

BROYDEN'S METHOD FOR OPERATORS WITH REGULARLY CONTINUOUS DIVIDED DIFFERENCES

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ABSTRACT. We present a new convergence analysis of popular Broyden's method in the Banach/Hilbert space setting which is applicable to non-smooth operators. Moreover, we do not assume a priori solvability of the equation under consideration. Nevertheless, without these simplifying assumptions our convergence theorem implies existence of a solution and superlinear convergence of Broyden's iterations. To demonstrate practical merits of Broyden's method, we use it for numerical solution of three nontrivial infinite-dimensional problems.

1. Introduction

Trying to overcome disadvantages of Newton's method

$$(1.1) \quad x_+ := x - \mathbf{f}'(x)^{-1}\mathbf{f}(x)$$

for solving systems of nonlinear equations

$$(1.2) \quad \mathbf{f}(x) = 0, \mathbf{f} : \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n,$$

C. Broyden proposed in [2] to replace the inverse Jacobian $\mathbf{f}'(x)^{-1}$ in (1.1) by some its approximation (an $n \times n$ matrix) \mathbf{A} satisfying the equation

$$(1.3) \quad \mathbf{A}(\mathbf{f}(x) - \mathbf{f}(x_-)) = x - x_-,$$

where x_- denotes the iteration preceding the current one x . Since this equation does not define \mathbf{A} uniquely, it determines a class of iterative methods that Broyden refers to as *quasi-Newton* methods. Particular methods of this class differ by the choice of \mathbf{A} among solutions of (1.3). Broyden's choice advocated in [2] is

$$\mathbf{A}_+ := \mathbf{A} - \frac{\mathbf{A}\mathbf{f}(x_+)}{\mathbf{f}(x)^T\mathbf{A}^T\mathbf{A}(\mathbf{f}(x_+) - \mathbf{f}(x))}\mathbf{f}(x)^T\mathbf{A}^T\mathbf{A}.$$

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Broyden's method has proved to be successful in practice for solving (1.2). Being paraphrased for operator equations

$$(1.4) \quad \mathbf{f}(x) = 0, \mathbf{f}: \mathbb{X} \supset D \rightarrow \mathbb{H},$$

where \mathbf{f} acts from a subset D of a Banach space \mathbb{X} into a Hilbert one \mathbb{H} , it becomes

$$(1.5) \quad x_+ := x - \mathbf{A}\mathbf{f}(x), \mathbf{A}_+ := \mathbf{A} - \frac{\mathbf{A}\mathbf{f}(x_+)}{\langle \mathbf{A}^*\mathbf{A}\mathbf{f}(x), \mathbf{f}(x_+) - \mathbf{f}(x) \rangle} \langle \mathbf{A}^*\mathbf{A}\mathbf{f}(x), \cdot \rangle.$$

Here \mathbf{A}^* stands for the adjoint of \mathbf{A} and $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{H} . Broyden's method has since been applied to infinite-dimensional problems [13, 14, 15] under various simplifying assumptions. There are also convergence analyses of the method in general Hilbert and Banach spaces [10, 21]. In all these studies, operators under discussion are assumed to be Lipschitz smooth. Similar assumptions are made in [22]. Since the method does not involve evaluation of derivatives of the operator, it can be used for solving equations with nondifferentiable operators. In this case, the above mentioned analyses are unapplicable. Another limiting postulate present (explicitly or implicitly) in existing analyses of Broyden's method is solvability of the equation (1.4). This assumption is often difficult to verify a priori. So, it should be dropped too. Moreover, the focus of the analysis is most often on conditions that ensure superlinear convergence property of Broyden's method, well known from its analyses in finite-dimensional spaces and their extensions to Hilbert ones [8, 16]. This property is proved also [20], when Lipschitz smoothness of the operator is not assumed.

The purpose of the present paper is to offer a new analysis of Broyden's method in Banach/Hilbert space setting free of unnecessary assumptions restricting applicability of other analyses available in literature. Our analysis is based on Kantorovich's majorization principle, which stems from Kantorovich's investigation of Newton's method in Banach space setting [11, Ch. XVIII]. However, the analysis we are going to present here differs from the others not only in methodology, but in several other aspects too:

- It is applicable to nonsmooth operators, since differentiability of \mathbf{f} is not assumed. Instead, we require that the selected divided difference operator $[x_1, x_2 | \mathbf{f}]$ of \mathbf{f} be *regularly continuous* [5, 7].

- No assumption about the existence of solution is made. On the contrary, our convergence condition, which involves only the starter (x_0, \mathbf{A}_0) , guarantees its existence.

- We show that if the sequence (x_n, \mathbf{A}_n) generated by the method (1.5) converges (to $(x_\infty, \mathbf{A}_\infty)$), it does to a solution of the system $\mathbf{f}(x) = 0$ & $\mathbf{A}[x, x | \mathbf{f}] = \mathbf{I}$, where \mathbf{I} is the identity operator: $\mathbf{I}x = x, \forall x$. This property of the method is very helpful to those interested in solution's sensitivity to small perturbations in the data.

- The bound on the error $\|x_\infty - x_n\|$ obtained implies superlinear rate of convergence. This bound, however, allows to estimate rate of convergence not only asymptotically, but beginning with initial iterations. Moreover, it suggests a new criterion for comparison of efficiency of various iterative methods reflecting their common goal: locate a solution of the equation (1.4).

- The existence condition, the existence and uniqueness radii are shown to be exact.

- The approach to the analysis of the method (1.5) adopted in this paper provides yet another example (in addition to those considered in [6]) of the role invariants of systems of difference equations can play in convergence analysis of iterative methods.

The use of regularly continuous divided differences makes necessary recalling some definitions and basic facts from [7] needed for understanding subsequent developments. This material makes up the next section. Section 3 contains derivation of the majorant generator and the convergence lemma preparing convergence theorem in Section 4. The short fifth section contains brief discussion of rate of convergence of Broyden's method. Then we test the method numerically on three nontrivial infinite-dimensional (including nonsmooth) problems to demonstrate method's practicability.

2. Regularly continuous divided differences

Though Broyden's method (1.5) is defined for any operator \mathbf{f} acting from a convex subset D of a Banach space (say \mathbb{X}) into a Hilbert one, the concept of divided difference operator (briefly dd) makes sense in Banach spaces. So, recalling its definition we are speaking in this section about Banach spaces.

Definition 2.1. A linear bounded operator \mathbf{A} from \mathbb{X} into \mathbb{Y} is called a divided difference operator, if for any given pair (x_1, x_2) of points of D it satisfies the (secant) equation

$$\mathbf{A}(x_1 - x_2) = \mathbf{f}(x_1) - \mathbf{f}(x_2).$$

To emphasize its dependence on x_1, x_2 , and \mathbf{f} , such an operator is denoted by the symbol $[x_1, x_2 | \mathbf{f}]$.

For given $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, linear operators satisfying the equation $\mathbf{A}x = y$ constitute an affine manifold in the space $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ of all bounded linear operators between \mathbb{X} and \mathbb{Y} :

$$\mathbf{A}_0 x = y \ \& \ \mathbf{A} x = y \implies (\mathbf{A} - \mathbf{A}_0)x = 0 \implies \mathbf{A} \in \mathbf{A}_0 + \mathcal{L}_x,$$

where $\mathcal{L}_x \subset \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is the subspace of operators vanishing on x . So, the symbol $[x_1, x_2 | \mathbf{f}]$ should be understood as the notation for this manifold or, more precisely, as its particular representative selected from it according to a certain rule specified in advance. If $[x_1, x_2 | \mathbf{f}]$ is selected to be continuous at x_1 with respect to x_2 , then it is easy to see that $[x_1, x_1 | \mathbf{f}] = \mathbf{f}'(x_1)$. Otherwise, the limit

$$\lim_{t \rightarrow +0} [x, x + th | \mathbf{f}]h = \lim_{t \rightarrow +0} t^{-1}[\mathbf{f}(x + th) - \mathbf{f}(x)]$$

(if it exists) may vary depending on h , $\|h\| = 1$. In this case, the limit is the directional derivative $\mathbf{f}'(x, h)$ of \mathbf{f} in the direction h . The following proposition lists for the record obvious properties of the set-valued map $\mathbf{f} \mapsto [x_1, x_2 | \mathbf{f}]$.

Proposition 2.2. (1) $[x_1, x_2 | \mathbf{f}] = \mathbf{f}$ if and only if \mathbf{f} is linear.

(2) $[x_1, x_2 | \alpha \mathbf{f}_1 + \beta \mathbf{f}_2] = \alpha [x_1, x_2 | \mathbf{f}_1] + \beta [x_1, x_2 | \mathbf{f}_2]$.

(3) If \mathbf{f} is a composition of the operators $\mathbf{f}_1, \mathbf{f}_2$: $\mathbf{f} = \mathbf{f}_1 \circ \mathbf{f}_2$, then

$$[x_1, x_2 | \mathbf{f}] = [\mathbf{f}_2(x_1), \mathbf{f}_2(x_2) | \mathbf{f}_1] [x_1, x_2 | \mathbf{f}_2].$$

Let \mathcal{N} be the class of nondecreasing concave (and so continuous) functions $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$. The typical representatives of this class are the functions $t \mapsto ct^p$, $0 < p \leq 1$.

