Korean J. Math. **23** (2015), No. 4, pp. 637–642
http://dx.doi.org/10.11568/kjm.2015.23.4.637

CUBIC PARTITION PAIRS WEIGHTED BY THE PARITY OF THE CRANK

Byungchan Kim

Abstract. We study congruence properties of the number of cubic partition pairs weighted by the parity of the crank. If we define such number to be $c(n)$, then

$$c(5n+4) \equiv 0 \pmod 5 \quad \text{and} \quad c(7n+2) \equiv 0 \pmod 7,$$

for all nonnegative integers n.

1. Introduction and Statements of Results

In a series of papers ([3], [4], [5]) H.-C. Chan studied congruence properties for a kind of bi-partition function $a(n)$, which arises from Ramanujan's cubic continued fraction, and thus $a(n)$ is known as the number of cubic partitions. Cubic partition function $a(n)$ is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Received July 31, 2015. Revised December 7, 2015. Accepted December 10, 2015.
2010 Mathematics Subject Classification: 11P82.
Key words and phrases: Partitions, Partition crank, cubic partitions, cubic crank, congruences.
This study was supported by the Research Program funded by the Seoul National University of Science and Technology.

Here and in the sequel, we will use the following standard q-product notation:

$$(a)_\infty := (a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}),\ |q|<1,$$
$$(a_1,a_2,\ldots,a_k;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty.$$

We can interpret $a(n)$ as the number of 2-color partitions of n with colors r and b subject to the restriction that the color b appears only in even parts. Recently, H. Zhao and Z. Zhong [9] investigated congruences for the partition function

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2}.$$

Here $b(n)$ counts the number of partition pairs $(\lambda_1,\ \lambda_2)$, where λ_1 and λ_2 are cubic partitions such that the sum of parts in λ_1 and λ_2 equals to n. In this sense, we will call $b(n)$ the number of cubic partition pairs. In particular, Zhao and Zhong proved that

THEOREM 1.1 (Theorem 3.2 of [9]). *For all* $n \geq 0$,

$$b(5n+4) \equiv 0 \pmod 5,$$
$$b(7n+a) \equiv 0 \pmod 7, \text{ if } a = 2,\ 3,\ 4, \text{ or } 6.$$

Once congruence properties of a certain type of partition function are known, it is natural to seek a partition statistic to give a combinatorial explanation of the known congruences. By using analogy of the crank of ordinary partitions introduced by G.E. Andrews and F.G. Garvan [1] and the crank of cubic partitions introduced by the author [7], the author introduce crank statistics to explain congruences modulo 7. This crank generating function is given by

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)z^m q^n = \left(\frac{(q)_\infty(q^2;q^2)_\infty}{(zq,q/z;q)_\infty(zq^2,q^2/z;q^2)_\infty}\right)^2. \tag{1.1}$$

The precise definition of the crank is quite lengthy, so we omit it here. Interested reader can find it in [8]. The function $M(m,n)$ is a crank function in a sense of that

$$\sum_{m\in\mathbb{Z}} M(m,n) = b(n) \quad \text{by letting } z=1 \text{ in (1.1)},$$

and for all nonnegative integers n,

$$M(i,7,7n+a) \equiv M(j,7,7n+a) \pmod 7 \quad \text{by [8, Theorem 3]},$$

where $M(i,7,n)$ is the number of cubic partition paris of n with crank $\equiv i \pmod 7$, $0 \le i < j \le 6$ and $a = 2$, 3, 4, or 6.

On the other hand, D. Choi, S-Y. Kang, and J. Lovejoy [6] studied congruence properties of the ordinary partition function weighted by the parity of crank. In this paper, as an analog of Choi, Kang, and Lovejoy's work [6], we study the number of cubic partition pairs weighted by the parity of the crank, i.e.

$$\begin{aligned}\sum_{n\ge 0} c(n)q^n &:= \sum_{n\ge 0}\left(\sum_{m\in\mathbb{Z}}(-1)^m M(m,n)\right)q^n \\ &= \left(\frac{(q)_\infty (q^2;q^2)_\infty}{(-q,;q)_\infty^2(-q^2;q^2)_\infty^2}\right)^2 \\ &= \frac{(q;q)_\infty^6 (q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^2}.\end{aligned}$$

Interestingly, $c(n)$ also satisfies the following congruences.

