphism and J is a weakly 2-absorbing primary ideal of R containing* $\mathrm{Ker}(f)$, *then $f(J)$ is a weakly 2-absorbing primary ideal of R'.*

Proof. (1) Let $a, b, c \in R$ such that $0 \neq abc \in f^{-1}(J')$. Since $\mathrm{Ker}(f) = 0$, we get $0 \neq f(abc) = f(a)f(b)f(c) \in J'$. Hence we have $f(a)f(b) \in J'$ or $f(b)f(c) \in \sqrt{J'}$ or $f(a)f(c) \in \sqrt{J'}$, and thus $ab \in f^{-1}(J')$ or $bc \in f^{-1}(\sqrt{J'})$ or $ac \in f^{-1}(\sqrt{J'})$. Since $f^{-1}(\sqrt{J'}) = \sqrt{f^{-1}(J')}$, we conclude that $f^{-1}(J')$ is a weakly 2-absorbing primary ideal of R.

(2) Let $a', b', c' \in R'$ and $0 \neq a'b'c' \in f(J)$. Then there exist $a, b, c \in R$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$ and $0 \neq f(abc) = a'b'c' \in f(J)$. Since $\mathrm{Ker}(f) \subseteq J$, we have $0 \neq abc \in J$. It implies that $ab \in J$ or $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. It means that $a'b' \in f(J)$ or $a'c' \in f(\sqrt{J}) \subseteq \sqrt{f(J)}$ or $b'c' \in f(\sqrt{J}) \subseteq \sqrt{f(J)}$. Thus $f(J)$ is a weakly 2-absorbing primary ideal of R'. □

Theorem 2.18. *Let I, J be proper ideals of R with $I \subseteq J$. Then the followings statements hold.*

1. *If J is a weakly 2-absorbing primary ideal of R, then J/I is a weakly 2-absorbing primary ideal of R/I.*
2. *If I is a 2-absorbing primary ideal of R and J/I is a weakly 2-absorbing primary ideal of R/I, then J is a 2-absorbing primary ideal of R.*
3. *If I is a weakly 2-absorbing primary ideal of R and J/I is a weakly 2-absorbing primary ideal of R/I, then J is a weakly 2-absorbing primary ideal of R.*

Proof. (1) It is obtained from Theorem 2.17.

(2) Let $a, b, c \in R$ and $abc \in J$. If $abc \in I$, then $ab \in I \subseteq J$ or $bc \in \sqrt{I} \subseteq \sqrt{J}$ or $ac \in \sqrt{I} \subseteq \sqrt{J}$. So we may assume that $abc \notin I$. Then we have $I \neq (a+I)(b+I)(c+I) \in J/I$. Since J/I is a weakly 2-absorbing primary ideal of R/I, we conclude $(a+I)(b+I) = ab+I \in J/I$ or $(a+I)(c+I) = ac+I \in \sqrt{J/I}$ or $(b+I)(c+I) = bc+I \in \sqrt{J/I}$. It follows that $ab \in J$ or $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. Thus J is a 2-absorbing primary ideal of R.

(3) Let $a, b, c \in R$ and $0 \neq abc \in J$. Then by a similar argument as in (2), J is a weakly 2-absorbing primary ideal of R. □

If I, J are weakly 2-absorbing primary ideals of a ring R such that $\sqrt{I} = \sqrt{J}$, then $I + J$ need not be a weakly 2-absorbing primary ideal of R. We have the following example.

Example 2.19. Let $A = \mathbb{Z}_2[T, U, W, X, Y]$, $H = (T^2, U^2, WXY + T + U, TU, TW, TX, TY, UW, UX, UY)A$ be an ideal of A, and $R = A/H$. Then by construction of R, $I = (TA + H)/H = \{0, T + H\}$ and $J = (UA + H)/H = \{0, U + H\}$ are weakly 2-absorbing primary ideals of R such that $|I| = |J| = 2$ and $\sqrt{I} = \sqrt{J} = \sqrt{0}$ (in R) $= (T, U, WXY)A/H$. Let $L = I + J = (H + (T, U)A)/H$. Then $\sqrt{L} = \sqrt{0}$ (in R) and L is not a weakly 2-absorbing primary ideal of R. For $0 \neq (W + H)(X + H)(Y + H) = WXY + H = T + U + H \in L$, but $WX + H \notin L$, $WY + H \notin \sqrt{L}$, and $XY + H \notin \sqrt{L}$.

For a commutative ring with $1 \neq 0$, let $Z(R)$ be the set of all zero-divisors of R.