Definition 2.3. The dd $[x_1, x_2 | \mathbf{f}]$ is said to be ω -regularly continuous on D , if an $\omega \in \mathcal{N}$ (call it *regularity modulus*) is known such that

$$(2.1) \quad \begin{aligned} & \omega^{-1}(\min\{\|[x_1, x_2 | \mathbf{f}]\|, \|[u_1, u_2 | \mathbf{f}]\|\}) + \|[x_1, x_2 | \mathbf{f}] - [u_1, u_2 | \mathbf{f}]\| \\ & - \omega^{-1}(\min\{\|[x_1, x_2 | \mathbf{f}]\|, \|[u_1, u_2 | \mathbf{f}]\|\}) \leq \|x_1 - u_1\| + \|x_2 - u_2\| \end{aligned}$$

for all $x_1, x_2, u_1, u_2 \in D$. We say also that it is *regularly continuous* on D , if it has there a regularity modulus.

Concavity is understood in the sense of convex analysis [17], that is ω is concave if it has convex subgraph $\{(s, t) \mid s \geq 0 \ \& \ t \leq \omega(s)\}$. Analogously, ω^{-1} denotes the function whose closed epigraph $cl\{(s, t) \mid s \geq 0 \ \& \ t \geq \omega^{-1}(s)\}$ is symmetrical to closure of the subgraph of ω with respect to the axis $t = s$. Clearly, ω^{-1} is a convex function on $[0, \infty)$ vanishing at zero, increasing in $[0, \omega(\infty)]$, and equal to ∞ for all $s > \omega(\infty)$ (if any).

Being monotone, the function ω has left- and right-hand derivatives at each $s \geq 0$ (they coincide everywhere except, perhaps, for a countable subset of $[0, \infty)$). In what follows, we denote the right-hand and two-sided derivatives by the usual prime ($'$), whereas the backprime ($'\backslash$) will symbolize the left-hand derivative.

If ω is linear: $\omega(t) = ct$, then (2.1) becomes

$$(2.2) \quad \|[x_1, x_2 | \mathbf{f}] - [u_1, u_2 | \mathbf{f}]\| \leq c(\|x_1 - u_1\| + \|x_2 - u_2\|), \quad \forall x_1, x_2, u_1, u_2 \in D.$$

The dd $[x_1, x_2 | \mathbf{f}]$ possessing this property is called *Lipschitz continuous*. For a discussion on how a regularity modulus can be found for a selected dd the reader is referred to [7].

We conclude this section with an immediate consequence of ω -regular continuity of a dd $[x_1, x_2 | \mathbf{f}]$, which will be of use in the next section.

Lemma 2.4 ([5]). *If dd $[x_1, x_2 | \mathbf{f}]$ is ω -regularly continuous on D , then*

$$\begin{aligned} & \left| \omega^{-1}(\|[x_1, x_2 | \mathbf{f}]\|) - \omega^{-1}(\|[u_1, u_2 | \mathbf{f}]\|) \right| \\ & \leq \|x_1 - u_1\| + \|x_2 - u_2\|, \quad \forall x_1, x_2, u_1, u_2 \in D. \end{aligned}$$

It follows that

$$(2.3) \quad \omega^{-1}(\|[x_1, x_2 | \mathbf{f}]\|) \geq \left(\omega^{-1}(\|[u_1, u_2 | \mathbf{f}]\|) - \|x_1 - u_1\| - \|x_2 - u_2\| \right)^+,$$

where the superscript $+$ denotes the nonnegative part of a real number:

$$r^+ := \max\{r, 0\}.$$

3. Majorant generator and convergence lemma

As seen from (1.5), $\mathbf{A}_+(f(x_+) - f(x)) = -\mathbf{A}f(x) = x_+ - x$ and \mathbf{A}_+ is invertible provided \mathbf{A} is invertible and $x_+ \neq x$ (that is $f(x) \neq 0$). So, invertibility of \mathbf{A}_0 implies invertibility of all \mathbf{A}_n generated by Broyden's method and the equality

$$(3.1) \quad \mathbf{A}_n = [x_n, x_{n-1} | \mathbf{f}]^{-1}.$$

If \mathbf{f} is differentiable at x_0 and the derivative $\mathbf{f}'(x_0)$ is boundedly invertible, then the natural choice for \mathbf{A}_0 is $\mathbf{f}'(x_0)^{-1}$. However, we cannot recommend this choice in general, since we do not assume here differentiability of \mathbf{f} . What we do assume is that \mathbf{A}_0 is invertible, so that \mathbf{A} and \mathbf{f} in (1.5) can be replaced by their normalizations $\mathbf{A}\mathbf{A}_0^{-1}$ and $\mathbf{A}_0\mathbf{f}$ without affecting the method. To save indexation, let us assume (with no loss in generality) that \mathbf{A} and \mathbf{f} are already normalized:

$$(3.2) \quad \mathbf{A}_0 = [x_0, x_{-1} | \mathbf{f}]^{-1} = \mathbf{I}.$$

The current iteration (x, \mathbf{A}) induces the triple of reals

$$\bar{t} := \|x - x_0\|, \quad \bar{\gamma} := \|x - x_{-1}\|, \quad \bar{\delta} := \|x_+ - x\|.$$

Clearly, $\bar{t}_+ := \|x_+ - x_0\| \leq \bar{t} + \bar{\delta}$. Also, $\bar{\gamma}_+ = \bar{\delta}$. The next lemma relates $\bar{\delta}_+ := \|x_{++} - x_+\| = \|\mathbf{A}_+\mathbf{f}(x_+)\|$ with the triple $(\bar{t}, \bar{\gamma}, \bar{\delta})$. To make its statement shorter, we use the abbreviations

$$(3.3) \quad \alpha := \omega^{-1}(1 - \underline{h}), \quad \bar{\gamma}_0 := \|x_0 - x_{-1}\|, \quad a := \alpha - \bar{\gamma}_0.$$

Lemma 3.1. *Let the selected $dd [x_1, x_2 | \mathbf{f}]$ of \mathbf{f} be ω -regularly continuous on D .*

If $\bar{t}_+ + \bar{t} < a$, then

$$\bar{\delta}_+ \leq \bar{\delta} \left(\frac{\omega(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma})}{\omega(a - \bar{t}_+ - \bar{t})} - 1 \right).$$

Proof. $\bar{\delta}_+ \leq \|\mathbf{A}_+\| \cdot \|\mathbf{f}(x_+)\|$. By the Banach lemma on perturbations and in view of (3.2),

$$(3.4) \quad \|\mathbf{A}_+\|^{-1} \geq \|\mathbf{A}_0\|^{-1} - \|\mathbf{A}_+^{-1} - \mathbf{A}_0^{-1}\| \geq 1 - \underline{h} - \|[x_+, x | \mathbf{f}] - [x_0, x_{-1} | \mathbf{f}]\|,$$

where by (2.1)

$$\begin{aligned} & \|[x_+, x | \mathbf{f}] - [x_0, x_{-1} | \mathbf{f}]\| \\ & \leq \omega \left(\min\{\omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}), \omega^{-1}(\|[x_0, x_{-1} | \mathbf{f}]\| - \underline{h})\} + \|x_+ - x_0\| \right) \end{aligned}$$

$$+ \|x - x_{-1}\|) - \omega\left(\min\{\omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}), \omega^{-1}(\|[x_0, x_{-1} | \mathbf{f}]\| - \underline{h})\}\right).$$

Because of (2.3) and (3.2),

$$(3.5) \quad \begin{aligned} \omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}) &\geq (\omega^{-1}(1 - \underline{h}) - \|x_+ - x_0\| - \|x - x_{-1}\|)^+ \\ &\geq (\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+. \end{aligned}$$

So, due to concavity and monotonicity of ω ,

$$(3.6) \quad \begin{aligned} &\|[x_+, x | \mathbf{f}] - [x_0, x_{-1} | \mathbf{f}]\| \\ &\leq \omega\left(\min\{(\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+, \alpha\} + \bar{t}_+ + \bar{t} + \bar{\gamma}_0\right) \\ &\quad - \omega\left(\min\{(\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+, \alpha\}\right) \\ &= \omega\left((\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+ + \bar{t}_+ + \bar{t} + \bar{\gamma}_0\right) - \omega\left((\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+\right). \end{aligned}$$

If this difference $< 1 - \underline{h}$, then (3.4) yields

$$\|\mathbf{A}_+\| \leq \frac{1}{1 - \underline{h} - \omega\left((\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+ + \bar{t}_+ + \bar{t} + \bar{\gamma}_0\right) + \omega\left((\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+\right)}.$$

The difference (3.6) $< 1 - \underline{h} = \omega(\alpha)$ if and only if $\bar{t}_+ + \bar{t} < a$. Hence, this assumption implies

$$(3.7) \quad \|\mathbf{A}_+\| \leq \frac{1}{1 - \underline{h} - \omega(\alpha) + \omega(a - \bar{t}_+ - \bar{t})} = \frac{1}{\omega(a - \bar{t}_+ - \bar{t})}.$$

As to $\|\mathbf{f}(x_+)\|$, note that the application of the classical Sherman-Morrison formula to (1.5) gives

$$\mathbf{A}_+^{-1} = \mathbf{A}^{-1} + \frac{\mathbf{f}(x_+)}{\|x_+ - x\|^2} \langle x_+ - x, \cdot \rangle,$$

so that $\|\mathbf{A}_+^{-1} - \mathbf{A}^{-1}\| = \|\mathbf{f}(x_+)\|/\|x_+ - x\|$ and $\|\mathbf{f}(x_+)\| = \bar{\delta}\|[x_+, x | \mathbf{f}] - [x, x_- | \mathbf{f}]\|$ (because of (3.1)). As follows from (2.1), the last norm

$$\begin{aligned} &\leq \omega\left(\min\{\omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}), \omega^{-1}(\|[x, x_- | \mathbf{f}]\| - \underline{h})\} + \|x_+ - x\| \right. \\ &\quad \left. + \|x - x_-\|) - \omega\left(\min\{\omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}), \omega^{-1}(\|[x, x_- | \mathbf{f}]\| - \underline{h})\}\right). \end{aligned}$$

