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# APPROXIMATE ANALYSIS OF $M/M/c$ RETRIAL QUEUE WITH SERVER VACATIONS

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ABSTRACT. We consider the $M/M/c/c$ queues in which the customers blocked to enter the service facility retry after a random amount of time and some of idle servers can leave the vacation. The vacation time and retrial time are assumed to be of phase type distribution. Approximation formulae for the distribution of the number of customers in service facility and the mean number of customers in orbit are presented. We provide an approximation for $M/M/c/c$ queue with general retrial time and general vacation time by approximating the general distribution with phase type distribution. Some numerical results are presented.

## 1. INTRODUCTION

In the classical $M/M/c/c$ queueing system, it is assumed that the customers blocked to enter the system are lost and servers are always available. However, in many practical situations the blocked customers repeat their requests until the customers get into the service facility and the servers may unavailable for a period of time due to a variety of reasons such as maintenance, taking breaks and doing secondary job. For example, in call center with multiple agents that answer the customer calls an arriving call joins the service facility if there is an available agent or line. Otherwise, that is, if all the lines (buffer) are seized with other calls, the customer will hang-up and retry to access the call center after random amount of time. Some of idle agents may take a break or work secondary job like outbound calls. This type of call center with multi-task agents can be modeled by the queueing system with retrials and vacations (RVQ). Variety of queueing systems such as retrial queue and vacation queue has been introduced to reduce the assumptions of classical model.

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Retrial queue is described by the feature that a customer enters the service facility if the service facility is not full upon arrival, otherwise the customer joins the orbit and repeats its request after random amount of time, called retrial time until the customer gets into the service facility. Many efforts have been devoted to derive performance measures such as queue length distribution, waiting times distribution, busy period distribution etc. in retrial queues. The detailed overviews of the related references with retrial queues can be found in the monographs [1, 2] and references therein.

Vacation queues reflect the situation that the servers may be temporary unavailable. The time period that the servers do not provide their service is considered as the servers take a vacation and is called a vacation time. Vacation queue has been studied extensively for modeling and analyzing the practical problems such as computer and communications systems, manufacturing systems and call centers with multitask employees. The detailed overview of single server vacation queues can be found in the monograph [3] and for the multi-server vacation queues, one can refer the recent monograph [4].

Retrial queues and vacation queues have been studied separately for last several decades. Recently, the interests on the retrial queues with vacations is growing rapidly. However, almost all the literature deals with the system with single-server and constant retrial policy that only one customer in orbit can retry e.g., see [5, 6, 7, 8, 9]. The single server queue with Bernoulli vacation schedule and linear retrial policy is considered in [10]. The call center with outgoing calls and the call center with after-call work introduced in [11] can be considered as RVQ with special vacation policy. An algorithmic solution for the $MAP/M/c/K$ queue with PH-vacation time and exponential retrial time is developed in [12].

The literature about the retrial queues with non-exponential retrial time is very limited. The main difficulty for analyzing the system with non-exponential retrial times is due to the fact that the model must keep track of the elapsed retrial time for each of possibly a very large number of customers as stated in [1]. Recently, approximations for $M/M/c/c$ retrial queue and $PH/PH/c$ retrial queue with $PH$-retrial time have been developed in [13, 14]. The stability condition for $MAP/PH/c/K$ queue with phase type distribution (PH) of vacation time and PH-retrial time is given in [15].

In this paper, we consider the $M/M/c/c$ queue with customer retrials and server vacations in which the retrial times and vacation times are PH-distributions and develop an approximation of the system. We also show the method can be applied to an approximation of the $M/M/c/c$ queue with general distributions of retrial time and vacation time by using the method in [14].

In Section 2, the mathematical model and stationary results are described. The approximation method is proposed in Section 3. Numerical results are given in Section 4. Application of the method developed in Section 3 to the system with general vacation time and retrial time is presented in Section 5. Conclusions are given in Section 6.

## 2. MODEL AND STATIONARY RESULTS

2.1. **The model.** Consider the $M/M/c/c$ queue which consists of an orbit with infinite capacity and a service facility with $c$ identical servers in parallel and no waiting positions. Customers arrive from outside according to a Poisson process with rate $\lambda$. The service time distribution of a customer is exponential with rate $\mu$. We adopt the following vacation policy called $(a, b)$-vacation policy [16]. If any $a$ $(1 \leq a < c)$ or more servers are idle at a service completion, that is, the number of customers at the service facility is less than or equal to $a^* = c - a$ upon a service completion, then $b$ $(b \leq a)$ servers among idle servers take a vacation and the remaining $b^* = c - b$ servers are available. The vacation time distribution is assumed to be of a phase type $PH(\boldsymbol{\delta}, \boldsymbol{V})$, where $\boldsymbol{V} = (v_{ij})$ is a nonsingular $w \times w$ matrix with $v_{ii} = -v_i < 0$, $1 \leq i \leq w$ and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_w) \geq 0$ with $\boldsymbol{\delta}\mathbf{e} = 1$ and $\mathbf{e}$ is the column vector of appropriate size whose components are all 1. Let $\boldsymbol{V}^0 = -\boldsymbol{V}\mathbf{e} = (v_1^0, \cdots, v_w^0)^T$ and $m_v = \boldsymbol{\delta}(-\boldsymbol{V})^{-1}\mathbf{e}$ be the mean vacation time. We consider the single vacation policy under which the servers take only one vacation and after the vacation the servers either serves the waiting customer in service facility if any or stays idle.

If a customer finds that the number of customers in service facility is less than $c$ upon arrival, the customer enters the service facility, otherwise the customer joins to orbit and repeats its request until the customer gets into the service facility. The customers in orbit retry independently of other customers and retrial times of each customer are assumed to be independent and identically distributed. We assume that the retrial time distribution of a customer in orbit is of phase type $PH(\boldsymbol{\theta}, \boldsymbol{U})$ whose distribution function is $F(t) = 1 - \boldsymbol{\theta}\exp(\boldsymbol{U}t)\mathbf{e}$, $t \geq 0$, where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_g) \geq 0$ with $\boldsymbol{\theta}\mathbf{e} = 1$ and $\boldsymbol{U} = (u_{ij})$ is a nonsingular $g \times g$ matrix with $u_{ii} = -u_i < 0$, $1 \leq i \leq g$. Let $\boldsymbol{u} = (u_1, \cdots, u_g)$, $\boldsymbol{\gamma} = -\boldsymbol{U}\boldsymbol{e} = (\gamma_1, \cdots, \gamma_g)^T$ and $m_r = \boldsymbol{\theta}(-\boldsymbol{U})^{-1}\mathbf{e}$ be the mean retrial time. We assume that $U^* = U + \boldsymbol{\gamma}\boldsymbol{\theta}$ is irreducible. For detailed description of the $PH$-distribution and $PH$-renewal process, see [17, Chapter 2].