THEOREM 1.2. *For all non-negative integers n,*

$$c(5n+4) \equiv 0 \pmod 5, \tag{1.2}$$
$$c(7n+2) \equiv 0 \pmod 7. \tag{1.3}$$

These congruence are interesting in that $b(5n+4) \equiv 0 \pmod 5$ and $b(7n+2) \equiv 0 \pmod 7$ also hold. Not every partitions weighted by the parity of crank satisfies the same type of Ramanujan congruences. For example, the number of cubic partitions weighted by the parity of crank does not seem to have simple congruences. The case original partition function and weighted counts by the parity of its crank have the same Ramanujan type congruencies seems to be very rare.

2. Proof of Theorem 1.2

Before starting the proof, we need to introduce basic results on q-series. For details, one might consult [2] :

$$(-q)_\infty = \frac{1}{(q;q^2)_\infty},$$

$$(-a,ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},$$

$$(q)_\infty^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2},$$

and

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n\geq 0} q^{n(n+1)/2}.$$

From the first identity, these are known as Euler identity, Jacobi triple product identity, Jacobi identity, and Gauss identity, respectively.

We start with the first congruence (1.2). First note that the generating function is congruent to

$$\frac{(q^5;q^5)_\infty(q;q)_\infty(q^2;q^2)_\infty^2(q^4;q^4)_\infty}{(q^{20};q^{20})_\infty} \pmod 5$$

$$\equiv \frac{(q^5;q^5)_\infty(q;q)_\infty(q^2;q^2)_\infty^3(q^4;q^4)_\infty}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty} \pmod 5$$

$$\equiv \frac{(q^5;q^5)_\infty \sum_{k=-\infty}^{\infty}(-1)^k q^{k(2k+1)} \sum_{j\geq 0}(-1)^j(2j+1)q^{j(j+1)}}{(q^{20};q^{20})_\infty} \pmod 5.$$

Here, we have applied Jacobi identity, and

$$\begin{aligned}\frac{(q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty} &= \frac{(q)_\infty}{(q^2;q^4)_\infty} \\ &= (q)_\infty(-q^2;q^2)_\infty \\ &= (q;q^2)_\infty(q^4;q^4)_\infty \\ &= (q,q^3,q^4;q^4)_\infty \\ &= \sum_{k=-\infty}^{\infty}(-1)^k q^{k(2k+1)},\end{aligned}$$

where the final equality follows form Jacobi triple product formula. Therefore, to contribute the coefficient of q^{5n+4}, $(k,j) \equiv (1,2) \pmod 5$, and thus the contribution toward the coefficient of q^{5n+4} is a multiple of 5.

Now we turn to prove the second congruence (1.3). First note that the generating function is congruent to

$$\frac{(q;q)_\infty^6 (q^2;q^2)_\infty^2 (q^4;q^4)_\infty^3}{(q^{28};q^{28})_\infty} \pmod 7$$
$$\equiv \frac{(q^7;q^7)_\infty (q^2;q^2)_\infty^2 (q^4;q^4)_\infty^3}{(q;q)_\infty (q^{28};q^{28})_\infty} \pmod 7$$
$$\equiv \frac{(q^7;q^7)_\infty (q^2;q^2)_\infty (q^4;q^4)_\infty^3}{(q;q^2)_\infty (q^{28};q^{28})_\infty} \pmod 7$$
$$\equiv \frac{(q^7;q^7)_\infty \sum_{k\geq 0} q^{k(k+1)/2} \sum_{j\geq 0} (-1)^j (2j+1) q^{2j(j+1)}}{(q^{28};q^{28})_\infty} \pmod 7,$$

where we have use Gauss identity and Jacobi identity. To contribute the coefficient of q^{7n+2}, $(k,j) \equiv (3,3) \pmod 7$, and thus the coefficients of q^{7n+2} are multiples of 7.

3. Concluding Remark

Beside the congruences, it seems that $c(n)$ has an interesting sign pattern, namely,

$$c(4n+1) < 0. \;\; c(4n+2) > 0, \;\; c(4n+3) > 0, \;\; c(4n+4) < 0,$$

for all nonnegative integers n. Asymptotical proof for the above should be available via the classical circle method. An unconditional q-theoretic proof of the above sign pattern would be interesting.

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Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
Seoul 01811, Republic of Korea
E-mail: `bkim4@seoultech.ac.kr`