As in (3.5),

$$\begin{aligned} \omega^{-1}(\|[x_+, x | \mathbf{f}]\| - \underline{h}) &\geq (\alpha - \bar{t}_+ - \bar{t} - \bar{\gamma}_0)^+ = a - \bar{t}_+ - \bar{t}, \\ \omega^{-1}(\|[x, x_- | \mathbf{f}]\| - \underline{h}) &\geq (\alpha - \bar{t} - \bar{t}_- - \bar{\gamma}_0)^+ = a - \bar{t} - \bar{t}_-, \end{aligned}$$

and so

$$(3.8) \quad \begin{aligned} \|[x_+, x | \mathbf{f}] - [x, x_- | \mathbf{f}]\| &\leq \omega\left(\min\{a - \bar{t}_+ - \bar{t}, a - \bar{t} - \bar{t}_-\} + \bar{\delta} + \bar{\gamma}\right) \\ &\quad - \omega\left(\min\{a - \bar{t}_+ - \bar{t}, a - \bar{t} - \bar{t}_-\}\right) \\ &= \omega(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma}) - \omega(a - \bar{t}_+ - \bar{t}). \end{aligned}$$

Hence,

$$(3.9) \quad \|\mathbf{f}(x_+)\| \leq \bar{\delta} \left(\omega(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma}) - \omega(a - \bar{t}_+ - \bar{t}) \right).$$

Together with (3.7) this results in

$$\bar{\delta}_+ \leq \bar{\delta} \left(\frac{\omega(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma})}{\omega(a - \bar{t}_+ - \bar{t})} - 1 \right),$$

as claimed. \square

The lemma suggests the following majorant generator $\mathbf{g}(t, \gamma, \delta) = (t_+, \gamma_+, \delta_+)$:

$$(3.10) \quad \begin{aligned} t_+ &:= t + \delta, \quad \gamma_+ := \delta, \\ \delta_+ &:= \delta \left(\frac{\omega(a - t_+ - t + \delta + \gamma)}{\omega(a - t_+ - t)} - 1 \right) = \delta \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - \delta)} - 1 \right). \end{aligned}$$

We say that the triple $q' = (t', \gamma', \delta')$ majorizes $q = (t, \gamma, \delta)$ (briefly $q \prec q'$), if

$$t \leq t' \quad \& \quad \gamma \leq \gamma' \quad \& \quad \delta \leq \delta'.$$

The lemma states that $\bar{q}_+ \prec \mathbf{g}(\bar{q})$.

Being fed with the initial triple q_0 , the generator iterates producing a majorant sequence as long as the denominator in (3.10) remains defined:

$$\& \quad \underset{n}{2t_n} + \delta_n < a.$$

Under this condition, we can ensure convergence of the sequence (x_n, \mathbf{A}_n) generated by the method (1.5) from the starter (x_0, \mathbf{A}_0) to a solution of the system

$$(3.11) \quad \mathbf{f}(x) = 0 \quad \& \quad \mathbf{A}[x, x | \mathbf{f}] = \mathbf{I}.$$

Lemma 3.2. *If q_0 is such that $\bar{q}_0 \prec q_0$ & $\& \quad \underset{n}{2t_n} + \delta_n < a$, then*

- (1) $\& \quad \underset{n}{\bar{q}_n} \prec q_n$;
- (2) $\gamma_\infty = \delta_\infty = 0$ & $\& \quad \underset{n}{t_n} \leq 0.5(a - \delta_n)$;
- (3) *the sequence (x_n, \mathbf{A}_n) remains in the ball $B((x_0, \mathbf{A}_0), (t_\infty, r_{\mathbf{A}}))$, where*

$$r_{\mathbf{A}} := \frac{\omega(a - \delta_0) - \omega(a - 2t_\infty)}{\omega(a - \delta_0) \omega(a - 2t_\infty)} + \frac{\omega(a + \gamma_0) - \omega(a - \delta_0)}{1 - \omega(a + \gamma_0) + \omega(a - \delta_0)},$$

and converges to a solution $(x_\infty, \mathbf{A}_\infty)$ of the system (3.11);

- (4) x_∞ *is the only solution of the equation $\mathbf{f}(x) = 0$ in the ball $B(x_0, a - t_\infty)$;*
- (5) *for all $n = 0, 1, \dots$,*

$$\|\mathbf{f}(x_{n+1})\| \leq \delta_n (\omega(a - 2t_n + \gamma_n) - \omega(a - 2t_n - \delta_n)),$$

$$\Delta_n := \|x_\infty - x_n\| \leq t_\infty - t_n,$$

$$\frac{\Delta_{n+1}}{\Delta_n} \leq \frac{\omega(\Delta_{n-1})}{\omega(a - 2t_n + \gamma_n)};$$

- (6) *all these inequalities are exact in the sense that they hold as equalities for a scalar quadratic polynomial.*

Proof. (1) The generator (3.10) is monotone in the sense that $q \prec q' \implies \mathbf{g}(q) \prec \mathbf{g}(q')$. Besides, we have noted above that $\bar{q}_+ \prec \mathbf{g}(\bar{q})$. Hence, $\bar{q}_+ \prec \mathbf{g}(q) = q_+$. By induction, $\bar{q}_0 \prec q_0 \implies \bar{q}_n \prec q_n$.

(2) $2t_n + \delta_n < a \iff t_n < 0.5(a - \delta_n)$. Thus, the sequence t_n is increasing and bounded, so that t_∞ is finite. Taking the limits in $t_{n+1} = t_n + \delta_n$ & $\gamma_{n+1} = \delta_n$ yields $\gamma_\infty = \delta_\infty = 0$.

(3) By (1), $\bar{\delta}_n \leq \delta_n$ and so

$$\|x_{n+m} - x_n\| \leq \sum_{k=n}^{n+m-1} \|x_{k+1} - x_k\| = \sum_{k=n}^{n+m-1} \bar{\delta}_k \leq \sum_{k=n}^{n+m-1} \delta_k < \sum_{k=n}^{\infty} \delta_k = t_\infty - t_n.$$

This shows that x_n is a Cauchy sequence and so converges: $\exists \lim x_n =: x_\infty$. Setting $n = 0$ results in $x_n \in B(x_0, t_\infty)$, while forcing m to ∞ yields $\|x_\infty - x_n\| \leq t_\infty - t_n$. Moreover, in view of (3.9) and because of concavity of ω ,

$$\begin{aligned} \|\mathbf{f}(x_{n+1})\| &\leq \bar{\delta}_n \left(\omega(a - \bar{t}_{n+1} - \bar{t}_n + \bar{\delta}_n + \bar{\gamma}_n) - \omega(a - \bar{t}_{n+1} - \bar{t}_n) \right) \\ &\leq \delta_n \left(\omega(a - t_{n+1} - t_n + \delta_n + \gamma_n) - \omega(a - t_{n+1} - t_n) \right) \\ (3.12) \quad &= \delta_n \left(\omega(a - 2t_n + \gamma_n) - \omega(a - 2t_n - \delta_n) \right) \rightarrow 0, \end{aligned}$$

so that $\mathbf{f}(x_\infty) = 0$. Consider now the operators $\mathbf{A}_{m+n} - \mathbf{A}_n$. By the classic Banach lemma (see, for example, [12, Theorem 1.2.1]),

$$(3.13) \quad \|\mathbf{A}_{m+n} - \mathbf{A}_n\| \leq \frac{\|\mathbf{A}_n\|^2 \|\mathbf{A}_{m+n}^{-1} - \mathbf{A}_n^{-1}\|}{1 - \|\mathbf{A}_n\| \cdot \|\mathbf{A}_{m+n}^{-1} - \mathbf{A}_n^{-1}\|},$$

where $\|\mathbf{A}_{m+n}^{-1} - \mathbf{A}_n^{-1}\| \leq \sum_{k=n}^{m+n-1} \|\mathbf{A}_{k+1}^{-1} - \mathbf{A}_k^{-1}\|$. By (3.1) and (3.8),

$$\begin{aligned} \|\mathbf{A}_{k+1}^{-1} - \mathbf{A}_k^{-1}\| &= \|[x_{k+1}, x_k | \mathbf{f}] - [x_k, x_{k-1} | \mathbf{f}]\| \\ &\leq \omega(a - 2t_k + \gamma_k) - \omega(a - 2t_k - \delta_k). \end{aligned}$$

According to (3.10), $t_k = t_{k-1} + \delta_{k-1}$ and $\gamma_k = \delta_{k-1}$. Therefore, $a - 2t_k + \gamma_k = a - 2t_{k-1} - \delta_{k-1}$, $\|\mathbf{A}_{k+1}^{-1} - \mathbf{A}_k^{-1}\| \leq \omega(a - 2t_{k-1} - \delta_{k-1}) - \omega(a - 2t_k - \delta_k)$ and

$$\begin{aligned} \|\mathbf{A}_{m+n}^{-1} - \mathbf{A}_n^{-1}\| &\leq \sum_{k=n}^{m+n-1} \left(\omega(a - 2t_{k-1} - \delta_{k-1}) - \omega(a - 2t_k - \delta_k) \right) \\ &= \omega(a - 2t_{n-1} - \delta_{n-1}) - \omega(a - 2t_{m+n-1} - \delta_{m+n-1}) \\ (3.14) \quad &< \omega(a - 2t_{n-1} - \delta_{n-1}) - \omega(a - 2t_\infty). \end{aligned}$$

