Korean J. Math. **23** (2015), No. 4, pp. 631–636
http://dx.doi.org/10.11568/kjm.2015.23.4.631

A NEW CHARACTERIZATION OF PRÜFER v-MULTIPLICATION DOMAINS

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Abstract. Let D be an integral domain and w be the so-called w-operation on D. In this note, we introduce the notion of $*(w)$-domains: D is a $*(w)$-domain if $((\cap(x_i))(\cap(y_j)))_w = \cap(x_iy_j)$ for all nonzero elements $x_1, \dots, x_m; y_1, \dots, y_n$ of D. We then show that D is a Prüfer v-multiplication domain if and only if D is a $*(w)$-domain and A^{-1} is of finite type for all nonzero finitely generated fractional ideals A of D.

1. Introduction

A *Prüfer v-multiplication domain* (PvMD) D is an integral domain in which each nonzero finitely generated ideal I is t-invertible, i.e., $(II^{-1})_t = D$. (Definitions related to the t-operation will be reviewed in the sequel.) PvMDs include Prüfer domains, GCD-domains, and Krull domains. There are many interesting characterizations of PvMDs in the literature. Among them, Prüfer domains are PvMDs whose maximal ideals are t-ideals, and D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D, if and only if the polynomial ring $D[X]$ over D is a PvMD. The purpose of this note is to give another new characterization of PvMDs.

We first review definitions related to the t-operation. Let D be an integral domain with quotient field K, $F(D)$ be the set of nonzero

Received July 5, 2015. Revised December 3, 2015. Accepted December 10, 2015.
2010 Mathematics Subject Classification: 13A15, 13F05.
Key words and phrases: Prüfer v-multiplication domain; (t, v)-Dedekind domain; $*(w)$-domain.

fractional ideals of D, and $f(D)$ be the set of nonzero finitely generated fractional ideals of D; so $f(D) \subseteq F(D)$, and $f(D) = F(D)$ if and only if D is a Noetherian domain. For $I \in F(D)$, if we let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, then $I^{-1} \in F(D)$, and so we can define $I_v = (I^{-1})^{-1}$. Also, let $I_t = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in f(D)\}$ and $I_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D) \text{ with } J_v = D\}$. Let $* = v, t$, or w. It is well known that $*$ is a map from $F(D)$ into $F(D)$ such that, for all $0 \neq a \in K$ and $I, J \in F(D)$; (i) $(aD)_* = aD$ and $(aI)_* = aI_*$, (ii) $I \subseteq I_*$ and if $I \subseteq J$, then $I_* \subseteq J_*$, and (iii) $(I_*)_* = I_*$. Clearly, $I_w \subseteq I_t \subseteq I_v$, and if I is finitely generated, then $I_t = I_v$. An $I \in F(D)$ is said to be $*$-*invertible* if $(II^{-1})_* = D$. We say that $I \in F(D)$ is a $*$-ideal if $I_* = I$, while a $*$-ideal is a maximal $*$-ideal if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) be the set of maximal $*$-ideals. Clearly, if D is a rank-one nondiscrete valuation domain, then v-Max$(D) = \emptyset$. However, if D is not a field and $\star = t$ or w, then $\star$-Max$(D) \neq \emptyset$, each maximal $\star$-ideal is a prime ideal, and $D = \bigcap_{\star\text{-Max}(D)} D_P$, t-Max$(D) = w$-Max(D), and $I_w = \bigcap_{P \in t\text{-Max}(D)} ID_P$; so $I_w D_P = ID_P$ for each $P \in t$-Max(D) and for all $I \in F(D)$ [2]. The equality of t-Max$(D) = w$-Max(D) leads to the conclusion that $I \in F(D)$ is t-invertible if and only if I is w-invertible. A v-ideal I of D is said to be *of finite type* if $I = J_v$ for some $J \in f(D)$.

Following [6], we say that D is a $*$-*domain* if for all $x_1, \ldots, x_m; y_1, \ldots, y_n \in D - \{0\}$, we have $(\cap(x_i))(\cap(y_j)) = \cap(x_i y_j)$. In [6], it was shown that D is a $*$-domain if and only if $(\cap(x_i))(\cap(y_j)) = \cap(x_i y_j)$ for all $x_1, \ldots, x_m; y_1, \ldots, y_n \in K - \{0\}$, if and only if D_M is a $*$-domain for all maximal ideals M of D and that a Prüfer domain and a GCD domain are $*$-domains. As a w-operation analogue of $*$-domains, we will call D a $*(w)$-*domain* if for all $x_1, \ldots, x_m; y_1, \ldots, y_n \in D - \{0\}$, we have $((\cap(x_i))(\cap(y_j)))_w = \cap(x_i y_j)$. Clearly, a $*$-domain is a $*(w)$-domain. In this paper, we prove that D is a $*(w)$-domain if and only if D_P is a $*$-domain for all $P \in t$-Max(D). We then use this notion to show that D is a PvMD if and only if D is a $*(w)$-domain and A^{-1} is of finite type for all $A \in f(D)$.