Theorem 2.20. *Let S be a multiplicatively closed subset of R. Then*

(1) *If I is a weakly 2-absorbing primary ideal of R with $I \cap S = \varnothing$, then $S^{-1}I$ is a weakly 2-absorbing primary ideal of $S^{-1}R$.*
(2) *If $S^{-1}I$ is a weakly 2-absorbing primary ideal of $S^{-1}R$ such that $S \cap Z_I(R) = \varnothing$ and $S \cap Z(R) = \varnothing$, then I is a weakly 2-absorbing primary ideal of R.*

Proof. (1) Let $a, b, c \in R$, $s, t, k \in S$ such that $0 \neq \frac{a}{s}\frac{b}{t}\frac{c}{k} \in S^{-1}I$. Then there exists $u \in S$ such that $0 \neq uabc \in I$. Since I is a weakly 2-absorbing primary ideal, we get either $uab \in I$ or $bc \in \sqrt{I}$ or $uac \in \sqrt{I}$. If $uab \in I$, then $\frac{a}{s}\frac{b}{t} = \frac{uab}{ust} \in S^{-1}I$. If $bc \in \sqrt{I}$, then $\frac{b}{t}\frac{c}{k} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$. If $uac \in \sqrt{I}$, then $\frac{a}{s}\frac{c}{k} = \frac{uac}{usk} \in \sqrt{S^{-1}I}$.

(2) Let $a, b, c \in R$ such that $0 \neq abc \in I$. Since $S \cap Z(R) = \varnothing$, we have $0 \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$. It follows either $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ or $\frac{b}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$ or $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$. If $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in S^{-1}I$, then $uab \in I$ for some $u \in S$. Since $u \in S$ and $S \cap Z_I(R) = \varnothing$, we conclude $ab \in I$. If $\frac{b}{1}\frac{c}{1} = \frac{bc}{1} \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$, then there exists $v \in S$ and a positive integer n such that $(vbc)^n = v^n b^n c^n \in I$. Since $v \in S$, we have $v^n \notin Z_I(R)$. Thus $b^n c^n \in I$, and so $bc \in \sqrt{I}$. If $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$, then similarly we obtain $ac \in \sqrt{I}$, and it completes the proof. □

Theorem 2.21. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I be a proper ideal of R_1, and $R = R_1 \times R_2$. Then the following statements are equivalent.*

(1) *$I \times R_2$ is a weakly 2-absorbing primary ideal of R.*
(2) *$I \times R_2$ is a 2-absorbing primary ideal of R.*
(3) *I is a 2-absorbing primary ideal of R_1.*

Proof. (1)⇒(2) Since $I \times R_2 \not\subseteq \sqrt{0}$, we conclude that $I \times R_2$ is a 2-absorbing primary ideal of R by Corollary 2.11.

(2)⇒(3) Suppose that I is not a 2-absorbing primary ideal of R_1. Then there exist $a, b, c \in R_1$ such that $abc \in I$, but $ab \notin I$, $bc \notin \sqrt{I}$, and $ac \notin \sqrt{I}$. Since $(a, 1)(b, 1)(c, 1) \in I \times R_2$, we have $(a, 1)(b, 1) = (ab, 1) \in I \times R_2$ or $(a, 1)(c, 1) = (ac, 1) \in \sqrt{I \times R_2} = \sqrt{I} \times R_2$ or $(b, 1)(c, 1) = (bc, 1) \in \sqrt{I \times R_2} = \sqrt{I} \times R_2$. It follows that $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, a contradiction. Thus I is a 2-absorbing primary ideal of R_1.

(3)⇒(1) Let I be a 2-absorbing primary ideal of R_1. Then $I \times R_2$ is a 2-absorbing primary ideal of R by [7, Theorem 2.23], and therefore (1) holds. □

Theorem 2.22. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1, I_2 be nonzero ideals of R_1 and R_2, respectively, and $R = R_1 \times R_2$. If $I_1 \times I_2$ is a proper ideal of R, then the following statements are equivalent.*

(1) *$I_1 \times I_2$ is a weakly 2-absorbing primary ideal of R.*

(2) *$I_1 = R_1$ and I_2 is a 2-absorbing primary ideal of R_1 or $I_2 = R_2$ and I_1 is a 2-absorbing primary ideal of R_1 or I_1, I_2 are primary ideals of R_1, R_2, respectively.*
(3) *$I_1 \times I_2$ is a 2-absorbing primary ideal of R.*

Proof. (1)⇒(2) Assume that $I_1 \times I_2$ is a weakly 2-absorbing primary ideal of R. If $I_1 = R_1$ ($I_2 = R_2$), then I_2 is a 2-absorbing primary ideal of R_2 (I_1 is a 2-absorbing primary ideal of R_1) by Theorem 2.21. So we may assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Let $a, b \in R_2$ such that $ab \in I_2$ and let $0 \neq x \in I_1$. Then $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \in I_1 \times I_2$. Since I_1 is proper, $(1, a)(1, b) = (1, ab) \notin \sqrt{I_1 \times I_2}$. Hence we have $(x, 1)(1, a) = (x, a) \in I_1 \times I_2$ or $(x, 1)(1, b) = (x, b) \in \sqrt{I_1 \times I_2}$, and so $a \in I_2$ or $b \in \sqrt{I_2}$. Thus I_2 is a primary ideal of R_2. Similarly, it can be easily shown that I_1 is a primary ideal of R_1.