Let $X_i(t)$ the number of customers in orbit whose service phase is of $i$, $1 \leq i \leq g$ and $Y(t)$ be the number of customers at service facility and $J(t)$ be the server state at time $t$ defined by

$$J(t) = \begin{cases} 0, & c \text{ servers are available} \\ j, & \text{the phase of vacation time is of } j,\ 1 \leq j \leq w. \end{cases}$$

Then $\boldsymbol{\Psi} = \{(\boldsymbol{X}(t), Y(t), J(t)), t \geq 0\}$ with $\boldsymbol{X}(t) = (X_1(t), \cdots, X_g(t))$ is a continuous time Markov chain on the state space

$$\mathcal{S} = \{(\boldsymbol{n}, k, j) \in \mathbb{Z}_+^{g+2} : \boldsymbol{n} \geq 0, 0 \leq k \leq c, 0 \leq j \leq w\}$$

where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\boldsymbol{n} = (n_1, \cdots, n_g) \geq 0$ means $n_i \geq 0$, $i = 1, 2, \cdots, g$.

In the following, denote the $M/M/c/c$ queue with retrials and vacations in which the retrial times and vacation times are PH-distributions by $M/M/c/c$ RVQ with (Ret, Vac)=(PH, PH). We assume that the stability condition $\rho = \frac{\lambda}{c\mu} < 1$ for $a > 1$ and $\rho < \frac{(c-1)\mu m_v + 1}{c\mu m_v + 1}$ for $a = 1$, see [15].

2.2. **Stationary equations.** Let $(\boldsymbol{X}, Y, J)$ be the stationary version of $\boldsymbol{\Psi}$ and $P(\boldsymbol{n}, k, j) = P(\boldsymbol{X} = \boldsymbol{n}, Y = k, J = 0) = 0$ for $(\boldsymbol{n}, k, j) \in \mathcal{S}$ and $P(\boldsymbol{n}, k, j) = 0$ for $(\boldsymbol{n}, k, j) \notin \mathcal{S}$. Denote the indicator function of $A$ by $I_A$ that is, $I_A = 1$ if $A$ is true and $I_A = 0$ otherwise. The balance equations for $P(\boldsymbol{n}, k, j)$ are as follows:

$$
\begin{aligned}
& (\lambda + \mu_k + \boldsymbol{n} \cdot \boldsymbol{u})\, P(\boldsymbol{n}, k, 0) \\
= \;& \lambda P(\boldsymbol{n}, k-1, 0) + \lambda \sum_{i=1}^{g} \theta_i P(\boldsymbol{n} - \boldsymbol{e}_i, k, 0) I_{\{k=c\}} \\
+ \;& \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i + 1) u_{ih} P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h, k, 0) + \sum_{i=1}^{g} (n_i + 1) \gamma_i P(\boldsymbol{n} + \boldsymbol{e}_i, k-1, 0) I_{\{k \geq 1\}} \\
+ \;& \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i + 1) \gamma_i \theta_h P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h, k, 0) I_{\{k=c\}} + \sum_{i=1}^{g} n_i \gamma_i \theta_i P(\boldsymbol{n}, k, 0) I_{\{k=c\}} \\
+ \;& \mu_{k+1} P(\boldsymbol{n}, k+1, 0) I_{\{a^*+1 \leq k \leq c-1\}} + \sum_{j=1}^{w} v_j^0 P(\boldsymbol{n}, k, j), \qquad (2.1)
\end{aligned}
$$

where $\mu_k = k\mu$ and $\boldsymbol{n} \cdot \boldsymbol{u} = \sum_{i=1}^{g} n_i u_i$ is the inner product of two vectors $\boldsymbol{n}$ and $\boldsymbol{u}$, and for $1 \leq j \leq w$,

$$
\begin{aligned}
& (\lambda + \mu_k^* + v_j + \boldsymbol{n} \cdot \boldsymbol{u})\, P(\boldsymbol{n}, k, j) \\
= \;& \lambda P(\boldsymbol{n}, k-1, j) + \sum_{i=1}^{g} \lambda \theta_i P(\boldsymbol{n} - \boldsymbol{e}_i, k, j) I_{\{k=c\}} \\
+ \;& \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i + 1) u_{ih} P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h, k, j) + \sum_{i=1}^{g} (n_i + 1) \gamma_i P(\boldsymbol{n} + \boldsymbol{e}_i, k-1, j) I_{\{k \geq 1\}} \\
+ \;& \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i + 1) \gamma_i \theta_h P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h, k, j) I_{\{k=c\}} + \sum_{i=1}^{g} n_i \gamma_i \theta_i P(\boldsymbol{n}, k, j) I_{\{k=c\}} \\
+ \;& \mu_{k+1}^* P(\boldsymbol{n}, k+1, j) I_{\{0 \leq k \leq c-1\}} + \mu_{k+1} \delta_j P(\boldsymbol{n}, k+1, 0) I_{\{0 \leq k \leq a^*\}} \\
+ \;& \sum_{h=1, h\neq j}^{w} P(\boldsymbol{n}, k, h) v_{hj}, \qquad (2.2)
\end{aligned}
$$

where $\mu_k^* = \min(k, b^*)\mu$.