2. Main Result

Let D be an integral domain with quotient field K. It is easy to see that D is a $*(w)$-domain if and only if $((\cap(x_i))(\cap(y_j)))_w = \cap(x_i y_j)$ for all $x_1, \dots, x_m; y_1, \dots, y_n \in K - \{0\}$. In this section, we use this notion to give new characterizations of PvMDs and related domains.

LEMMA 1. *The following statements are equivalent for an integral domain D.*

1. *D is a $*(w)$-domain.*
2. *D_P is a $*$-domain for all $P \in t$-Max(D).*
3. *$(AB)^{-1} = (A^{-1}B^{-1})_w$ for all $A, B \in f(D)$.*

Proof. (1) $\Leftrightarrow$ (2) Let $x_1, \dots, x_m; y_1, \dots, y_n \in K - \{0\}$. Note that $(IJ)D_P = (ID_P)(JD_P)$ and $(I \cap J)D_P = ID_P \cap JD_P$ for all $I, J \in F(D)$ and $P \in t\text{-Max}(D)$ [3, Theorems 4.3 and 4.4]. Also, $I_w = \bigcap_{P \in t\text{-Max}(D)} ID_P$ and $I_w D_P = ID_P$ for all $P \in t\text{-Max}(D)$. Hence $((\cap(x_i))(\cap(y_j)))_w = \cap(x_i y_j)$ if and only if $(\cap(x_i)D_P)(\cap(y_j)D_P) = \cap(x_i y_j)D_P$ for all $P \in t\text{-Max}(D)$. Thus, D is a $*(w)$-domain if and only if D_P is a $*$-domain for all $P \in t\text{-Max}(D)$.

(1) $\Rightarrow$ (3) Let $A = (x_1, \dots, x_m)$ and $B = (y_1, \dots, y_n)$ be nonzero finitely generated fractional ideals of D. Then $AB = (\{x_i y_j\})$, and hence $(A^{-1}B^{-1})_w = ((\cap(\frac{1}{x_i}))(\cap(\frac{1}{y_j})))_w = \cap(\frac{1}{x_i y_j}) = (AB)^{-1}$.

(3) $\Rightarrow$ (1) Let $x_1, \dots, x_m; y_1, \dots, y_n \in K - \{0\}$, and put $A = (\frac{1}{x_1}, \dots, \frac{1}{x_m})$ and $B = (\frac{1}{y_1}, \dots, \frac{1}{y_n})$. Then $A, B \in f(D)$, and hence, $((\cap(x_i))(\cap(y_j)))_w = (A^{-1}B^{-1})_w = (AB)^{-1} = \cap(x_i y_j)$ by (3). □

Recall from [6, Theorem 2.1] that D is a $*$-domain if and only if D_M is a $*$-domain for every maximal ideal M of D. Hence, if each maximal ideal of D is a t-ideal (e.g., D is a Prüfer domain or D is one-dimensional), then D is a $*$-domain if and only if D is a $*(w)$-domain by Lemma 1.

COROLLARY 2. *Let S be a multiplicative subset of D. If D is a $*(w)$-domain, then D_S is also a $*(w)$-domain.*

Proof. If Q is a maximal t-ideal of D_S, then $Q \cap D$ is a t-ideal of D and $Q = (Q \cap D)D_S$. Hence, there is a maximal t-ideal M of D with $Q \cap D \subseteq M$, and so $D_{Q \cap D} = (D_M)_{(Q \cap D)D_M}$. By Lemma 1, D_M is a $*$-domain, and hence $D_{Q \cap D} = (D_M)_{(Q \cap D)D_M}$ is a $*$-domain (see the proof of [6, Theorem 2.1]). Again, by Lemma 1, D_S is a $*(w)$-domain. □

We next give a new characterization of PvMDs.

THEOREM 3. *An integral domain D is a PvMD if and only if D is a $*(w)$-domain and A^{-1} is of finite type for all $A \in f(D)$.*

Proof. (⇒) Let $P \in t\text{-Max}(D)$. Then D_P is a valuation domain, and hence D_P is a $*$-domain. Thus D is a $*(w)$-domain by Lemma 1. Also, if $A \in f(D)$, then $(AA^{-1})_t = D$, and hence A^{-1} is t-invertible. Thus, A^{-1} must be of finite type.