(2)⇒(3) The proof is clear by [7, Theorem 2.23].

(3)⇒(1) It is clear. □

Theorem 2.23. *Let R_1 and R_2 be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$. Then a nonzero proper ideal I of R is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary if and only if one of the following conditions holds.*

(1) *$I = I_1 \times I_2$, where $I_1 \neq R_1$ is a nonzero weakly primary ideal of R_1 that is not primary and $I_2 = 0$ is a primary ideal of R_2.*
(2) *$I = I_1 \times I_2$, where $I_2 \neq R_2$ is a nonzero weakly primary ideal of R_2 that is not primary and $I_1 = 0$ is a primary ideal of R_1.*

Proof. Suppose that I is a nonzero weakly 2-absorbing primary ideal of R that is not 2-absorbing primary ideal. Then $I = I_1 \times I_2$ for some ideals I_1, I_2 of R_1 and R_2, respectively. Assume that $I_1 \neq 0$ and $I_2 \neq 0$. Then I is a 2-absorbing primary ideal of R by Theorem 2.22, a contradiction. Therefore $I_1 = 0$ or $I_2 = 0$. Without loss of generality we may assume that $I_2 = 0$. We show that $I_2 = 0$ is a primary ideal of R_2. Let $a, b \in R_2$ such that $ab \in I_2$, and let $0 \neq x \in I_1$. Since $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \in I$ and $(1, ab) \notin \sqrt{I}$, we obtain $(x, a) = (x, 1)(1, a) \in I$ or $(x, b) = (x, 1)(1, b) \in \sqrt{I}$, and so $a \in I_2$ or $b \in \sqrt{I_2}$. Thus $I_2 = 0$ is a primary ideal of R_2. Next, we show that I_1 is a weakly primary ideal of R_1. Observe that $I_1 \neq R_1$. For if $I_1 = R_1$, then $R_1 \times 0$ is a 2-absorbing primary ideal of R by [7, Theorem 2.23]. Let $0 \neq ab \in I_1$ for some $a, b \in R_1$. Since $0 \neq (a, 1)(b, 1)(1, 0) \in I_1 \times 0$ and $(ab, 1) \notin I_1 \times 0$, we conclude $(a, 0) = (a, 1)(1, 0) \in \sqrt{I_1 \times 0} = \sqrt{I}$ or $(b, 0) = (b, 1)(1, 0) \in \sqrt{I_1 \times 0} = \sqrt{I}$. Thus $a \in I_1$ or $b \in \sqrt{I_1}$, and therefore I_1 is a weakly primary ideal of R_1. Now, we show that I_1 is not primary. Suppose that I_1 is a primary ideal of R_1. Since $I_2 = \{0\}$ is a primary ideal of R_2, we conclude that $I = I_1 \times I_2$ is a 2-absorbing primary ideal of R by [7, Theorem 2.23], a contradiction. Thus I_1 is a weakly primary ideal of R_1 that is not primary.

Conversely, suppose that (1) holds. Assume that $(0, 0) \neq (a, a')(b, b')(c, c') \in I = I_1 \times 0$. Since $a'b'c' = 0$ and $(0, 0) \neq (a, a')(b, b')(c, c') \in I_1 \times 0$, we conclude that $abc \neq 0$. Assume $(a,a')(b,b') \notin I$. We consider three cases. **Case one**: Suppose that $ab \notin I_1$, but $a'b' = 0$. Since I_1 is a weakly primary ideal of R_1, we have $c \in \sqrt{I_1}$. Since $I_2 = 0$ is a primary ideal of R_2, we have $a' = 0$ or $b' \in \sqrt{I_2}$. Thus $(a,a')(c,c') \in \sqrt{I}$ or $(b,b')(c,c') \in \sqrt{I}$. **Case two**: Suppose that $ab \notin I_1$ and $a'b' \neq 0$. Then $(c,c') \in \sqrt{I_1} \times \sqrt{0} = \sqrt{I}$. Thus $(a,a')(c,c') \in \sqrt{I}$ or $(b,b')(c,c') \in \sqrt{I}$. **Case three**: Suppose that $ab \in I_1$, but $a'b' \neq 0$. Since $0 \neq ab \in I_1$ and I_1 is a weakly primary ideal of R_1, we have $a \in I_1$ or $b \in \sqrt{I_1}$. Since $a'b' \neq 0$ and $I_2 = 0$ is a primary ideal of R_2, we have $c' \in \sqrt{I_2}$. Thus $(a,a')(c,c') \in \sqrt{I}$ or $(b,b')(c,c') \in \sqrt{I}$. Hence I is a weakly 2-absorbing primary ideal of R. Since I_1 is not a primary ideal of R_1, I is not a 2-absorbing primary ideal of R by [7, Theorem 2.23]. □