Let $\pi(k, j) = P(Y = k, J = j)$, $(k, j) \in \mathcal{Y} = \{(k, j) : 0 \leq k \leq c,\ 0 \leq j \leq w\}$ and $\gamma(k, j)$ be retrial rate from orbit given that $(Y, J) = (k, j)$, that is,

$$
\gamma(k, j) = \sum_{i=1}^{g} \gamma_i L_i(k, j), \ \ (k, j) \in \mathcal{Y},
$$

where $L_i(k, j) = \mathbb{E}[X_i | Y = k, J = j]$. Summing over $\boldsymbol{n} \in \mathbb{Z}_+^g$ in (2.1) and (2.2), we have

$$\begin{aligned}
& \left[(\lambda+\gamma(k,0))I_{\{k\le c-1\}}+\mu_k\right]\pi(k,0) \\
= \;& (\lambda+\gamma(k-1,0))\pi(k-1,0)+\mu_{k+1}\pi(k+1,0)I_{\{a^*+1\le k\le c-1\}} \\
+ \;& \sum_{h=1}^{w} v_h^0\pi(k,h)
\end{aligned} \tag{2.3}$$

and for $j=1,2,\cdots,w$,

$$\begin{aligned}
& \left[(\lambda+\gamma(k,j))I_{\{k\le c-1\}}+\mu_k^*+v_j\right]\pi(k,j) \\
= \;& (\lambda+\gamma(k-1,j))\pi(k-1,j)+\mu_{k+1}^*\pi(k+1,j)I_{\{0\le k\le c-1\}} \\
& +\mu_{k+1}\delta_j\pi(k+1,0)I_{\{0\le k\le a^*\}}+\sum_{h=1,h\neq j}^{w} v_{hj}\pi(k,h).
\end{aligned} \tag{2.4}$$

Equations (2.3) and (2.4) can be represented as $\boldsymbol{\pi}Q=0$, where $\boldsymbol{\pi}=(\pi(k,j),(k,j)\in\mathcal{Y})$ and

$$Q=\begin{pmatrix}
B_0 & A_0 & & & & \\
C_1 & B_1 & A_1 & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & C_{c-1} & B_{c-1} & A_{c-1} \\
 & & & C_c & B_c
\end{pmatrix}. \tag{2.5}$$

The matrix $A_k=Diag[\lambda+\gamma(k,j),j=0,1,\cdots,w]$ is the diagonal matrix of size $w+1$, $0\le k\le c-1$. The matrix $B_k$ is a square matrix of size $w+1$ whose $(i,j)$-component $B_k(i,j)$ is as follows: for $0\le i,j\le w$,

$$B_k(i,j)=\begin{cases}
v_i^0, & 1\le i\le w, j=0\\
v_{ij}, & 1\le i\neq j\le w\\
-\Delta_k(i), & i=j\\
0, & \text{otherwise}
\end{cases}$$

where $\Delta_k(i)$ is the positive number that makes $Q\mathbf{e}=0$ and the components not stated above are all zero. The $(i,j)$-component of $C_k$, $1\le k\le c$ is as follows:

$$C_k(i,j)=\begin{cases}
\mu_k\delta_j, & 1\le k\le a^*+1, i=0, 1\le j\le w\\
\mu_k, & a^*+2\le k\le c, i=j=0\\
\mu_k^*, & 1\le i=j\le w
\end{cases}$$

and the components not stated above are all zero. The stationary distribution $\boldsymbol{\pi}=(\boldsymbol{\pi}(k),0\le k\le c)$ with $\boldsymbol{\pi}(k)=(\pi(k,0),\pi(k,1),\cdots,\pi(k,w))$ of $Q$ can be computed as

$$\boldsymbol{\pi}(n)=\pi(0)\mathcal{R}_1\cdots\mathcal{R}_n,\ n=1,2,\cdots,K,$$

where $\boldsymbol{\pi}(0)$ is the unique solution of the equations $\boldsymbol{\pi}(0)(B_0 + \mathcal{R}_1 C_1) = 0$ and

$$\boldsymbol{\pi}(0)\left(\mathbf{e} + \sum_{n=1}^{c} \mathcal{R}_1 \cdots \mathcal{R}_n \mathbf{e}\right) = 1,$$

and the matrices $\mathcal{R}_n$ of size $w+1$ are given recursively as

$$\begin{aligned} \mathcal{R}_c &= A_{c-1}(-B_c)^{-1}, \\ \mathcal{R}_n &= A_{n-1}[-(B_n + \mathcal{R}_{n+1} C_{n+1})]^{-1}, \ n = c-1, c-2, \cdots, 1. \end{aligned}$$

**Lemma 2.1.** *The marginal distributions* $P_i(n) = P(X_i = n)$, $P_i(n,c) = P(X_i = n, Y = c)$ *and* $P_i(n,k,j) = P(X_i = n, Y = k, J = j)$, $1 \le i \le g$ *satisfy the following relation :*

$$\begin{aligned} & (n+1)u_i P_i(n+1) - (n+1)\gamma_i \theta_i P_i(n+1, c) \\ = \ & \lambda \theta_i P_i(n,c) + \sum_{j=1, j\neq i}^{g} u_{ji} \mathbb{E}[X_j I_{\{X_i = n\}}] + \sum_{j=1, j\neq i}^{g} \gamma_j \theta_i \mathbb{E}[X_j I_{\{X_i = n, Y = c\}}], \ n \ge 0 \end{aligned} \tag{2.6}$$

*Proof.* Let $P(\boldsymbol{n}) = P(\boldsymbol{X} = \boldsymbol{n})$ and $P(\boldsymbol{n}, c) = P(\boldsymbol{X} = \boldsymbol{n}, Y = c)$. Summing over $k = 0, 1, \cdots, c$ and $j = 0, 1, \cdots, w$ in (2.1) and (2.2), we have

$$\begin{aligned} & (\boldsymbol{n} \cdot \boldsymbol{u}) P(\boldsymbol{n}) + \lambda P(\boldsymbol{n}, c) - \lambda \sum_{i=1}^{g} \theta_i P(\boldsymbol{n} - \boldsymbol{e}_i, c) \\ = \ & \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i + 1) u_{ih} P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h) \\ + \ & \sum_{i=1}^{g} (n_i+1)\gamma_i P(\boldsymbol{n} + \boldsymbol{e}_i) - \sum_{i=1}^{g} (n_i+1)\gamma_i P(\boldsymbol{n} + \boldsymbol{e}_i, c) \\ + \ & \sum_{i=1}^{g} \sum_{h=1, h\neq i}^{g} (n_i+1)\gamma_i \theta_h P(\boldsymbol{n} + \boldsymbol{e}_i - \boldsymbol{e}_h, c) + \sum_{i=1}^{g} n_i \gamma_i \theta_i P(\boldsymbol{n}, c). \end{aligned} \tag{2.7}$$

Summing over $n_j$ $(j \neq i)$ for each $n_i = n$ in (2.7), we have after tedious algebra that

$$\begin{aligned} & (n+1)u_i P_i(n+1) - n u_i P_i(n) - \lambda \theta_i (P_i(n,c) - P_i(n-1,c)) \\ = \ & \gamma_i \theta_i ((n+1) P_i(n+1, c) - n P_i(n,c)) + \sum_{j=1, j\neq i} u_{ji} (\mathbb{E}[X_j I_{\{X_i = n\}}] - \mathbb{E}[X_j I_{\{X_i = n-1\}}]) \\ + \ & \sum_{j=1, j\neq i} \gamma_j \theta_i (\mathbb{E}[X_j I_{\{X_i = n, Y = c\}}] - \mathbb{E}[X_j I_{\{X_i = n-1, Y = c\}}]), \ n \ge 0. \end{aligned} \tag{2.8}$$

Equation (2.6) is immediate from (2.8). □

Note that (2.6) is the local balance equations for $X_i$. Indeed, the left hand side of (2.6) is the rate at which $X_i$ enters state $n$ from $n+1$ and the right hand side of (2.6) is the rate at which $X_i$ enters state $n+1$ from $n$.