This shows that \mathbf{A}_n^{-1} is a Cauchy sequence in the Banach space $\mathcal{L}(\mathbb{X}, \mathbb{H})$ of bounded linear operators from \mathbb{X} to \mathbb{H} and so converges to a limit \mathbf{A}_∞^{-1} . As $\mathbf{A}_n^{-1} = [x_n, x_{n-1} | \mathbf{f}]$, taking limits yields $\mathbf{A}_\infty^{-1} = [x_\infty, x_\infty | \mathbf{f}]$ or $\mathbf{A}_\infty[x_\infty, x_\infty | \mathbf{f}] = \mathbf{I}$. Besides, (3.14) implies $\|\mathbf{A}_{m+1}^{-1} - \mathbf{A}_1^{-1}\| < \omega(a - \delta_0) - \omega(a - 2t_\infty)$ and so

$$\|\mathbf{A}_{m+1} - \mathbf{A}_1\| < \frac{\|\mathbf{A}_1\|^2 (\omega(a - \delta_0) - \omega(a - 2t_\infty))}{1 - \|\mathbf{A}_1\| (\omega(a - \delta_0) - \omega(a - 2t_\infty))},$$

where $\|\mathbf{A}_1\| \leq 1/\omega(a - \delta_0)$ by (3.7). It follows that

$$\|\mathbf{A}_{m+1} - \mathbf{A}_1\| < \frac{\omega(a - \delta_0) - \omega(a - 2t_\infty)}{\omega(a - \delta_0) \omega(a - 2t_\infty)}.$$

Also, as seen from (3.13) and (3.2),

$$\|\mathbf{A}_1 - \mathbf{A}_0\| \leq \frac{\|\mathbf{A}_1^{-1} - \mathbf{A}_0^{-1}\|}{1 - \|\mathbf{A}_1^{-1} - \mathbf{A}_0^{-1}\|},$$

where $\|\mathbf{A}_1^{-1} - \mathbf{A}_0^{-1}\| \leq \omega(a + \gamma_0) - \omega(a - \delta_0)$ by (3.8). Therefore,

$$\|\mathbf{A}_1 - \mathbf{A}_0\| \leq \frac{\omega(a + \gamma_0) - \omega(a - \delta_0)}{1 - \omega(a + \gamma_0) + \omega(a - \delta_0)}$$

and

$$\begin{aligned} \|\mathbf{A}_{m+1} - \mathbf{A}_0\| &\leq \|\mathbf{A}_{m+1} - \mathbf{A}_1\| + \|\mathbf{A}_1 - \mathbf{A}_0\| \\ &\leq \frac{\omega(a - \delta_0) - \omega(a - 2t_\infty)}{\omega(a - \delta_0) \omega(a - 2t_\infty)} + \frac{\omega(a + \gamma_0) - \omega(a - \delta_0)}{1 - \omega(a + \gamma_0) + \omega(a - \delta_0)} =: r_{\mathbf{A}}. \end{aligned}$$

(4) Let x_* be another solution of the equation $\mathbf{f}(x) = 0$. Then $0 = \mathbf{f}(x_*) - \mathbf{f}(x_\infty) = [x_*, x_\infty | \mathbf{f}](x_* - x_\infty)$, which shows that the dd $[x_*, x_\infty | \mathbf{f}]$ is not invertible and so $1 \leq \|\mathbf{I} - [x_*, x_\infty | \mathbf{f}]\| = \|[x_0, x_{-1} | \mathbf{f}] - [x^*, x_\infty | \mathbf{f}]\|$. By (2.1), this norm

$$\begin{aligned} &\leq \omega(\min\{\omega^{-1}(1 - \underline{h}), (\omega^{-1}(\|[x_*, x_\infty | \mathbf{f}]\| - \underline{h}))\} + \|x^* - x_0\| + \bar{t}_\infty + \bar{\gamma}_0) - \\ &\omega(\min\{\omega^{-1}(1 - \underline{h}), (\omega^{-1}(\|[x_*, x_\infty | \mathbf{f}]\| - \underline{h}))\}). \end{aligned}$$

Here,

$$\begin{aligned} \omega^{-1}(\|[x_*, x_\infty | \mathbf{f}]\| - \underline{h}) &\geq (\omega^{-1}(1 - \underline{h}) - \|x_* - x_0\| - \bar{t}_\infty - \bar{\gamma}_0)^+ \\ &\geq (\alpha - \|x_* - x_0\| - t_\infty - \bar{\gamma}_0)^+ \end{aligned}$$

by (2.3). Therefore,

$$\begin{aligned} 1 &\leq \omega(\min\{\alpha, (\alpha - \|x_* - x_0\| - t_\infty - \bar{\gamma}_0)^+\} + \|x_* - x_0\| + t_\infty + \bar{\gamma}_0) \\ &\quad - \omega(\min\{\alpha, (\alpha - \|x_* - x_0\| - t_\infty - \bar{\gamma}_0)^+\}) \\ &= \omega((a - \|x_* - x_0\| - t_\infty)^+ + \|x_* - x_0\| + t_\infty + \bar{\gamma}_0) - \omega((a - \|x_* - x_0\| - t_\infty)^+) \\ &\leq \omega(\|x_* - x_0\| + t_\infty + \bar{\gamma}_0) \end{aligned}$$

(because of concavity of ω) and $\|x_* - x_0\| \geq \omega^{-1}(1) - \bar{\gamma}_0 - t_\infty \geq a - t_\infty \geq 0.5a$.

(5) The bounds for $\mathbf{f}(x_{n+1})$ and Δ_n have been proved above. It remains to obtain one for the ratio Δ_{n+1}/Δ_n . As

$$\begin{aligned} x_{n+1} - x_\infty &= x_n - x_\infty - \mathbf{A}_n \mathbf{f}(x_n) \\ &= \mathbf{A}_n[x_n, x_{n-1} | \mathbf{f}](x_n - x_\infty) - \mathbf{A}_n(\mathbf{f}(x_n) - \mathbf{f}(x_\infty)) \\ &= \mathbf{A}_n[x_n, x_{n-1} | \mathbf{f}](x_n - x_\infty) - \mathbf{A}_n[x_n, x_\infty | \mathbf{f}](x_n - x_\infty) \\ &= \mathbf{A}_n([x_n, x_{n-1} | \mathbf{f}] - [x_n, x_\infty | \mathbf{f}])(x_n - x_\infty), \end{aligned}$$

we have the inequality $\Delta_{n+1} \leq \Delta_n \|\mathbf{A}_n\| \| [x_n, x_{n-1} | \mathbf{f}] - [x_n, x_\infty | \mathbf{f}] \|$. By (3.7),

$$\|\mathbf{A}_n\| \leq \frac{1}{\omega(a - \bar{t}_n - \bar{t}_{n-1})} \leq \frac{1}{\omega(a - t_n - t_{n-1})} = \frac{1}{\omega(a - 2t_n + \gamma_n)}.$$

Besides, by (2.1)

$$\begin{aligned} & \| [x_n, x_{n-1} | \mathbf{f}] - [x_n, x_\infty | \mathbf{f}] \| \\ & \leq \omega \left(\min \{ \omega^{-1}(\| [x_n, x_{n-1} | \mathbf{f}] - \underline{h} \|), \omega^{-1}(\| [x_n, x_\infty | \mathbf{f}] - \underline{h} \|) \} + \|x_{n-1} - x_\infty\| \right) \\ & \quad - \omega \left(\min \{ \omega^{-1}(\| [x_n, x_{n-1} | \mathbf{f}] - \underline{h} \|), \omega^{-1}(\| [x_n, x_\infty | \mathbf{f}] - \underline{h} \|) \} \right), \end{aligned}$$

where $\omega^{-1}(\| [x_n, x_\infty | \mathbf{f}] - \underline{h} \|) \geq \left(\omega^{-1}(\| [x_n, x_{n-1} | \mathbf{f}] - \underline{h} \|) - \|x_{n-1} - x_\infty\| \right)^+$ by (2.3). So,

$$\begin{aligned} & \| [x_n, x_{n-1} | \mathbf{f}] - [x_n, x_\infty | \mathbf{f}] \| \\ & \leq \omega \left((\omega^{-1}(\| [x_n, x_{n-1} | \mathbf{f}] - \underline{h} \|) - \Delta_{n-1})^+ + \Delta_{n-1} \right) \\ & \quad - \omega \left((\omega^{-1}(\| [x_n, x_{n-1} | \mathbf{f}] - \underline{h} \|) - \Delta_{n-1})^+ \right) \\ & \leq \omega(\Delta_{n-1}). \end{aligned}$$

Thus, $\Delta_{n+1} \leq \Delta_n \frac{\omega(\Delta_{n-1})}{\omega(a - 2t_n + \gamma_n)}$.