Theorem 2.24. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 < n < \infty$, and R_1, R_2, …, R_n are commutative rings with $1 \neq 0$. Let I be a nonzero proper ideal of R. Then the following statements are equivalent.*

(1) *I is a weakly 2-absorbing primary ideal of R.*
(2) *I is a 2-absorbing primary ideal of R.*
(3) *Either $I = \times_{j=1}^{n} I_j$ such that for some $k \in \{1,\dots,n\}$, I_k is a 2-absorbing primary ideal of R_k, and $I_j = R_j$ for every $j \in \{1,\dots,n\} - \{k\}$, or $I = \times_{j=1}^{n} I_j$ such that for some k, $m \in \{1,\dots,n\}$, I_k is a primary ideal of R_k, I_m is a primary ideal of R_m, and $I_j = R_j$ for every $j \in \{1,\dots,n\} - \{k,m\}$.*

Proof. (1)⇔(2) Since I is a proper ideal of R, we have $I = I_1 \times \cdots \times I_n$, where every I_i is an ideal of R_i, and $I_j \neq R_j$ for some $j \in \{1,\dots,n\}$. Suppose that $I = I_1 \times I_2 \times \cdots \times I_n \neq 0$ is a weakly 2-absorbing primary ideal of R. Then there is an element $0 \neq (a_1,a_2,\dots,a_n) \in I$. Hence $0 \neq (a_1,a_2,\dots,a_n) = (a_1,1,1,\dots,1)(1,a_2,1,\dots,1)\cdots(1,1,\dots,a_n) \in I$ implies there is a $j \in \{1,\dots,n\}$ such that $b_j = 1$ and $(b_1,\dots,b_n) \in \sqrt{I} = \sqrt{I_1} \times \cdots \times \sqrt{I_n}$, where $b_1,\dots,b_n \in \{1,a_1,\dots,a_n\}$. Hence $\sqrt{I_j} = R_j$, and so $I_j = R_j$. Thus $\sqrt{I} \neq \sqrt{0}$, and hence by Corollary 2.11, I is a 2-absorbing primary ideal. The converse is obvious.

(2)⇔(3) It is clear by [7, Theorem 2.24]. □

Theorem 2.25. *Let R_1, R_2 and R_3 be commutative rings with $1 \neq 0$, and let $R = R_1 \times R_2 \times R_3$. Then every proper ideal of R is a weakly 2-absorbing primary ideal of R if and only if R_1, R_2, and R_3 are fields.*

Proof. Suppose that every proper ideal of R is a weakly 2-absorbing primary ideal of R. Without loss of generality, we may assume that R_1 is not a field. Then R_1 has a nonzero proper ideal I. Thus $J = I \times 0 \times 0$ is a weakly 2-absorbing primary ideal of R, which is impossible by Theorem 2.24.

Conversely, suppose that R_1, R_2, R_3 are fields. Then every nonzero proper ideal of R is a 2-absorbing ideal by [5, Theorem 3.4]. Since 0 is always weakly 2-absorbing primary, the proof is completed. □

Theorem 2.26. *Suppose that every proper ideal of R is a weakly 2-absorbing primary ideal. Then R has at most three incomparable* (*under inclusion*) *prime ideals.*

Proof. Deny. Then there are M_1, M_2, M_3, and M_4 incomparable prime ideals of R. Let $I = M_1 \cap M_2 \cap M_3$. Hence $\sqrt{I} = \sqrt{M_1} \cap \sqrt{M_2} \cap \sqrt{M_3}$. Thus $\sqrt{I}$ is not a 2-absorbing ideal of R by [3, Theorem 2.5]. So I is not a 2-absorbing primary ideal of R by [7, Theorem 2.2]. Hence $I^3 = 0$ by Theorem 2.10. Thus $I^3 = M_1^3 M_2^3 M_3^3 = 0 \subseteq M_4$ implies that $M_1 \subseteq M_4$ or $M_2 \subseteq M_4$ or $M_3 \subseteq M_4$, a contradiction. Thus R has at most three incomparable (under inclusion) prime ideals. □

In view of Theorem 2.26, we have the following result.