**Proposition 2.2.** *Let* $L_i = \mathbb{E}[X_i]$ *and* $\boldsymbol{L} = (L_1, \cdots, L_g)$*. Then*

$$\boldsymbol{L} = \Lambda\boldsymbol{\theta}(-\boldsymbol{U})^{-1}, \tag{2.9}$$

$$L = \sum_{i=1}^{g} L_i = \Lambda m_r, \tag{2.10}$$

*where*

$$\Lambda = \lambda P(Y = c) + \sum_{i=1}^{g} \gamma_i \mathbb{E}[X_i I_{\{Y=c\}}], \tag{2.11}$$

*is the total arrival rate to orbit.*

*Proof.* Summing over $n$ in (2.6) yields

$$u_i L_i = \sum_{j=1, j\neq i}^{g} u_{ji} L_j + \theta_i \Lambda, \;\; 1 \leq i \leq \nu$$

and hence $\boldsymbol{L}(-\boldsymbol{U}) = \Lambda\boldsymbol{\theta}$ and (2.10) is immediate from (2.9) and $\boldsymbol{\theta}(-\boldsymbol{U})^{-1}\mathbf{e} = m_r$. □

Note from (2.9) and $\boldsymbol{\theta}(-\boldsymbol{U})^{-1}\boldsymbol{\gamma} = 1$ that $\sum_{i=1}^{g} \gamma_i L_i = \Lambda$. The proportion $R(k, j)$ of returning customers from orbit who find the arrival phase and service facility in state $(k, j)$ is given by (see [1, 2, 18])

$$\begin{aligned} R(k,j) &= \frac{\sum_{i=1}^{g} \gamma_i \mathbb{E}[X_i I_{\{Y=k,J=j\}}]}{\sum_{i=1}^{g} \gamma_i L_i} \\ &= \frac{1}{\Lambda}\gamma(k,j)\pi(k,j), \;\; (k,j) \in \mathcal{Y}. \end{aligned} \tag{2.12}$$

It can be seen from (2.12) and (2.11) that the blocking probability $R_B = \sum_{j=0}^{w} R(c, j)$ of a returning customer is given by

$$R_B = \frac{1}{\Lambda}\sum_{i=1}^{g} \gamma_i \mathbb{E}[X_i I_{\{Y=c\}}] = 1 - \frac{\lambda}{\Lambda}P(Y = c). \tag{2.13}$$

Once $R(k, j)$ is given, it follows from (2.10) and (2.13) that

$$L = \frac{\lambda P_B}{1 - R_B} m_r, \tag{2.14}$$

where $P_B = P(Y = c)$.

## 3. APPROXIMATIONS

As noted in Section 2, $Q$ is represented in terms of $\gamma(k, j)$ that is closely related with $R(k, j)$. In this section, we present an approximation formula for $R(k, j)$ and then propose an algorithm for computing $\boldsymbol{\pi}$. We adopt the following approximation assumption about the behavior of service facility based on the observation that $Q$ is the generator of the $M/M/c/c$ vacation queue with state dependent arrival rates.

**Assumption A.** *The service facility behaves like a level dependent quasi-birth-and-death process with generator $Q$ and is independent of the retrials.*

Let $\boldsymbol{\xi} = \{\xi(t),\ t \geq 0\}$ be the Markov chain with generator $Q$. We approximate $R(k,j)$ with the conditional probability that given a customer joins orbit at time 0, the customer finds that $\boldsymbol{\xi}$ is in state $(k,j)$ at the retrial instant. That is, $R(k,j)$ is approximated by

$$\begin{aligned} R(k,j) &\approx \int_0^\infty P(\xi(t) = (k,j) \mid \text{a customer joins orbit at time } t=0)\, dF(t) \\ &= \frac{1}{\Lambda}\sum_{i=0}^{w}(\lambda+\gamma(c,i))\pi(c,i)\left[\int_0^\infty e^{Qt}\boldsymbol{\theta}e^{\boldsymbol{U}t}\boldsymbol{\gamma}dt\right]_{(c,i),(k,j)}, \end{aligned} \tag{3.1}$$

where

$$Q_U = \int_0^\infty e^{Qt}\boldsymbol{\theta}e^{\boldsymbol{U}t}\boldsymbol{\gamma}dt$$

and $[\cdot]_{z,z'}$ is the $(z,z')$-component of the matrix $[\cdot]$.

Once initial value of $\gamma(k,j)$ is given, $\Lambda R(k,j)$ is can be approximated by $(3.1)$ using the stationary distribution $\boldsymbol{\pi}$ and $\gamma(k,j)$ is updated from $\Lambda R(k,j)$ by the formula $(2.12)$. The following algorithm summarizes the results above. We write $Q$ as $Q(\gamma)$ to highlight the dependence of $\gamma$.

**Algorithm for computing $Q$**

1. Initialization. Let $\gamma^{(0)}(k,j) = 0$ and compute the stationary distribution $\boldsymbol{\pi}^{(0)}$ of $Q^{(0)}$.
2. (Repeating procedure) Repeat the following steps until a stopping criterion is satisfied. For $n = 1, 2, \cdots,$
   (1) Set
   $$\Lambda_j^{(n)} = (\lambda + \gamma^{(n-1)}(c,j))\pi^{(n-1)}(c,j), j = 0, 1, \cdots, w$$
   and compute $\Lambda R^{(n)}(k,j)$ using $(3.1)$. That is,
   $$\Lambda R^{(n)}(k,j) = \sum_{i=0}^{w}\Lambda_i^{(n)}[Q_U^{(n-1)}]_{(c,i),(k,j)}.$$
   (2) Calculate
   $$\gamma^{(n)}(k,j) = \frac{\Lambda R^{(n)}(k,j)}{\pi^{(n-1)}(k,j)}$$
   and update $Q^{(n)} = Q(\gamma^{(n)}(k,j))$ and compute the stationary distribution $\boldsymbol{\pi}^{(n)}$ of $Q^{(n)}$.
   (3) Check the stopping criterion. Let
   $$TOL = \max_{(k,j)\in\mathcal{Y}} |\gamma^{(n)}(k,j) - \gamma^{(n-1)}(k,j)|.$$
   If $TOL < \epsilon$ for a given tolerance $\epsilon > 0$, then stop the iteration. Otherwise, continue iteration.