(⇐) Let $A \in f(D)$. Then $A^{-1} = B_v$ for some $B \in f(D)$, and hence by Lemma 1, $D \subseteq (AA^{-1})^{-1} = (AB_v)^{-1} = (AB)^{-1} = (A^{-1}B^{-1})_w = (A^{-1}A_v)_w \subseteq (A^{-1}A_v)_t = (A^{-1}A_t)_t = (A^{-1}A)_t \subseteq D$. Thus, $(AA^{-1})_t = D$. □

A *Mori domain* is an integral domain which satisfies the ascending chain condition on the set of integral v-ideals. Mori domains contain Krull domains and Noetherian domains. Also, it is well known that D is a Krull domain if and only if D is a Mori PvMD.

Corollary 4. *A Mori domain D is a Krull domain if and only if D is a $*(w)$-domain.*

Proof. This is an immediate consequence of Theorem 3 because (i) a Mori domain is a Krull domain if and only if it is a PvMD, (ii) every v-ideal of a Mori domain is of finite type, and A^{-1} is a v-ideal for all $A \in F(D)$. □

An integral domain D is called a (t, v)-*Dedekind domain* (or *pre-Krull domain* as in [6]) if A_v is t-invertible for all $A \in F(D)$. Clearly, a (t, v)-Dedekind domain is a PvMD. Also, if D is a (t, v)-Dedekind domain, then $(A_vA^{-1})_t = D$, and so $(AA^{-1})_v = (A_vA^{-1})_v = D$ for all $A \in F(D)$. Thus, a (t, v)-Dedekind domain is completely integrally closed. Hence, Krull domains ⇒ (t, v)-Dedekind domains ⇒ completely integrally closed PvMDs ⇒ PvMDs. The (t, v)-Dedekind domains were studied in [1, 4, 7].

Lemma 5. (cf. [5, Lemma 1.2]) *If $A \in F(D)$, then A_v is t-invertible if and only if $(AB)^{-1} = (A^{-1}B^{-1})_w$ for all $B \in F(D)$.*

Proof. (⇒) If $x \in (AB)^{-1}$, then $xAB \subseteq D$, and so $xA \subseteq B^{-1}$. Hence $xA_v = (xA)_v \subseteq (B^{-1})_v = B^{-1}$, and thus $x \in xD = x(A_vA^{-1})_w = (xA_vA^{-1})_w \subseteq (B^{-1}A^{-1})_w$. For the reverse containment, let $y \in (B^{-1}A^{-1})_w$. Then $yA_v \subseteq A_v(A^{-1}B^{-1})_w \subseteq (A_v(A^{-1}B^{-1})_w)_w = (A_vA^{-1}B^{-1})_w = ((A_vA^{-1})_wB^{-1})_w = B^{-1}$. Hence $yAB \subseteq yA_vB \subseteq B^{-1}B \subseteq D$, and thus $y \in (AB)^{-1}$.

(⇐) Let $B = A^{-1}$. Then $B \in F(D)$, and hence $D \subseteq (AA^{-1})^{-1} = (A^{-1}A_v)_w \subseteq D$. Thus, $(A^{-1}A_v)_w = D$. □

We next give a new characterization of (t, v)-Dedekind domains via $*(w)$-domains.

Corollary 6. *The following statements are equivalent for an integral domain D.*

1. *D is a (t, v)-Dedekind domain.*
2. *D is a $*(w)$-domain and A_v is of finite type for all $A \in F(D)$.*
3. *D is completely integrally closed and $(AB)_v = (A_v B_v)_t$ for all $A, B \in F(D)$.*
4. *$(AB)^{-1} = (A^{-1}B^{-1})_t$ for all $A, B \in F(D)$.*
5. *$(AB)^{-1} = (A^{-1}B^{-1})_w$ for all $A, B \in F(D)$.*

Proof. (1) $\Rightarrow$ (2) Since A_v is t-invertible, A_v is of finite type. Also, a (t, v)-Dedekind domain is a PvMD, and so by Theorem 3, D is a $*(w)$-domain.

(2) $\Rightarrow$ (1) Let $A \in F(D)$. Then $A^{-1} \in F(D)$ with $(A^{-1})_v = A^{-1}$, and hence both A_v and A^{-1} are of finite type. Hence, $A_v = I_v$ and $A^{-1} = J_v$ for some $I, J \in f(D)$. Thus, by (2) and Lemma 1, $D \supseteq (A_v A^{-1})_t = (I_v J_v)_t = (I_t J_t)_t = (IJ)_t = (IJ)_v = ((IJ)^{-1})^{-1} = ((I^{-1}J^{-1})_w)^{-1} = (A^{-1}A_v)^{-1} \supseteq D$. Thus, $(A_v A^{-1})_t = D$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) [7, Proposition 4.1].

(1) $\Leftrightarrow$ (5) Lemma 5. □

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