Corollary 2.27. *Suppose that every proper ideal of R is a weakly 2-absorbing primary ideal. Then R has at most three maximal ideals.*

Definition 2.28. Let I be a weakly 2-absorbing primary ideal of R and suppose that $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. We say I is *free triple-zero with respect to $I_1 I_2 I_3$* if (a, b, c) is not a triple-zero of I for every $a \in I_1, b \in I_2$, and $c \in I_3$.

Conjecture 1. *Let I be a weakly 2-absorbing primary ideal of R and suppose that $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. Then I is free triple-zero with respect to $I_1 I_2 I_3$.*

Lemma 2.29. *Let I be a weakly 2-absorbing primary ideal of a ring R and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal J of R such that (a, b, c) is not a triple-zero of I for every $c \in J$. If $ab \notin I$, then $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.*

Proof. Suppose that $aJ \nsubseteq \sqrt{I}$ and $bJ \nsubseteq \sqrt{I}$. Then $aj_1 \notin \sqrt{I}$ and $bj_2 \notin \sqrt{I}$ for some $j_1, j_2 \in J$. Since (a, b, j_1) is not a triple-zero of I and $abj_1 \in I$ and $ab \notin I$ and $aj_1 \notin \sqrt{I}$, we have $bj_1 \in \sqrt{I}$. Since (a, b, j_2) is not a triple-zero of I and $abj_2 \in I$ and $ab \notin I$ and $bj_2 \notin \sqrt{I}$, we have $aj_2 \in \sqrt{I}$. Now, since $(a, b, j_1 + j_2)$ is not a triple-zero of I and $ab(j_1 + j_2) \in I$ and $ab \notin I$, we have $a(j_1 + j_2) \in \sqrt{I}$ or $b(j_1 + j_2) \in \sqrt{I}$. Suppose that $a(j_1 + j_2) = aj_1 + aj_2 \in \sqrt{I}$. Since $aj_2 \in \sqrt{I}$, we have $aj_1 \in \sqrt{I}$, a contradiction. Suppose that $b(j_1 + j_2) = bj_1 + bj_2 \in \sqrt{I}$. Since $bj_1 \in \sqrt{I}$, we have $bj_2 \in \sqrt{I}$, a contradiction again. Thus $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$. □

Remark 1. Let I be a weakly 2-absorbing primary ideal of R and suppose that $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is free triple-zero with respect to $I_1 I_2 I_3$. Then if $a \in I_1, b \in I_2$, and $c \in I_3$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Let I be a weakly 2-absorbing primary ideal of R. In view of the below result, one can see that Conjecture 1 is valid if and only if whenever $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq \sqrt{I}$ or $I_1 I_3 \subseteq \sqrt{I}$.

Theorem 2.30. *Let I be a weakly 2-absorbing primary ideal of R and suppose that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$. Then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.*

Proof. Suppose that I is a weakly 2-absorbing primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$. Suppose that $I_1I_2 \not\subseteq I$. By Remark 1, we proceed with the same argument as in the proof of [7, Theorem 2.19]. We show that $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. Suppose that neither $I_1I_3 \subseteq \sqrt{I}$ nor $I_2I_3 \subseteq \sqrt{I}$. Then there are $q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$. Since $q_1q_2I_3 \subseteq I$ and neither $q_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we have $q_1q_2 \in I$ by Lemma 2.29.