**Remarks.** 1. Computing procedure can be interpreted as follows. Setting $\gamma^{(0)}(k,j) = 0$ denotes that the retrial phenomena is ignored and $Q^{(0)}$ is the generator of $M/M/c/c$ vacation queue. Since $\gamma^{(n)}(k,j) = \sum_{i=1}^{g} \gamma_i L_i^{(n-1)}(k,j)$ is the arrival rate from the group of blocking customers in the system $Q^{(n-1)}$, the system $Q^{(n)}$ is the $M/M/c/c$ vacation queue with extra arrivals with rates $\gamma^{(n)}(k,j)$ that depend on the system state. Although the convergence of the iteration scheme for $\gamma^{(n)}(k,j)$ is not proved analytically, extensive numerical experiments show the convergence of the sequence $\{\gamma^{(n)}\}_{n=0}^{\infty}$.

2. The integration in (3.1) can be computed using the method in [13]. Here we briefly sketch the method. Note that the LST $\tilde{F}(\omega) = \boldsymbol{\theta}(\omega I - \boldsymbol{U})^{-1}\boldsymbol{\gamma}$ of the retrial time distribution $F(t)$ is a rational function and the probability density function $f(t)$ of $F(t)$ can be expressed by a linear combination of the function of the form $t^n e^{-\eta t}$ [19, Appendix E]. It can be easily seen that

$$H(n,\eta) = \int_0^{\infty} \exp[Qt] t^n e^{-\eta t}\, dt = n![(\eta I - Q)^{-1}]^{n+1},\ n = 0, 1, \cdots .$$

Thus $R(k,j)$ is the linear combination of $H(n,\eta)$. Using the algorithm in [20], one can calculate the inversion $(\eta I - Q)^{-1}$ by computing inversions of the matrices of size $w+1$.

3. The complexity of the algorithm for computing $Q$ method is as follows. The most time consuming parts in the algorithm are to compute $\pi(n)$ and $Q_U$. In one iteration, we need to invert $c$ matrices of size $w+1$. Thus the time complexity of one iteration becomes $O((w+1)^3)$. The number of iterations needed is difficult to predict because it depends on the tolerance $\epsilon$ and the system parameters such as mean retrial time $m_r$ and traffic intensity $\rho$. We could see from numerical experiments that the iteration increases as $m_r$ or $\rho$ increases. For example, the algorithm was run on a laptop computer at 1.20 GHz with 2.96 GB RAM using the Mathematica®6 and the command `TimeUsed` is used to capture the CPU time [21] for $M/M/10/10$ queue with $w = 2$ and $g = 4$, $m_r = 1.0$. The run times and the number of iterations are 1.95 seconds with 12 iterations for $\rho = 0.4$ and 3.91 second with 38 iterations for $\rho = 0.8$ when $\epsilon = 10^{-5}$ is used.

## 4. Numerical examples and refinement

In this section, some numerical results are presented for the accuracy check of approximations. The blocking probability $P_B = P(Y = c)$, the probability $P_V = 1 - P(J = 0)$ that the servers are in vacation, the mean $\mathbb{E}[Y]$ and standard deviation $SD[Y]$ of the number of customers in service facility and the mean number of customers in orbit $L = \sum_{i=1}^{g} \mathbb{E}[X_i]$ are considered.

Throughout this section, we consider the $M/M/c/c$ queue with service rate $\mu = 1.0$ and the mean of vacation time is $m_v = 1.5$. The arrival rate $\lambda$ is determined by the formula $\rho = \frac{\lambda}{c\mu}$, that is, $\lambda = c\rho\mu$ given traffic intensity $\rho$. In the numerical tables, denote the vacation time and retrial time by Vac and Ret, respectively.

In Table 1, approximation results (App) for $P_B$ and $P_V$ are compared with the exact results (Exact) for the system with exponential (EXP) vacation time and exponential retrial time with mean $m_r = 0.1, 1.0, 10.0$. For the number $c$ of servers and the parameters $(a, b)$ for vacation

TABLE 1. $P_B$, $P_V$, $\mathbb{E}[Y]$ and $SD[Y]$ with (Vac,Ret)=(EXP,EXP)