(6) For any scalar function f with ω -regularly continuous dd's, the application of the method (1.5) produces the sequence of triples

$$(\bar{t}_n, \bar{\gamma}_n, \bar{\delta}_n) = (x_n - x_0, x_n - x_{n-1}, x_{n+1} - x_n).$$

If $(t, \gamma, \delta) = (x - x_0, x - x_-, x_+ - x)$, then $t_+ := t + \delta = x - x_0 + x_+ - x = x_+ - x_0$, $\gamma_+ := \delta = x_+ - x$, and

$$\delta_+ := \delta \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - \delta)} - 1 \right) = (x_+ - x) \left(\frac{\omega(a + 2x_0 - x - x_-)}{\omega(a + 2x_0 - x_+ - x)} - 1 \right).$$

So, $(t, \gamma, \delta) = (x - x_0, x - x_-, x_+ - x)$ implies $(t_+, \gamma_+, \delta_+) = (x_+ - x_0, x_+ - x, x_{++} - x_+)$ if and only if

$$(x_+ - x) \left(\frac{\omega(a + 2x_0 - x - x_-)}{\omega(a + 2x_0 - x_+ - x)} - 1 \right) = -\frac{f(x_+)}{[x_+, x | f]}.$$

For linear ω (the case of Lipschitz continuity of the dd), this equation becomes

$$(3.15) \quad \frac{(x_+ - x)(x_+ - x_-)}{b - x - x_+} = -\frac{f(x_+)}{[x_+, x | f]}, \quad b := a + 2x_0.$$

This functional equation is solved by the quadratic polynomial $f(x) := x^2 - bx + c$. Indeed, for it $[x_+, x | f] = x_+ + x - b$,

$$x_+ - x = -\frac{f(x)}{[x, x_- | f]} = -\frac{f(x)}{x + x_- - b} = \frac{f(x)}{b - 2x + \gamma},$$

and $x_+ - x_- = \gamma + \frac{f(x)}{b - 2x + \gamma}$, so that (3.15) can be rewritten as

$$\begin{aligned} & \frac{f(x)}{b - 2x + \gamma} \left(\gamma + \frac{f(x)}{b - 2x + \gamma} \right) \\ &= f \left(x + \frac{f(x)}{b - 2x + \gamma} \right) \\ &= \left(x + \frac{f(x)}{b - 2x + \gamma} \right)^2 - b \left(x + \frac{f(x)}{b - 2x + \gamma} \right) + c \\ &= \frac{(f(x) + x(b - 2x + \gamma))^2 - b(f(x) + x(b - 2x + \gamma))(b - 2x + \gamma) + c(b - 2x + \gamma)^2}{(b - 2x + \gamma)^2} \end{aligned}$$

or, equivalently,

$$\begin{aligned} & f(x)[f(x) + \gamma(b - 2x + \gamma)] \\ &= [f(x) + x(b - 2x + \gamma)]^2 - b[f(x) + x(b - 2x + \gamma)](b - 2x + \gamma) \\ & \quad + c(b - 2x + \gamma)^2. \end{aligned}$$

Expanding the expression on the right yields

$$\begin{aligned} & f(x) + 2xf(x)(b - 2x + \gamma) + x^2(b - 2x + \gamma)^2 - bf(x)(b - 2x + \gamma) \\ & \quad - bx(b - 2x + \gamma)^2 + c(b - 2x + \gamma)^2 \\ &= (x^2 - bx + c)(b - 2x + \gamma)^2 - f(x)(b - 2x)(b - 2x + \gamma) + f(x)^2 \\ &= f(x) [(b - 2x + \gamma)^2 - (b - 2x)(b - 2x + \gamma) + f(x)] \\ &= f(x)[\gamma(b - 2x + \gamma) + f(x)], \end{aligned}$$

the expression on the left. Therefore, $(t, \gamma, \delta) = (\bar{t}, \bar{\gamma}, \bar{\delta})$ implies $(t_+, \gamma_+, \delta_+) = (\bar{t}_+, \bar{\gamma}_+, \bar{\delta}_+)$ and by induction

$$\begin{aligned} & \gamma_0 = x_0 - x_{-1} \quad \& \quad \delta_0 = -\frac{f(x_0)}{[x_0, x_{-1} | f]} \\ \implies & \&_n(t_n, \gamma_n, \delta_n) = (x_n - x_0, x_n - x_{n-1}, x_{n+1} - x_n). \end{aligned}$$

Hence the claim. \square

4. Convergence theorem

As Lemma 3.2 shows, in order to guarantee convergence of x_n , one has to choose such starters $x_0, x_{-1}, \mathbf{A}_0, q_0$ that $\bar{q}_0 \prec q_0$ & $\&_n 2t_n + \delta_n < a$. So, our next task is to get a precise description of the set of all q_0 that result in $\&_n 2t_n + \delta_n < a$ (the convergence domain of the generator (3.10)).

Proposition 4.1. *Let $(t_n, \gamma_n, \delta_n)$ be the sequence generated by the generator (3.10) from the starter $(t_0, \gamma_0, \delta_0)$. Then*

(1) $\& 2t_n + \delta_n < a \iff \delta_0 \leq f_\infty(t_0, \gamma_0)$, where

$$f_0(t, \gamma) := \omega^{-1}(1 - \underline{h}) - 2t - \gamma$$

for all (t, γ) with $2t + \gamma \leq \omega^{-1}(1 - \underline{h})$ and $f_{n+1}(t, \gamma)$ the unique solution for δ of the equation

$$f_n(t + \delta, \delta) = \delta \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - \delta)} - 1 \right).$$

(2) The function f_∞ is a solution of the system
(4.1)

$$x(t + x(t, \gamma), x(t, \gamma)) = x(t, \gamma) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - x(t, \gamma))} - 1 \right) \quad \& \quad x(t_\infty, \gamma) = 0.$$

(3) The function $I(t, \gamma, \delta) := f_\infty(t, \gamma) - \delta$ is an invariant of the generator
(3.10).

Proof. (1) $2t_n + \delta_n < a \iff \delta_n < a - 2t_n =: f_0(t_n, \gamma_n)$. Suppose that for some $k \geq 0$ $\delta_{n-k} < f_k(t_{n-k}, \gamma_{n-k})$, where f_k is decreasing with respect to the first argument to $f_k(t_\infty, \gamma) = 0$ and not increasing with respect to the second. Using (3.10), rewrite this inequality as

$$\delta_{n-k-1} \left(\frac{\omega(a - 2t_{n-k-1} + \gamma_{n-k-1})}{\omega(a - 2t_{n-k-1} - \delta_{n-k-1})} - 1 \right) < f_k(t_{n-k-1} + \delta_{n-k-1}, \delta_{n-k-1})$$

or, equivalently, as $F_k(t_{n-k-1}, \gamma_{n-k-1}, \delta_{n-k-1}) > 0$, where

$$(4.2) \quad F_k(t, \gamma, \delta) := f_k(t + \delta, \delta) - \delta \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - \delta)} - 1 \right).$$

The function F_k is decreasing in $\delta \in [0, t_\infty - t]$ from $F_k(t, \gamma, 0) = f_k(t, 0) > 0$ to

$$\begin{aligned} F_k(t, \gamma, t_\infty - t) &= f_k(t + t_\infty - t, t_\infty - t) - (t_\infty - t) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - t_\infty + t)} - 1 \right) \\ &= f_k(t_\infty, t_\infty - t) - (t_\infty - t) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - t - t_\infty)} - 1 \right) \\ &= -(t_\infty - t) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - t - t_\infty)} - 1 \right) < 0. \end{aligned}$$

Therefore, the equation $F_k(t, \gamma, \delta) = 0$ is uniquely solvable for $\delta \in (0, t_\infty - t)$. We denote the solution $f_{k+1}(t, \gamma)$:

$$(4.3) \quad F_k(t, \gamma, f_{k+1}(t, \gamma)) = 0.$$

The function f_{k+1} is decreasing in $t \in [0, t_\infty)$. To see it, note that it follows from the induction hypothesis and properties of ω , that F_k is decreasing in both t and δ . So,

$$\begin{aligned} t < t' &\implies F_k(t, \gamma, f_{k+1}(t, \gamma)) = 0 = F_k(t', \gamma, f_{k+1}(t', \gamma)) < F_k(t, \gamma, f_{k+1}(t', \gamma)) \\ &\implies f_{k+1}(t', \gamma) < f_{k+1}(t, \gamma). \end{aligned}$$

Similarly,

$$\begin{aligned}\gamma < \gamma' &\implies F_k(t, \gamma, f_{k+1}(t, \gamma)) = 0 = F_k(t, \gamma', f_{k+1}(t, \gamma')) < F_k(t, \gamma, f_{k+1}(t, \gamma')) \\ &\implies f_{k+1}(t, \gamma') < f_{k+1}(t, \gamma).\end{aligned}$$

Besides, by the induction hypothesis, $F_k(t_\infty, \gamma, 0) = f_k(t_\infty, 0) = 0$, which means according to (4.3) that $f_{k+1}(t_\infty, \gamma) = 0$. Thus,

$$\delta_{n-k} < f_k(t_{n-k}, \gamma_{n-k}) \implies \delta_{n-k-1} < f_{k+1}(t_{n-k-1}, \gamma_{n-k-1}).$$

By induction, $2t_n + \delta_n < a \iff \delta_0 < f_n(t_0, \gamma_0)$ and

$$\&_n 2t_n + \delta_n < a \iff \delta_0 \leq \inf_n f_n(t_0, \gamma_0).$$

The sequence f_n is pointwise decreasing:

$$(4.4) \quad \&_n f_{n+1}(t, \gamma) < f_n(t, \gamma).$$

This is proved inductively. By definition,

$$f_1(t, \gamma) \in (0, t_\infty - t) \subset (0, 0.5a - t) = (0, 0.5f_0(t, \gamma)),$$

so that $f_1(t, \gamma) < f_0(t, \gamma)$. Suppose now that $f_n(t, \gamma) < f_{n-1}(t, \gamma)$ for some $n \geq 1$. Then by (4.3) and (4.2)

$$\begin{aligned}F_{n-1}(t, \gamma, f_n(t, \gamma)) &= 0 = F_n(t, \gamma, f_{n+1}(t, \gamma)) \\ &= f_n(t + f_{n+1}(t, \gamma), f_{n+1}(t, \gamma)) - f_{n+1}(t, \gamma) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - f_{n+1}(t, \gamma))} - 1 \right) \\ &< f_{n-1}(t + f_{n+1}(t, \gamma), f_{n+1}(t, \gamma)) - f_{n+1}(t, \gamma) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - f_{n+1}(t, \gamma))} - 1 \right) \\ &= F_{n-1}(t, \gamma, f_{n+1}(t, \gamma)) \implies f_{n+1}(t, \gamma) < f_n(t, \gamma),\end{aligned}$$

because F_{n-1} is decreasing with respect to the third argument. Thus,

$$f_n(t, \gamma) < f_{n-1}(t, \gamma) \implies f_{n+1}(t, \gamma) < f_n(t, \gamma).$$