Since $I_1I_2 \not\subseteq I$, we have $ab \notin I$ for some $a \in I_1$, $b \in I_2$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq \sqrt{I}$ or $bI_3 \subseteq \sqrt{I}$ by Lemma 2.29. We consider three cases. **Case one**: Suppose that $aI_3 \subseteq \sqrt{I}$, but $bI_3 \not\subseteq \sqrt{I}$. Since $q_1bI_3 \subseteq I$ and neither $bI_3 \subseteq \sqrt{I}$ nor $q_1I_3 \subseteq \sqrt{I}$, we conclude that $q_1b \in I$ by Lemma 2.29. Since $(a+q_1)bI_3 \subseteq I$ and $aI_3 \subseteq \sqrt{I}$, but $q_1I_3 \not\subseteq \sqrt{I}$, we conclude that $(a+q_1)I_3 \not\subseteq \sqrt{I}$. Since neither $bI_3 \subseteq \sqrt{I}$ nor $(a+q_1)I_3 \subseteq \sqrt{I}$, we conclude that $(a+q_1)b \in I$ by Lemma 2.29. Since $(a+q_1)b = ab + q_1b \in I$ and $q_1b \in I$, we conclude that $ab \in I$, a contradiction. **Case two**: Suppose that $bI_3 \subseteq \sqrt{I}$, but $aI_3 \not\subseteq \sqrt{I}$. Since $aq_2I_3 \subseteq I$ and neither $aI_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we conclude that $aq_2 \in I$. Since $a(b+q_2)I_3 \subseteq I$ and $bI_3 \subseteq \sqrt{I}$, but $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b+q_2)I_3 \not\subseteq \sqrt{I}$. Since neither $aI_3 \subseteq \sqrt{I}$ nor $(b+q_2)I_3 \subseteq \sqrt{I}$, we conclude that $a(b+q_2) \in I$ by Lemma 2.29. Since $a(b+q_2) = ab + aq_2 \in I$ and $aq_2 \in I$, we conclude that $ab \in I$, a contradiction. **Case three**: Suppose that $aI_3 \subseteq \sqrt{I}$ and $bI_3 \subseteq \sqrt{I}$. Since $bI_3 \subseteq \sqrt{I}$ and $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b+q_2)I_3 \not\subseteq \sqrt{I}$. Since $q_1(b+q_2)I_3 \subseteq I$ and neither $q_1I_3 \subseteq \sqrt{I}$ nor $(b+q_2)I_3 \subseteq \sqrt{I}$, we conclude that $q_1(b+q_2) = q_1b + q_1q_2 \in I$ by Lemma 2.29. Since $q_1q_2 \in I$ and $q_1b+q_1q_2 \in I$, we conclude that $bq_1 \in I$. Since $aI_3 \subseteq \sqrt{I}$ and $q_1I_3 \not\subseteq \sqrt{I}$, we conclude that $(a+q_1)I_3 \not\subseteq \sqrt{I}$. Since $(a+q_1)q_2I_3 \subseteq I$ and neither $q_2I_3 \subseteq \sqrt{I}$ nor $(a+q_1)I_3 \subseteq \sqrt{I}$, we conclude that $(a+q_1)q_2 = aq_2 + q_1q_2 \in I$ by Lemma 2.29. Since $q_1q_2 \in I$ and $aq_2+q_1q_2 \in I$, we conclude that $aq_2 \in I$. Now, since $(a+q_1)(b+q_2)I_3 \subseteq I$ and neither $(a+q_1)I_3 \subseteq \sqrt{I}$ nor $(b+q_2)I_3 \subseteq \sqrt{I}$, we conclude that $(a+q_1)(b+q_2) = ab + aq_2 + bq_1 + q_1q_2 \in I$ by Lemma 2.29. Since $aq_2, bq_1, q_1q_2 \in I$, we have $aq_2 + bq_1 + q_1q_2 \in I$. Since $ab + aq_2 + bq_1 + q_1q_2 \in I$ and $aq_2 + bq_1 + q_1q_2 \in I$, we conclude that $ab \in I$, a contradiction. Hence $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. □

3. A visit to weakly prime ideals and weakly 2-absorbing ideals

Definition 3.1. Let I be a weakly prime ideal of R. We say (a, b) is a *twin-zero* of I if $ab = 0$, $a \notin I$, and $b \notin I$.

In this section, we use the concept "twin-zero" in order to give alternative proofs to some results in [2].

Note that if I is a weakly prime ideal of R that is not a prime ideal, then I has a twin-zero (a, b) for some $a, b \in R$.

Theorem 3.2. *Let I be a weakly prime ideal of R and suppose that (a, b) is a twin-zero of I for some $a, b \in R$. Then $aI = bI = 0$.*

Proof. Suppose that $aI \neq 0$. Then there exists $i \in I$ such that $ai \neq 0$. Hence $a(b + i) \neq 0$. Since $a \notin I$ and I is weakly prime, we have $b + i \in I$, and hence $b \in I$, a contradiction. Thus $aI = 0$. Similarly, it can be easily verified that $bI = 0$. □

Theorem 3.3 ([2, Theorem 1]). *Let I be a weakly prime ideal of R. If I is not prime, then $I^2 = 0$.*

Proof. Let (a, b) be a twin-zero of I. Suppose that $i_1 i_2 \neq 0$ for some $i_1, i_2 \in I$. Then by Theorem 3.2, we have $(a + i_1)(b + i_2) = i_1 i_2 \neq 0$. Thus $(a + i_1) \in I$ or $(b + i_2) \in I$, and hence $a \in I$ or $b \in I$, a contradiction. Therefore $I^2 = 0$. □

Theorem 3.4 ([2, Theorem 4]). *Let I be a weakly prime ideal of R. If I is not prime, then $I \subseteq \sqrt{0}$ and $I\sqrt{0} = 0$.*