| | | | $(c,a,b)=(5,5,3)$ | | $(c,a,b)=(10,7,5)$ | |
|---|---|---|---|---|---|---|
| | $\rho$ | $m_r$ | Exact | App (Err(%)) | Exact | App (Err(%)) |
| $P_B$ | 0.4 | 0.1 | 0.0763 | 0.0763 (0.06) | 0.0216 | 0.0215 (0.53) |
| | | 1.0 | 0.0634 | 0.0622 (1.92) | 0.0160 | 0.0157 (2.11) |
| | | 10.0 | 0.0573 | 0.0570 (0.46) | 0.0138 | 0.0136 (0.37) |
| | 0.8 | 0.1 | 0.5465 | 0.5154 (5.70) | 0.4270 | 0.3896 (8.75) |
| | | 1.0 | 0.4732 | 0.4470 (5.53) | 0.3244 | 0.2975 (8.30) |
| | | 10.0 | 0.4384 | 0.4342 (0.96) | 0.2836 | 0.2798 (1.35) |
| $P_V$ | 0.4 | 0.1 | 0.2703 | 0.2700 (0.12) | 0.6305 | 0.6303 (0.03) |
| | | 1.0 | 0.2669 | 0.2664 (0.18) | 0.6294 | 0.6294 (0.00) |
| | | 10.0 | 0.2657 | 0.2656 (0.00) | 0.6302 | 0.6303 (0.02) |
| | 0.8 | 0.1 | 0.0523 | 0.0467 (10.9) | 0.1315 | 0.1217 (7.42) |
| | | 1.0 | 0.0374 | 0.0305 (18.6) | 0.1036 | 0.0915 (11.7) |
| | | 10.0 | 0.0282 | 0.0269 (4.33) | 0.0856 | 0.0834 (2.59) |
| $\mathbb{E}[Y]$ | 0.4 | 0.1 | 2.104 | 2.108 (0.18) | 4.230 | 4.234 (0.09) |
| | | 1.0 | 2.101 | 2.102 (0.05) | 4.224 | 4.225 (0.02) |
| | | 10.0 | 2.103 | 2.104 (0.01) | 4.224 | 4.224 (0.00) |
| | 0.8 | 0.1 | 4.063 | 4.062 (0.01) | 8.280 | 8.287 (0.08) |
| | | 1.0 | 4.049 | 4.046 (0.08) | 8.230 | 8.225 (0.06) |
| | | 10.0 | 4.046 | 4.046 (0.01) | 8.225 | 8.225 (0.00) |
| $SD[Y]$ | 0.4 | 0.1 | 1.427 | 1.428 (0.05) | 2.185 | 2.187 (0.12) |
| | | 1.0 | 1.395 | 1.391 (0.32) | 2.155 | 2.152 (0.14) |
| | | 10.0 | 1.371 | 1.370 (0.11) | 2.131 | 2.129 (0.05) |
| | 0.8 | 0.1 | 1.266 | 1.217 (3.88) | 2.031 | 1.958 (3.63) |
| | | 1.0 | 1.147 | 1.092 (4.79) | 1.830 | 1.742 (4.82) |
| | | 10.0 | 1.071 | 1.061 (0.90) | 1.690 | 1.673 (0.99) |

policy, two cases $(c,a,b)=(5,5,3)$ and $(10,7,5)$ are presented. Exact results for exponential retrial times are obtained by using the algorithm in Shin (2014a). The relative percentage errors Err(%) of approximations are given by $\text{Err}(\%) = |\text{App} - \text{Exact}| \times 100/\text{Exact}$. We can see from Table 1 that the relative percentage error Err(%) for $P_B$ is not sufficiently small for the case of $\rho = 0.8$, $m_r = 0.1, 1.0$ and $(c,a,b) = (10,7,5)$. The maximal absolute error is $0.0373$ for $\rho = 0.8$, $m_r = 0.1$ and $(c,a,b) = (10,7,5)$ and others seems to be small enough for practical sense. The approximation for $P_V$ works well especially for $\rho = 0.4$. The relative percentage error Err(%) does not seem to be sufficiently small for $\rho = 0.8$ and small $m_r$. However the values of $P_V$ for $\rho = 0.8$ are small and the maximal absolute error is 0.0121 for $\rho = 0.8$, $m_r = 1.0$ and $(c,a,b) = (10,7,5)$ which do not seem particularly meaningful as a measure of the practical accuracy of an approximation. Similar comparisons for $\mathbb{E}[Y]$ and $SD[Y]$ are given in Table 1 from which we see that $\mathbb{E}[Y]$ is less sensitive with respect to the

TABLE 2. $L$ with (Vac,Ret)=(EXP,EXP)

| | | $(c,a,b)=(5,5,3)$ | | $(c,a,b)=(10,7,5)$ | |
|---|---|---|---|---|---|
| $\rho$ | $m_r$ | Exact | App (Err(%)) | Exact | App (Err(%)) |
| 0.4 | 0.1 | 0.092 | 0.098 (5.87) | 0.039 | 0.043 (10.4) |
| | 1.0 | 0.201 | 0.214 (6.76) | 0.088 | 0.098 (10.6) |
| | 10.0 | 1.272 | 1.291 (1.44) | 0.578 | 0.591 (2.19) |
| | 20.0 | 2.466 | 2.485 (0.76) | 1.123 | 1.136 (1.15) |
| 0.8 | 0.1 | 2.857 | 3.022 (5.79) | 2.851 | 3.080 (8.03) |
| | 1.0 | 5.602 | 5.788 (3.32) | 5.428 | 5.855 (7.86) |
| | 10.0 | 33.035 | 33.232 (0.59) | 32.919 | 33.511 (1.80) |
| | 20.0 | 63.550 | 63.748 (0.31) | 63.681 | 64.290 (0.96) |

mean retrial time $m_r$ than other performance measures such as $P_B$ and the approximations for $\mathbb{E}[Y]$ and $SD[Y]$ works well. One can also see that the relative errors for $(c,a,b)=(10,7,5)$ are greater than those for $(c,a,b)=(5,5,3)$.

Denote by $L(m_r)$ and $L_{\text{App}}(m_r)$ the exact and approximation of $L$ as a function of $m_r$. In many numerical experiments, the approximation of $L$ does not provide satisfactory accuracy for small value of $m_r$. However, $L$ increases rapidly and the the differences $\text{Err}(m_r) = L(m_r) - L_{\text{App}}(m_r)$ varies slowly as $m_r$ increases. We adopt the modified formula $\hat{L}(m_r)$ for approximation of $L(m_r)$ in [13] as

$$\hat{L}(m_r) = L_{\text{App}}(m_r) + (L_V - L_{\text{Appr}}(m_r^*)), \tag{4.1}$$

where $L_V$ is the mean number of customers waiting in the queue for the system with $m_r = 0.0$ that is the ordinary $M/M/c$ vacation queue and $m_r^*$ is chosen to be small enough so that the variation of $L_{\text{App}}(m_r)$ is negligible for $m_r \le m_r^*$. In the numerical tables of this paper, $\hat{L}$ with $m_r^* = 10^{-3}$ are presented for the approximation of $L$. Numerical results for the approximation of $L$ in Table 2 show that $\hat{L}$ provides good approximation especially for large $m_r$ and all the absolute error $|\text{App} - \text{Exact}|$ are less than 1.0.

Approximations (App) for the system with PH-vacation time, PH-retrial times and $(c,a,b) = (10,7,5)$ are compared with simulations (Sim) in Tables 3-4. Let $C_v^2$ and $C_r^2$ be the squared coefficients of variation (SCV) of vacation time and retrial time, respectively. For numerical examples, we consider the Erlang distribution of order two $(E_2)$ for $SCV = 0.5$ and the hyperexponential distribution of order two $(H_2)$ for $SCV = 2.0$. The probability density function of $H_2$ distribution used in tables is $f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}$, $t \ge 0$, where the parameters $p$, $\mu_1$ and $\mu_2$ are determined by the mean $m$ and SCV of the distribution as $p = \frac{1}{2}(1 + \sqrt{(SCV-1)/(SCV+1)})$, $\mu_1 = \frac{2p}{m}$, $\mu_2 = \frac{2(1-p)}{m}$. Simulation models for the system with $E_2$ and $H_2$ retrial times are developed with ARENA [22]. Simulation run time is set to 110,000 unit times including 10,000 unit times of warm-up period, where the expected value of service time is one unit time. Ten replications are conducted for each case and the average value and the half length of 95% confidence interval (c.i.) are obtained.