By induction, (4.4) is proved. It follows that $\inf_n f_n(t_0, \gamma_0) = f_\infty(t_0, \gamma_0)$ and

$$\&_n 2t_n + \delta_n < a \iff \delta_0 \leq f_\infty(t_0, \gamma_0).$$

(2) Since the sequence f_n converges, we can take limits in (4.3) to get $F_\infty(t, \gamma, f_\infty(t, \gamma)) = 0$, that is

$$f_\infty(t + f_\infty(t, \gamma), f_\infty(t, \gamma)) = f_\infty(t, \gamma) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - f_\infty(t, \gamma))} - 1 \right).$$

Besides, we have seen above that $f_k(t_\infty, \gamma) = 0 \implies f_{k+1}(t_\infty, \gamma) = 0$, so that $f_\infty(t_\infty, \gamma) = 0$. Hence, f_∞ satisfies (4.1).

(3) It suffices to show that $I(t, \gamma, \delta) = 0 \implies I(t_+, \gamma_+, \delta_+) = 0$. Indeed, $I(t, \gamma, \delta) = 0 \implies f_\infty(t, \gamma) = \delta$, so that

$$\begin{aligned} I(t_+, \gamma_+, \delta_+) &= f_\infty(t_+, \gamma_+) - \delta_+ = f_\infty(t + \delta, \delta) - \delta \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - \delta)} - 1 \right) \\ &= f_\infty(t + f_\infty(t, \gamma), f_\infty(t, \gamma)) - f_\infty(t, \gamma) \left(\frac{\omega(a - 2t + \gamma)}{\omega(a - 2t - f_\infty(t, \gamma))} - 1 \right) = 0 \end{aligned}$$

by (2). \square

For linear ω ($\omega(t) = ct$), the generator (3.10) simplifies:

$$(4.5) \quad t_+ := t + \delta, \quad \gamma_+ := \delta, \quad \delta_+ := \delta \frac{\gamma + \delta}{a - 2t - \delta},$$

where $a = c^{-1} - \|x_0 - x_{-1}\|$. Consequently, the proposition can be stated more rigorously:

Proposition 4.2. (1) *The function $I(t, \gamma, \delta) := (a - 2t)^2 - 4\delta(a - 2t + \gamma)$ is an invariant of the generator (4.5).*

(2) *The sequence $(t_n, \gamma_n, \delta_n)$, generated by the generator (4.5) from the starter $(0, \gamma_0, \delta_0)$, converges ($t_\infty < \infty$) if and only if*

$$I_0 := I(0, \gamma_0, \delta_0) = (c^{-1} - \gamma_0)^2 - 4c^{-1}\delta_0 \geq 0.$$

In this case,

$$\&_n 2t_n + \delta_n < a \iff t_n = 0.5(c^{-1} - \gamma_0) - \delta_n - \sqrt{\delta_n(\gamma_n + \delta_n) + 0.25I_0} =: f(\gamma_n, \delta_n).$$

(3) *The function f is a solution of the system*

$$(4.6) \quad x \left(\delta, \delta \frac{\gamma + \delta}{a - 2x(\gamma, \delta) - \delta} \right) = x(\gamma, \delta) + \delta \quad \& \quad x(0, 0) = t_\infty.$$

Proof. (1) By (4.5),

$$\begin{aligned} I(t_+, \gamma_+, \delta_+) &= (a - 2t_+)^2 - 4\delta_+(a - 2t_+ + \gamma_+) \\ &= (a - 2t - 2\delta)^2 - 4\delta \frac{\delta + \gamma}{a - 2t - 2\delta + \delta} (a - 2t - 2\delta + \delta) \\ &= (a - 2t)^2 - 4\delta(a - 2t) + 4\delta^2 - 4\delta(\delta + \gamma) \\ &= (a - 2t)^2 - 4\delta(a - 2t) - 4\delta\gamma = I(t, \gamma, \delta). \end{aligned}$$

(2) If $t_\infty < \infty$, then $\gamma_{n+1} = \delta_n = t_{n+1} - t_n \rightarrow 0$. Then by (1), $I(0, \gamma_0, \delta_0) = I(t_\infty, 0, 0)$, i.e., $a^2 - 4\delta_0(a + \gamma_0) = (a - 2t_\infty)^2 \geq 0$. Conversely, if $I_0 \geq 0$, then $\&_n I(t_n, \gamma_n, \delta_n) = I_0 \geq 0$, whence

$$t_n = 0.5 \left(a - \delta_n - \sqrt{\delta_n^2 + \gamma_n \delta_n + 0.25I_0} \right) \leq 0.5 \left(a - \sqrt{I_0} \right)$$

and $t_\infty = 0.5(a - \sqrt{I_0}) = 0.5(c^{-1} - \gamma_0 - \sqrt{I_0}) < \infty$.

$$\begin{aligned}
(3) \text{ By (1), } I(t, \gamma, \delta) &= (a - 2t_\infty)^2 = I(t_+, \gamma_+, \delta_+) \\
\iff t &= 0.5a - \delta - \sqrt{\delta(\gamma + \delta) + (0.5a - t_\infty)^2} =: f(\gamma, \delta) \quad \& \quad t_+ = f(\gamma_+, \delta_+) \\
\implies f(\gamma, \delta) + \delta &= f\left(\delta, \delta \frac{\gamma + \delta}{a - 2f(\gamma, \delta) - \delta}\right).
\end{aligned}$$

Besides, $f(\gamma, \delta) = 0.5a - \delta - \sqrt{\delta(\gamma + \delta) + (0.5a - t_\infty)^2} \implies f(0, 0) = t_\infty$. \square

Now, to get the convergence theorem for operators with ω -regularly continuous dd's we need only to replace the condition $\&_n 2t_n + \delta_n < a$ in Lemma 3.2 by its equivalent in terms of the starter $(0, \gamma_0, \delta_0)$ provided by Proposition 4.1.

Theorem 4.3. *Let the selected dd $[x_1, x_2 | \mathbf{f}]$ of \mathbf{f} be ω -regularly continuous on D . If the starters $x_0, \mathbf{A}_0, \gamma_0, \delta_0$ are such that*

$$\|x_0 - x_{-1}\| \leq \gamma_0 \quad \& \quad \|\mathbf{A}_0 \mathbf{f}(x_0)\| \leq \delta_0 \leq f_\infty(0, \gamma_0),$$

where f_∞ is the function of Proposition 4.1, then

(1) $\gamma_\infty = \delta_\infty = 0$ and

$$\&_n \left(\|x_n - x_0\| \leq t_n \leq 0.5(a - \delta_n) \quad \& \quad \|x_n - x_{n-1}\| \leq \gamma_n \quad \& \quad \|x_{n+1} - x_n\| \leq \delta_n \right);$$

(2) the sequence (x_n, \mathbf{A}_n) generated by the method (1.5) from the starter (x_0, \mathbf{A}_0) converges to a solution $(x_\infty, \mathbf{A}_\infty)$ of the system (4.1);

(3) x_∞ is the only solution of the equation $\mathbf{f}(x) = 0$ in the ball $B(x_0, a - t_\infty)$;

(4) for all $n = 0, 1, \dots$,

$$\begin{aligned}
(4.7) \quad & \|\mathbf{f}(x_{n+1})\| \leq \delta_n (\omega(a - 2t_n + \gamma_n) - \omega(a - 2t_n - \delta_n)), \\
& \Delta_n := \|x_\infty - x_n\| \leq t_\infty - t_n, \\
& \frac{\Delta_{n+1}}{\Delta_n} \leq \frac{\omega(\Delta_{n-1})}{\omega(a - 2t_n + \gamma_n)}.
\end{aligned}$$

For operators with Lipschitz continuous dd's, we use in Lemma 3.2 the equivalent of the condition $\&_n 2t_n + \delta_n < a$ provided by Proposition 4.2:

$$\&_n 2t_n + \delta_n < c^{-1} - \gamma_0 \iff 4c^{-1}\delta_0 \leq (c^{-1} - \gamma_0)^2.$$

The result is the following:

Corollary 4.4. *Let the selected dd $[x_1, x_2 | \mathbf{f}]$ of \mathbf{f} be Lipschitz continuous on D in the sense of (2.2). If the starters $x_0, x_{-1}, \mathbf{A}_0, \gamma_0, \delta_0$ are such that*

$$\|x_0 - x_{-1}\| \leq \gamma_0 \quad \& \quad \|\mathbf{A}_0 \mathbf{f}(x_0)\| \leq \delta_0 \leq \frac{a^2}{4(a + \gamma_0)},$$

then

(1) $\gamma_\infty = \delta_\infty = 0$, $I_0 := (c^{-1} - \gamma_0)^2 - 4c^{-1}\delta_0 \geq 0$, and for all $n \geq 1$

$$\|x_n - x_0\| \leq t_n = 0.5 \left(c^{-1} - \gamma_0 - \delta_n - \sqrt{\delta_n^2 + \gamma_n \delta_n + 0.25I_0} \right),$$

$$\|x_n - x_{n-1}\| \leq \gamma_n,$$

$$\|\mathbf{A}_n \mathbf{f}(x_n)\| \leq \delta_n;$$

(2) the sequence (x_n, \mathbf{A}_n) generated by the method (1.5) from the starter (x_0, \mathbf{A}_0) remains in the ball $B((x_0, \mathbf{A}_0), (t_\infty, r_A))$, where

$$t_\infty = 0.5 \left(c^{-1} - \gamma_0 - \sqrt{I_0} \right), \quad r_A := \frac{c^{-1} - \gamma_0 - \delta_0 - \sqrt{I_0}}{c(c^{-1} - \gamma_0 - \delta_0)\sqrt{I_0}} + \frac{\gamma_0 + \delta_0}{1 - c(\gamma_0 + \delta_0)},$$

and converges to a solution $(x_\infty, \mathbf{A}_\infty)$ of the system

$$\mathbf{f}(x) = 0 \quad \& \quad \mathbf{A}[x, x | \mathbf{f}] = \mathbf{I};$$

(3) x_∞ is the only solution of the equation $\mathbf{f}(x) = 0$ in the ball

$$B(x_0, 0.5(a - t_\infty));$$

(4) for all $n = 0, 1, \dots$,

$$\|\mathbf{f}(x_{n+1})\| \leq c\delta_n(\gamma_n + \delta_n),$$

$$\Delta_n := \|x_\infty - x_n\| \leq \delta_n + \sqrt{\delta_n^2 + \gamma_n \delta_n + 0.25I_0} - 0.5\sqrt{I_0},$$

$$(4.8) \quad \frac{\Delta_{n+1}}{\Delta_n} \leq \frac{\Delta_{n-1}}{\gamma_n + 2\sqrt{\gamma_n^2 + \gamma_n \gamma_{n-1}} + 0.25I_0}.$$

5. Rate of convergence

The bound (4.7) shows that the sequence x_n converges (if and when) to x_∞ superlinearly. It should be stressed that this result applies also to non-smooth operators. Superlinear convergence alone, however, is insufficient to judge merits of an iterative method. Theorem 4.3 offers a more firm ground for comparison of various methods. By (4.7),

$$\Delta_{n+1} \leq \Delta_n \omega(\Delta_{n-1}) / \omega(a - 2t_\infty) = \Delta_n \omega_0(\Delta_{n-1}),$$

where $\omega_0(t) := \omega(t) / \omega(a - 2t_\infty)$. Consider the related system of difference equations:

$$(5.1) \quad u_+ := u \omega_0(v), \quad v_+ := u.$$

It is easy to see that this generator produces a majorant sequence u_n for Δ_n : $\Delta_0 \leq u_0 \implies \& \Delta_n \leq u_n$. This sequence can be described by a one-dimensional difference equation of the type $u_{n+1} = f(u_n)$. If such f exists, then by (5.1) $v = f(u)$ and $v_+ = u = f(u_+) = f(u \omega_0(v)) = f(u \omega_0(f(u)))$, so that f must satisfy the functional equation

$$(5.2) \quad x(u \omega_0(x(u))) = u.$$

Conversely, a solution f of this equation determines the one-dimensional difference equation $u_{n+1} = f(u_n)$ that, given u_0 , generates the sequence u_n of estimates for errors Δ_n : $\& \Delta_n \leq u_n$. By Theorem 4.3, $\Delta_0 \leq t_\infty$, so that t_∞ is a natural candidate for u_0 . Thus, if one knows a solution of the functional equation (5.2), he is able to get a priori estimations for Δ_n before running the

infinite-dimensional process (1.5). The equation (5.2) can be solved numerically by one of the iterative methods applicable to nondifferentiable operators, Broyden's method included.

The above remark suggests a novel approach to comparison of efficiency of iterative methods. Consider a situation when analyses of two iterative methods

$$x_+ := \mathbf{F}_i(x), \quad i = 1, 2,$$

produced two sequences $u_n^{(1)}$ and $u_n^{(2)}$ of upper bounds for the respective errors $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$. Clearly, if $\&_n u_n^{(1)} \leq u_n^{(2)}$, we would prefer the first method.

However, it is an ideal situation. More realistically, suppose that $\&_{n=0}^m u_n^{(1)} \leq u_n^{(2)}$ and $\&_{n=m+1}^{\infty} u_n^{(2)} \leq u_n^{(1)}$. Then, the obvious decision is to use \mathbf{F}_1 for the first $m + 1$ iterations and then switch to \mathbf{F}_2 . In general, $\&_n \Delta_n \leq \min \{u_n^{(1)}, u_n^{(2)}\}$ and one should use \mathbf{F}_1 , if $u_n^{(1)} \leq u_n^{(2)}$, and \mathbf{F}_2 , otherwise. We see that different methods can be combined to create new more efficient ones.

In the case of linear ω , (4.8) implies

$$\Delta_{n+1} \leq \alpha \Delta_n \Delta_{n-1}, \quad \alpha := I_0^{-1/2}$$

or, equivalently, $\alpha \Delta_{n+1} \leq \alpha \Delta_n \cdot \alpha \Delta_{n-1}$. The corresponding difference equation is $u_{n+1} = u_n u_{n-1}$. The variable change $v_n = \ln u_n$ yields $v_{n+1} = v_n + v_{n-1}$, the familiar Fibonacci difference equation [1]. Its solution is

$$v_n = Fib_{n-1} v_1 + Fib_n v_0,$$

where Fib_n is the n -th Fibonacci number:

$$Fib_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

It follows that

$$\begin{aligned} \alpha \Delta_n &\leq u_n = \exp(v_n) = \exp(Fib_{n-1} v_1 + Fib_n v_0) \\ &= \exp(Fib_{n-1} \ln u_1 + Fib_n \ln u_0) \\ &= \exp\left(\ln\left(u_1^{Fib_{n-1}} u_0^{Fib_n}\right)\right) = u_1^{Fib_{n-1}} u_0^{Fib_n}. \end{aligned}$$

By Corollary 4.4, $\Delta_0 \leq 0.5(a - \sqrt{I_0})$ and

$$\Delta_1 \leq \delta_1 + \sqrt{\delta_1^2 + \delta_1 \delta_0 + 0.25 I_0} - 0.5 \sqrt{I_0}.$$

So, we can take $u_0 = 0.5(a - \sqrt{I_0})$ and

$$u_1 = \delta_1 + \sqrt{\delta_1^2 + \delta_1 \delta_0 + 0.25 I_0} - 0.5 \sqrt{I_0},$$

where $\delta_1 = \delta_0(\gamma_0 + \delta_0)/(a - \delta_0)$ by (4.5). Hence,

$$\Delta_n \leq \sqrt{I_0} u_1^{Fib_{n-1}} u_0^{Fib_n}, \quad u_0 := 0.5(a - \sqrt{I_0}),$$

$$u_1 := \delta_1 + \sqrt{\delta_1^2 + \delta_1 \delta_0 + 0.25 I_0} - 0.5 \sqrt{I_0}.$$

6. Applications

As seen from (1.5), implementation of the method does not require the knowledge of the regularity modulus or the Lipschitz constant, which are needed for convergence analysis. One should also has in mind that the convergence conditions given by Theorem 4.3 and Corollary 4.4 are only sufficient and so the iterations (1.5) may converge even when these conditions are violated. Therefore, the lack of information about properties of the operator of interest should not prevent a practitioner from using the method. The applications presented in this section can serve as illustrations for this remark.

6.1. Complementarity problem

Let \mathbb{H}, C , and $\mathbf{g} : \mathbb{H} \rightarrow \mathbb{H}$ be a Hilbert space, a closed cone in it, and a (generally nonsmooth) operator acting on \mathbb{H} , respectively. The problem of finding a solution of the system

$$(6.1) \quad x \in C \ \& \ \mathbf{g}(x) \in C^* \ \& \ \langle x, \mathbf{g}(x) \rangle = 0,$$

where C^* is the dual cone: $C^* := \{y \in \mathbb{H} \mid \langle x, y \rangle \geq 0, \forall x \in C\}$, is called [9] complementarity problem (briefly CP(C, \mathbf{g})). Its special case

$$(6.2) \quad x \in \mathbb{E}^n \ \& \ \&_{i=1}^n (x_i \geq 0 \ \& \ g_i(x) \geq 0 \ \& \ x_i g_i(x) = 0)$$

is the subject of vast literature (see [4],[9]). In particular, Mangasarian in [18] proved a theorem establishing equivalence between the problem (6.2) and an operator equation. His result can be stated as:

Proposition 6.1 ([18]). *Let C and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the standard positive cone of \mathbb{E}^n and any strictly increasing function with $\varphi(0) = 0$. Define the operator $\mathbf{f} : \mathbb{E}^n \rightarrow \mathbb{E}^n$ by setting*

$$\mathbf{f}(x)_i := \varphi(|x_i - g_i(x)|) - \varphi(x_i) - \varphi(g_i(x)), \quad i = 1, \dots, n.$$

A vector $x = (x_1, \dots, x_n)$ solves CP(C, \mathbf{g}) if and only if $\mathbf{f}(x) = 0$.