Proof. Suppose that I is not prime. Then $I \subseteq \sqrt{0}$ by Theorem 3.3. Let $w \in \sqrt{0}$. If $w \in I$, then $wI = 0$ by Theorem 3.3. Thus assume that $w \notin I$ and $wI \neq 0$. Hence $wi \neq 0$ for some $i \in I$. Let m be the least positive integer such that $w^m = 0$. Since $w(w^{m-1} + i) = wi \neq 0$ and $w \notin I$, we have $w^{m-1} + i \in I$, and hence $w^{m-1} \in I$. Since $0 \neq w^{m-1} \in I$ and I is weakly prime, we conclude that $w \in I$, a contradiction. Thus $wI = 0$. Hence $I\sqrt{0} = 0$. □

Theorem 3.5. *Let I be a weakly prime ideal of R and suppose that (a, b) is a twin-zero of I. If $ar \in I$ for some $r \in R$, then $ar = 0$.*

Proof. Suppose that $0 \neq ar \in I$ for some $r \in R$. Then $r \in I$. Thus $ar = 0$ by Theorem 3.2, a contradiction. □

Theorem 3.6. *Let I be a weakly prime ideal of R and suppose that $AB \subseteq I$ for some ideals A, B of R. If I has a twin-zero (a, b) for some $a \in A$ and $b \in B$, then $AB = 0$.*

Proof. Suppose that I has a twin-zero (a, b) for some $a \in A$ and $b \in B$ and assume that $cd \neq 0$ for some $c \in A$ and $d \in B$. Then $c \in I$ or $d \in I$. Without loss of generality, we may assume that $c \in I$. Since $I^2 = 0$ by Theorem 3.2 and $0 \neq cd \in I$, we conclude that $d \notin I$. Since $ad \in I$, we have $ad = 0$ by Theorem 3.5. Since $(a + c)d = cd \neq 0$ and $d \notin I$, we have $a + c \in I$. Hence $a \in I$, a contradiction. Thus $AB = 0$. □

Theorem 3.7 ([2, Theorem 3(4)]). *Let I be a weakly prime ideal of R and suppose that $0 \neq AB \subseteq I$ for some ideals A, B of R. Then $A \subseteq I$ or $B \subseteq I$.*

Proof. Since $0 \neq AB \subseteq I$, we conclude that for every $a \in A$ and $b \in B$, we have $a \in I$ or $b \in I$ by Theorem 3.6. Without loss of generality, assume that $B \not\subseteq I$. Hence $b \notin I$ for some $b \in B$. Let $a \in A$. Since $ab \in I$ and $b \notin I$, we have $a \in I$. Thus $A \subseteq I$. □

We recall the following definition from [6].

Definition 3.8. Let I be a weakly 2-absorbing ideal of a ring R and $a, b, c \in R$. We say (a, b, c) is *a triple-zero of* I if $abc = 0$, $ab \notin I$, $bc \notin I$, and $ac \notin I$.

Definition 3.9. Let I be a weakly 2-absorbing ideal of R and suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. We say I is *free triple-zero with respect to* $I_1I_2I_3$ if (a, b, c) is not a triple-zero of I for every $a \in I_1, b \in I_2$, and $c \in I_3$.

Conjecture 2. *Let I be a weakly 2-absorbing ideal of R and suppose that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. Then I is free triple-zero with respect to $I_1I_2I_3$.*

Lemma 3.10. *Let I be a weakly 2-absorbing ideal of a ring R and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal J of R such that (a, b, c) is not a triple-zero of I for every $c \in J$. If $ab \notin I$, then $aJ \subseteq I$ or $bJ \subseteq I$.*

Proof. Suppose that $aJ \not\subseteq I$ and $bJ \not\subseteq I$. Then $aj_1 \notin I$ and $bj_2 \notin I$ for some $j_1, j_2 \in J$. Since (a, b, j_1) is not a triple-zero of I and $abj_1 \in I$ and $ab \notin I$ and $aj_1 \notin I$, we have $bj_1 \in I$. Since (a, b, j_2) is not a triple-zero of I and $abj_2 \in I$ and $ab \notin I$ and $bj_2 \notin I$, we have $aj_2 \in I$. Now, since $(a, b, j_1 + j_2)$ is not a triple-zero of I and $ab(j_1 + j_2) \in I$ and $ab \notin I$, we have $a(j_1 + j_2) \in I$ or $b(j_1 + j_2) \in I$. Suppose that $a(j_1 + j_2) = aj_1 + aj_2 \in I$. Since $aj_2 \in I$, we have $aj_1 \in I$, a contradiction. Suppose that $b(j_1 + j_2) = bj_1 + bj_2 \in I$. Since $bj_1 \in I$, we have $bj_2 \in I$, a contradiction again. Thus $aJ \subseteq I$ or $bJ \subseteq I$. □

Remark 2. Let I be a weakly 2-absorbing ideal of R and suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$. Then if $a \in I_1, b \in I_2$, and $c \in I_3$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Let I be a weakly 2-absorbing ideal of R. In view of the below result, one can see that Conjecture 2 is valid if and only if whenever $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.