TABLE 3. $P_B$ and $P_V$ with (Vac, Ret)=(PH, PH) and $(c, a, b) = (10, 7, 5)$

| | | | $P_B$ | | $P_V$ | |
|---|---|---|---|---|---|---|
| (Vac, Ret) | $\rho$ | $m_r$ | App | Sim (c.i.) | App | Sim (c.i.) |
| $(E_2, H_2)$ | 0.4 | 0.1 | 0.0170 | 0.0175 (±0.0003) | 0.6224 | 0.6216 (±0.0015) |
| | | 1.0 | 0.0132 | 0.0138 (±0.0003) | 0.6217 | 0.6218 (±0.0016) |
| | | 10.0 | 0.0117 | 0.0120 (±0.0002) | 0.6223 | 0.6214 (±0.0013) |
| | 0.8 | 0.1 | 0.3733 | 0.4103 (±0.0013) | 0.1167 | 0.1264 (±0.0009) |
| | | 1.0 | 0.2948 | 0.3192 (±0.0010) | 0.0909 | 0.1014 (±0.0010) |
| | | 10.0 | 0.2755 | 0.2790 (±0.0010) | 0.0818 | 0.0839 (±0.0009) |
| $(H_2, E_2)$ | 0.4 | 0.1 | 0.0280 | 0.0281 (±0.0006) | 0.6370 | 0.6363 (±0.0018) |
| | | 1.0 | 0.0188 | 0.0200 (±0.0004) | 0.6357 | 0.6358 (±0.0013) |
| | | 10.0 | 0.0156 | 0.0158 (±0.0003) | 0.6367 | 0.6363 (±0.0016) |
| | 0.8 | 0.1 | 0.4094 | 0.4468 (±0.0023) | 0.1268 | 0.1363 (±0.0013) |
| | | 1.0 | 0.2986 | 0.3321 (±0.0010) | 0.0909 | 0.1063 (±0.0013) |
| | | 10.0 | 0.2809 | 0.2848 (±0.0013) | 0.0835 | 0.0859 (±0.0009) |

TABLE 4. $L$ with (Vac, Ret)=(PH, PH) and $(c, a, b) = (10, 7, 5)$

| | | $\rho = 0.4$ | | $\rho = 0.8$ | |
|---|---|---|---|---|---|
| (Vac,Ret) | $m_r$ | App | Sim (c.i.) | App | Sim (c.i.) |
| $(E_2, H_2)$ | 0.1 | 0.031 | 0.030 (±0.003) | 2.807 | 2.807 (±0.021) |
| | 1.0 | 0.080 | 0.077 (±0.017) | 5.681 | 5.681 (±0.026) |
| | 10.0 | 0.502 | 0.501 (±0.008) | 32.736 | 32.736 (±0.199) |
| $(H_2, E_2)$ | 0.1 | 0.067 | 0.061 (±0.003) | 3.766 | 3.564 (±0.069) |
| | 1.0 | 0.124 | 0.111 (±0.004) | 6.189 | 5.725 (±0.048) |
| | 10.0 | 0.669 | 0.651 (±0.001) | 33.769 | 32.776 (±0.251) |

It can be seen from Table 3 that the absolute values of deviation $\text{Dev} = |\text{App} - \text{Sim}|$ for $P_B$ are all less than 0.01 except for the case $\rho = 0.8$ and $m_r = 0.1, 1.0$. The values of Dev are in $(0.037, 0.038)$ for $m_r = 0.1$ and $(0.024, 0.033)$ for $m_r = 0.1$. Table 3 also show that Dev for $P_V$ are all less than $0.01$ except for the case $\rho = 0.8$. The maximal value of Dev for $P_V$ is 0.0154 for the case $\rho = 0.8$ and $m_r = 1.0$ in the system with (Vac,Ret)=$(H_2, E_2)$. It can be seen from Table 4 for $L$ that Dev is less than 1.0 for all cases. Overall the accuracy of approximations in the system with the combinations of $E_2$ and $H_2$ distributions of vacation times and retrial times is of the similar order of magnitude as that in the case of exponential distributions of vacation time and retrial time.

## 5. Approximation of the System with General Distributions of Vacation Time and Retrial Time

In this section, we show that the method developed in the previous section can be applied to the system with general distributions of vacation time and retrial time by an example, the system with Weibul distribution (WEIB) of vacation time and lognormal distribution (LN) of retrial time. We fit the first three moments of Weibul distribution and lognormal distribution with PH-distributions and approximate the system with WEIB-vacation time and LN-retrial time by the system with PH-vacation time and PH-retrial time.

Let $m_k$, $k = 1, 2, 3$ be the $k$th moment of a positive random variable. If $m_k$, $k = 1, 2, 3$ satisfy that $m_2 > 2m_1^2$ and $m_1 m_3 > 1.5m_2^2$, then the probability density function $f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}$, $t \geq 0$ of the hyperexponential distribution $\mathrm{H}_2(p; \mu_1, \mu_2)$ with the preassigned moments $m_i$, $i = 1, 2, 3$ is determined by the parameters, see [19, 23],

$$\mu_1 = \frac{1}{2}\left(a_1 + \sqrt{a_1^2 - 4a_2}\right), \quad \mu_2 = \frac{1}{2}\left(a_1 - \sqrt{a_1^2 - 4a_2}\right), \quad p = \frac{\mu_1(1-\mu_2 m_1)}{\mu_1 - \mu_2}, \tag{5.1}$$

where

$$a_2 = \frac{6m_1^2 - 3m_2}{1.5m_2^2 - m_1 m_3}, \quad a_1 = \frac{1}{m_1}\left(1 + \frac{1}{2}m_2 a_2\right).$$

The parameters of $\mathrm{H}_2(p; t_1/m, t_2/m)$ for fitting the first three moments of lognormal distribution with mean $m$ and $SCV = 2.0$ are given by $p = 0.9714045207910316$, $t_1 = 1.1380711874576983$ and $t_2 = 0.19526214587563495$ and the parameters for the Weibul distribution with mean $m$ and $SCV = 2.0$ are $p = 0.6587280978333091$, $t_1 = 2.0364867918542$ and $t_2 = 0.5044393653326359$.