Mangasarian's theorem can be extended to any separable Hilbert space \mathbb{H} using the fact that it has an orthonormal basis $\{e_k\}_1^\infty$ (see, for example, [11, Ch.IV] or [3, Corollary 2.1.8]), so that each $x \in \mathbb{H}$ has the unique representation $x = \sum_1^\infty \langle x, e_k \rangle e_k$. Taking for C the standard positive cone

$$C := \left\{ x \in \mathbb{H} \mid \&_k \langle x, e_k \rangle \geq 0 \right\},$$

we obtain $C^* = C$ and $\mathbf{g}(x) \in C^* \iff \&_k \langle \mathbf{g}(x), e_k \rangle \geq 0$. With this choice of C , CP(C, \mathbf{g}) becomes: find $x \in \mathbb{H}$ such that

$$(6.3) \quad \&_k \left(\langle x, e_k \rangle \geq 0 \ \& \ \langle \mathbf{g}(x), e_k \rangle \geq 0 \ \& \ \langle x, e_k \rangle \langle \mathbf{g}(x), e_k \rangle = 0 \right).$$

Proposition 6.2. *Let \mathbb{H} and $\{e_k\}_1^\infty$ be a separable Hilbert space and an orthonormal basis in it, respectively. Let φ be any strictly increasing function on \mathbb{R} with $\varphi(0) = 0$. Define $\mathbf{f}: \mathbb{H} \rightarrow \mathbb{H}$ by setting coordinates $\langle \mathbf{f}(x), e_k \rangle$ of $\mathbf{f}(x)$ to*

$$\varphi(|\langle \mathbf{g}(x) - x, e_k \rangle|) - \varphi(\langle \mathbf{g}(x), e_k \rangle) - \varphi(\langle x, e_k \rangle).$$

Then $x \in \mathbb{H}$ solves (6.3) if and only if $\mathbf{f}(x) = 0$.

Proof. Fix a $k \in \mathbb{N}$ and let for short $a := \langle x, e_k \rangle$ and $b := \langle \mathbf{g}(x), e_k \rangle$. The claim reduces to

$$(6.4) \quad a \geq 0 \ \& \ b \geq 0 \ \& \ ab = 0 \iff \varphi(|a - b|) = \varphi(a) + \varphi(b).$$

Because of the symmetry $a \leftrightarrow b$, it suffices to consider the case $a \geq b$. If the left side of (6.4) is true, then $a \geq b = 0$ and so the right side is true also. Conversely, suppose that $\varphi(a - b) = \varphi(a) + \varphi(b)$. The following situations are conceivable:

- (i) $a \geq b > 0$, (ii) $a \geq b = 0$, (iii) $a > 0 > b$, (iv) $a = 0 > b$, (v) $0 > a \geq b$.

We must show that only (ii) is possible if $\varphi(a - b) = \varphi(a) + \varphi(b)$. Indeed, as φ is increasing and $\varphi(0) = 0$,

$$(i) \implies \varphi(a) \geq \varphi(b) > 0 \implies \varphi(a) > \varphi(a - b) = \varphi(a) + \varphi(b) > \varphi(a),$$

$$(iii) \implies \varphi(a) > 0 > \varphi(b) \implies \varphi(a) < \varphi(a - b) = \varphi(a) + \varphi(b) < \varphi(a),$$

$$(iv) \implies \varphi(a) = 0 > \varphi(b) \implies 0 = \varphi(a) < \varphi(|b|) = \varphi(a - b) = \varphi(a) + \varphi(b) \\ = \varphi(b) < 0,$$

$$(v) \implies 0 > \varphi(a) \geq \varphi(b) \implies 0 \leq \varphi(a - b) = \varphi(a) + \varphi(b) < 0.$$

Thus, in each of these four cases, $\varphi(a - b) \neq \varphi(a) + \varphi(b)$, while (ii) implies $a \geq 0 \ \& \ b \geq 0 \ \& \ ab = 0$. \square

Inasmuch as the Hilbert space $L_2[0, 1]$ of square integrable functions on $[0, 1]$ is separable [11], [3] (and so has an orthonormal basis) and because the function $t \mapsto t$ can be taken for φ , we have:

Corollary 6.3. *A function $x \in L_2[0, 1]$ solves the system*

$$(6.5) \quad \bigwedge_{0 \leq t \leq 1} \left(x(t) \geq 0 \ \& \ \mathbf{g}(x)(t) \geq 0 \ \& \ x(t)\mathbf{g}(x)(t) = 0 \right)$$

if and only if $\mathbf{f}(x)(t) := \min\{x(t), \mathbf{g}(x)(t)\} = 0$.

We have applied the method (1.5) to the problem (6.5) with

$$\mathbf{g}(x)(t) := x(t) - (t - c)(2 - t),$$

where c is some constant within $(0, 1)$. For this \mathbf{g} , the problem (6.5) has the only solution $x(t) = (t - c)^+(2 - t)$. The successive approximations x_n were represented in computer memory by the vector of their values at the knots of the

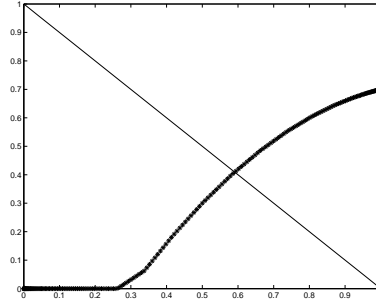


FIGURE 1. Complementarity problem

Tchebyshev mesh of 21 points on $[0, 1]$. Starting from $x_0(t) := 1 - t$, $x_{-1}(t) := 0.9x_0(t) + 0.001$, and

$$(6.6) \quad (\mathbf{A}_0 h)(t) := \frac{x_0(t) - x_{-1}(t)}{\mathbf{f}(x_0)(t) - \mathbf{f}(x_{-1})(t)} h(t),$$

the method attains the requested accuracy of approximation (measured by the max-norm of the function $\mathbf{f}(x)(t)$) of 10^{-12} after 8 iterations. The next table shows the errors $\|\mathbf{f}(x_n)(t)\|$ versus the iteration number n for $c = 0.3$. The figure captioned “Complementarity problem” shows the initial (the solid line) and the final (the asterisked line) approximations to the solution.

n	Error
0	$1.00e\ 0$
1	$7.00e-1$
2	$7.00e-1$
3	$1.37e\ 0$
4	$7.62e-2$
5	$7.63e-2$
6	$4.80e-12$
7	$1.28e-12$
8	$7.32e-13$

6.2. A functional equation

We apply here Broyden’s method to the functional equation (5.2) with $\omega_0(t) := \sqrt{t}$. With $x_0(t) := t$, $x_{-1}(t) := 0.9x_0(t)$, and \mathbf{A}_0 as in (6.6), the max-norm of the function $\mathbf{f}(x)(t) := x(t\omega_0(x(t))) - t$ has been reduced to 10^{-12} after 40 iterations. The figure captioned “Functional equation” shows the initial (the solid line) and the final (the dashed line) approximations to the solution.

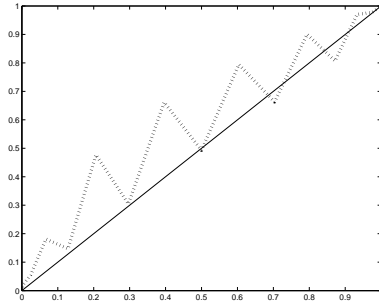


FIGURE 2. Functional equation

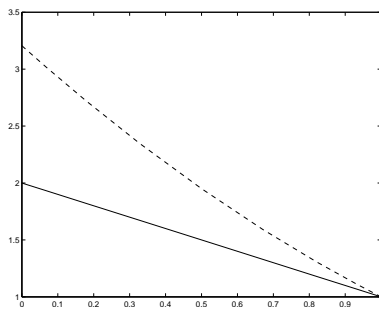


FIGURE 3. Integral equation

6.3. An integral equation

Pimbley [19] considered the following integral equation arising in a model of statistical mechanics:

$$(6.7) \quad \mathbf{f}(x)(t) := x(t) - \lambda \int_t^1 x(s-t)x(s) ds - 1 = 0, \quad 0 \leq t \leq 1, \quad x \in \mathbb{C}[0,1].$$

He has found that it has two positive decreasing twice differentiable solutions for each $\lambda \in (0, 0.5)$, one for $\lambda = 0.5$, and none for $\lambda > 0.5$, and has investigated their properties. However, he did not try to solve this equation numerically.

We apply the method (1.5) to the equation (6.7) with $\lambda = 0.5$ starting from $x_0(t) := x_{-1}(t) := 2 - t$ and $\mathbf{A}_0 := \mathbf{I}$. The successive approximations $x_n(t)$ were represented by cubic splines created on the Tchebyshev mesh of 16 points on $[0, 1]$. The quality of an approximation x_n is measured by the max-norm of the corresponding function (6.7). It is shown in the table below under the caption Error. The final approximation x_{15} is tabulated in columns 3 and 4 of the table. The figure captioned “Integral equation” shows the initial (the solid line) and the final (the dashed line) approximations.

Iteration	Error	t	$x_{15}(t)$
0	1.6597e-1	0.0000e-0	3.0410e+0
1	1.1076e-1	3.1439e-3	3.0329e+0
2	4.6124e-2	2.8058e-2	2.9698e+0
3	2.2092e-2	7.6638e-2	2.8485e+0
4	7.2765e-3	1.4645e-1	2.6781e+0
5	9.3770e-4	2.3398e-1	2.4714e+0
6	6.1311e-5	3.3486e-1	2.2428e+0
7	6.6470e-6	4.4402e-1	2.0071e+0
8	4.8785e-6	5.5598e-1	1.7782e+0
9	3.3993e-6	6.6514e-1	1.5677e+0
10	3.3742e-7	7.6602e-1	1.3843e+0
11	2.3144e-9	8.5355e-1	1.2338e+0
12	1.0981e-9	9.2336e-1	1.1196e+0
13	4.7155e-10	9.7194e-1	1.0431e+0
14	4.7808e-12	9.9686e-1	1.0048e+0
15	3.8636e-14	1.0000e+0	1.0000e+0

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