Theorem 3.11. *Let I be a weakly 2-absorbing ideal of R and suppose that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$. Then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.*

Proof. Suppose that I is a weakly 2-absorbing ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that such that I is free triple-zero with respect to $I_1I_2I_3$. Suppose that $I_1I_2 \not\subseteq I$. By Remark 2, we proceed with a similar argument as in the proof of [7, Theorem 2.19]. We show that $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Suppose that neither $I_1I_3 \subseteq I$ nor $I_2I_3 \subseteq I$. Then there are

$q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1I_3 \subseteq I$ nor $q_2I_3 \subseteq I$. Since $q_1q_2I_3 \subseteq I$ and neither $q_1I_3 \subseteq I$ nor $q_2I_3 \subseteq I$, we have $q_1q_2 \in I$ by Lemma 3.10.

Since $I_1I_2 \not\subseteq I$, we have $ab \notin I$ for some $a \in I_1$, $b \in I_2$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq I$ or $bI_3 \subseteq I$ by Lemma 3.10. We consider three cases. **Case one**: Suppose that $aI_3 \subseteq I$, but $bI_3 \not\subseteq I$. Since $q_1bI_3 \subseteq I$ and neither $bI_3 \subseteq I$ nor $q_1I_3 \subseteq I$, we conclude that $q_1b \in I$ by Lemma 3.10. Since $(a+q_1)bI_3 \subseteq I$ and $aI_3 \subseteq I$, but $q_1I_3 \not\subseteq I$, we conclude that $(a+q_1)I_3 \not\subseteq I$. Since neither $bI_3 \subseteq I$ nor $(a+q_1)I_3 \subseteq I$, we conclude that $(a+q_1)b \in I$ by Lemma 3.10. Since $(a+q_1)b = ab + q_1b \in I$ and $q_1b \in I$, we conclude that $ab \in I$, a contradiction. **Case two**: Suppose that $bI_3 \subseteq I$, but $aI_3 \not\subseteq I$. Since $aq_2I_3 \subseteq I$ and neither $aI_3 \subseteq I$ nor $q_2I_3 \subseteq I$, we conclude that $aq_2 \in I$. Since $a(b+q_2)I_3 \subseteq I$ and $bI_3 \subseteq I$, but $q_2I_3 \not\subseteq I$, we conclude that $(b+q_2)I_3 \not\subseteq I$. Since neither $aI_3 \subseteq I$ nor $(b+q_2)I_3 \subseteq I$, we conclude that $a(b+q_2) \in I$ by Lemma 3.10. Since $a(b+q_2) = ab + aq_2 \in I$ and $aq_2 \in I$, we conclude that $ab \in I$, a contradiction. **Case three**: Suppose that $aI_3 \subseteq I$ and $bI_3 \subseteq I$. Since $bI_3 \subseteq I$ and $q_2I_3 \not\subseteq I$, we conclude that $(b+q_2)I_3 \not\subseteq I$. Since $q_1(b+q_2)I_3 \subseteq I$ and neither $q_1I_3 \subseteq I$ nor $(b+q_2)I_3 \subseteq I$, we conclude that $q_1(b+q_2) = q_1b + q_1q_2 \in I$ by Lemma 3.10. Since $q_1q_2 \in I$ and $q_1b + q_1q_2 \in I$, we conclude that $bq_1 \in I$. Since $aI_3 \subseteq I$ and $q_1I_3 \not\subseteq I$, we conclude that $(a+q_1)I_3 \not\subseteq I$. Since $(a+q_1)q_2I_3 \subseteq I$ and neither $q_2I_3 \subseteq I$ nor $(a+q_1)I_3 \subseteq I$, we conclude that $(a+q_1)q_2 = aq_2 + q_1q_2 \in I$ by Lemma 3.10. Since $q_1q_2 \in I$ and $aq_2 + q_1q_2 \in I$, we conclude that $aq_2 \in I$. Now, since $(a+q_1)(b+q_2)I_3 \subseteq I$ and neither $(a+q_1)I_3 \subseteq I$ nor $(b+q_2)I_3 \subseteq I$, we conclude that $(a+q_1)(b+q_2) = ab + aq_2 + bq_1 + q_1q_2 \in I$ by Lemma 3.10. Since $aq_2, bq_1, q_1q_2 \in I$, we have $aq_2 + bq_1 + q_1q_2 \in I$. Since $ab + aq_2 + bq_1 + q_1q_2 \in I$ and $aq_2 + bq_1 + q_1q_2 \in I$, we conclude that $ab \in I$, a contradiction. Hence $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. □

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