Coxian distribution with Erlang node denoted by $\mathrm{CE}_{k,j}(p; \mu_1, \mu_2)$ is the composition of the mixture of two Erlang distributions of order $k$ and $j$ whose Laplace transform $f^*(s)$ of the probability density function is given by

$$f^*(s) = p\left(\frac{\mu_1}{\mu_1 + s}\right)^k \left(\frac{\mu_2}{\mu_2 + s}\right)^j + (1-p)\left(\frac{\mu_2}{\mu_2 + s}\right)^j, \ s \geq 0.$$

If the $n$th moment of $\mathrm{CE}_{k,j}(p; \mu_1, \mu_2)$ is $m_n$, then the $n$th moment of $\mathrm{CE}_{k,j}(p; \frac{\mu_1}{a}, \frac{\mu_2}{a})$ is $a^n m_n$. Following the method in [24], we fit the first three moments of LN with $m$ and $SCV = 0.5$ with $\mathrm{CE}_{1,3}(p\,; t_1/m, t_2/m)$ whose phase type representation $PH(\boldsymbol{\alpha}, T)$ is given by $\boldsymbol{\alpha} = (p, 1-p, 0, 0)$ with $p = 0.1167466452506409$ and

$$T = \frac{1}{m}\begin{pmatrix} -t_1 & t_1 & 0 & 0 \\ 0 & -t_2 & t_2 & 0 \\ 0 & 0 & -t_2 & t_2 \\ 0 & 0 & 0 & -t_2 \end{pmatrix},$$

where $t_1 = 0.9501281621$, $t_2 = 3.4202636232$. The first three moments of WEIB with $m_v$ and $C_v^2 = 0.5$ is fitted by $\mathrm{CE}_{2,1}(0.751282; 2.88098/m_v, 2.09007/m_v)$.

In Tables 5-6, the approximation results for $P_B$, $P_V$ and $L$ in the system with PH-vacation time and PH-retrial time are compared with the simulations of the system with WEIB-vacation

TABLE 5. $P_B$ and $P_V$ with (Vac, Ret)=(WEIB, LN) and $(c, a, b) = (10, 7, 5)$

| | | | $P_B$ | | $P_V$ | |
|---|---|---|---|---|---|---|
| $(C_v^2, C_r^2)$ | $\rho$ | $m_r$ | App | Sim (c.i.) | App | Sim (c.i.) |
| (0.5, 2.0) | 0.4 | 0.1 | 0.0171 | 0.0176 (±0.0003) | 0.6233 | 0.6228 (±0.0017) |
| | | 1.0 | 0.0131 | 0.0138 (±0.0003) | 0.6225 | 0.6215 (±0.0012) |
| | | 10.0 | 0.0118 | 0.0118 (±0.0003) | 0.6233 | 0.6226 (±0.0012) |
| | 0.8 | 0.1 | 0.3755 | 0.4133 (±0.0017) | 0.1184 | 0.1271 (±0.0007) |
| | | 1.0 | 0.2927 | 0.3192 (±0.0012) | 0.0904 | 0.1020 (±0.0012) |
| | | 10.0 | 0.2759 | 0.2777 (±0.0011) | 0.0820 | 0.0834 (±0.0007) |
| (2.0, 0.5) | 0.4 | 0.1 | 0.0289 | 0.0291 (±0.0006) | 0.6385 | 0.6397 (±0.0017) |
| | | 1.0 | 0.0192 | 0.0202 (±0.0004) | 0.6371 | 0.6387 (±0.0011) |
| | | 10.0 | 0.0159 | 0.0163 (±0.0003) | 0.6383 | 0.6394 (±0.0016) |
| | 0.8 | 0.1 | 0.4153 | 0.4536 (±0.0016) | 0.1281 | 0.1394 (±0.0011) |
| | | 1.0 | 0.2982 | 0.3344 (±0.0010) | 0.0908 | 0.1083 (±0.0013) |
| | | 10.0 | 0.2815 | 0.2878 (±0.0016) | 0.0838 | 0.0877 (±0.0013) |

TABLE 6. $L$ with (Vac, Ret)=(WEIB, LN) and $(c, a, b) = (10, 7, 5)$

| | | $\rho = 0.4$ | | $\rho = 0.8$ | |
|---|---|---|---|---|---|
| $(C_v^2, C_r^2)$ | $m_r$ | App | Sim (c.i.) | App | Sim (c.i.) |
| (0.5, 2.0) | 0.1 | 0.030 | 0.029 (±0.001) | 2.772 | 2.540 (±0.026) |
| | 1.0 | 0.077 | 0.073 (±0.002) | 5.554 | 5.177 (±0.040) |
| | 10.0 | 0.500 | 0.491 (±0.016) | 32.691 | 31.622 (±0.217) |
| (2.0, 0.5) | 0.1 | 0.067 | 0.061 (±0.003) | 3.756 | 3.545 (±0.073) |
| | 1.0 | 0.125 | 0.111 (±0.003) | 6.123 | 5.689 (±0.060) |
| | 10.0 | 0.680 | 0.667 (±0.015) | 33.821 | 33.260 (±0.308) |

time and LN-retrial time. It can be seen from the tables that the approximation works similarly to the case of the system with PH-vacation time and PH-retrial time investigated in Section 4.

## 6. CONCLUSIONS

Approximation method for some performance measures in $M/M/c/c$ queue with server vacations and customer retrials in which the vacation time and retrial time are of phase type distributions has been presented. A sufficient condition for the system to be positive recurrent is presented. We showed that the approach can be applied to the system with general distributions of vacation time and retrial time by fitting the general distribution with PH-distribution. Accuracy of the approximations has been numerically investigated by comparing them with exact results and simulation. The numerical examples show that the accuracy of approximation is good in practical sense and tends to improve as the mean retrial time $m_r$ increases. The approximate method has used vacation queue with finite capacity state dependent arrival rate and it is required to invert the matrix of size $w+1$ for numerical implementation. Thus the method can be applied to RVQ with many servers. Furthermore, the approach has potential applications to the RVQ with various vacation policies and more complex arrival process.

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