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SOME CLASSES OF 3-DIMENSIONAL NORMAL ALMOST PARACONTACT METRIC MANIFOLDS

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Abstract. The aim of present paper is to investigate 3-dimensional ξ-projectively flat and $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifolds. As a first step, we proved that if the 3-dimensional normal almost paracontact metric manifold is ξ-projectively flat then $\Delta\beta = 0$. If additionally β is constant then the manifold is β-para-Sasakian. Later, we proved that a 3-dimensional normal almost paracontact metric manifold is $\tilde{\varphi}$-projectively flat if and only if it is an Einstein manifold for α, β =const. Finally, we constructed an example to illustrate the results obtained in previous sections.

1. Introduction

Paracontact metric structures were introduced in [11], as a natural odd-dimensional counterpart to paraHermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ have been studied by many authors in the recent years, particularly since the appearance of [18]. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [8], [16], [18]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1],[2],[5],[6],[7],[12]).

Z. Olszak studied normal almost contact metric manifolds of dimension 3 [14]. He derive certain necessary and sufficient conditions for

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an almost contact metric structure on manifold to be normal and curvature properties of such structures and normal almost contact metric structures on a manifold of constant curvature are studied. Recently, J. Wełyczko studied curvature and torsion of Frenet-Legendre curves in 3-dimensional normal almost paracontact metric manifolds [15]. The structures of some classes of 3-dimensional normal almost contact metric manifolds were studied in [9]. C. L. Bejan and M. Crasmareanu considered second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry [3].

In this study, we make the first contribution to investigate under which conditions normal almost paracontact metric manifold of dimension 3 is ξ-projectively flat and $\tilde{\varphi}$-projectively flat.

The outline of the article goes as follows. In Section 2, we recall basic facts which we will need throughout the paper. In Section 3, we deal with some results related with 3-dimensional normal almost paracontact manifolds. Section 4 is devoted to 3-dimensional ξ-projectively flat normal almost paracontact manifolds. For such manifolds there are defined two scalar invariants α, β. Our first main result is that if the 3-dimensional normal almost paracontact metric manifold is ξ-projectively flat then $\Delta\beta = 0$, where Δ denotes the Beltrami operator. If additionally β is constant then the manifold is β-para-Sasakian. Section 5 is devoted to 3-dimensional $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifolds. Our second main result is that a 3-dimensional normal almost paracontact metric manifold is $\tilde{\varphi}$-projectively flat if and only if it is an Einstein manifold for $\alpha, \beta =$const. Finally, we would like to remark the construction of 3-dimensional normal almost paracontact metric manifold example which yields our results.

2. Preliminaries

In this section we collect the formulas and results we need on paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [11], [18] and references therein for more information about paracontact metric geometry.

An $(2n+1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1, \ \tilde{\varphi}^2 = I - \eta \otimes \xi,$

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the ± 1-eigendistributions, $\mathcal{D}^{\pm} = \mathcal{D}_{\tilde{\varphi}}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension n.

From the definition it follows that $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. We denote by $[\tilde{\varphi}, \tilde{\varphi}]$ the Nijenhius torsion

$$[\tilde{\varphi}, \tilde{\varphi}](X, Y) = \tilde{\varphi}^2[X, Y] + [\tilde{\varphi}X, \tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X, Y] - \tilde{\varphi}[X, \tilde{\varphi}Y].$$

When the tensor field $N_{\tilde{\varphi}} = [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y), \tag{1}$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n + 1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $\tilde{g}(X_i, X_j) = \delta_{ij}$, $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$, $\tilde{g}(X_i, Y_j) = 0$, $\tilde{g}(\xi, X_i) = g(\xi, Y_j) = 0$, and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\tilde{\varphi}$-basis.

We can now define the *fundamental form* of the almost paracontact metric manifold by $F(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$. If $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$, then $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be *paracontact metric manifold.* In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h} = \frac{1}{2}\mathcal{L}_{\xi}\tilde{\varphi}$, where $\mathcal{L}\xi$, denotes the Lie derivative. It is known [18] that $\tilde{h}$ anti-commutes with $\tilde{\varphi}$ and satisfies $\tilde{h}\xi = 0$, tr$\tilde{h}$ =tr$\tilde{h}\tilde{\varphi} = 0$ and

$$\tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}, \tag{2}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, \tilde{g})$.

Moreover $\tilde{h} = 0$ if and only if ξ is Killing vector field. In this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *K-paracontact manifold.* A normal paracontact metric manifold is called a *para-Sasakian manifold.* Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$\tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y). \tag{3}$$

Similarly as in the class of almost contact metric manifolds [4], a normal almost paracontact metric manifold will be called para-Sasakian if $F = d\eta$ [10] and quasi-para-Sasakian if $dF = 0$. Obviously, the class of para-Sasakian manifolds is contained in the class of quasi-para-Sasakian manifolds. The converse does not hold in general. A paracontact metric manifold will be called paracosymplectic if $dF = 0$, $d\eta = 0$ [8], more generally α-para-Kenmotsu if $dF = 2\alpha\eta \wedge F$, $d\eta = 0$, $\alpha = const. \neq 0$.

3. Normal almost paracontact metric manifolds

Proposition 3.1. [15] *For a* 3*-dimensional almost paracontact metric manifold M the following three conditions are mutually equivalent*

(a) *M is normal,*

(b) there exist functions α, β on M such that

$$(\tilde{\nabla}_X \tilde{\varphi})Y = \beta(\tilde{g}(X,Y)\xi - \eta(Y)X) + \alpha(\tilde{g}(\tilde{\varphi}X,Y)\xi - \eta(Y)\tilde{\varphi}X), \tag{4}$$

(c) there exist functions α, β on M such that

$$\tilde{\nabla}_X \xi = \alpha(X - \eta(X)\xi) + \beta\tilde{\varphi}X. \tag{5}$$

Corollary 3.2. *For a normal almost paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ on M, we have $\tilde{\nabla}_\xi \xi = 0$ and $d\eta = -\beta F$. The functions α, β realizing (4) as well as (5) are given by* [15]

$$2\alpha = \text{ Trace } \left\{X \longrightarrow \tilde{\nabla}_X \xi\right\}, \quad 2\beta = \text{Trace } \left\{X \longrightarrow \tilde{\varphi}\tilde{\nabla}_X \xi\right\}. \tag{6}$$

Proposition 3.3. [15] *For a* 3*-dimensional almost paracontact metric manifold M, the following three conditions are mutually equivalent*

(a) *M is quasi-para-Sasakian,*

(b)*there exists a function β on M such that*

$$(\tilde{\nabla}_X \tilde{\varphi})Y = \beta(\tilde{g}(X,Y)\xi - \eta(Y)X), \tag{7}$$

(c)*there exists a function β on M such that*

$$\tilde{\nabla}_X \xi = \beta\tilde{\varphi}X. \tag{8}$$

A 3-dimensional normal almost paracontact metric manifold is

- paracosymplectic if $\alpha = \beta = 0$ [8],
- quasi-para-Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [10], [15],
- β-para-Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ and β is constant, in particular, para-Sasakian if $\beta = -1$ [15], [18],
- α-para-Kenmotsu if $\alpha \neq 0$ and α is constant and $\beta = 0$.

Theorem 3.4. *Let $(M,\tilde{\varphi},\xi,\eta,g)$ be a 3-dimensional normal almost paracontact metric manifold. Then the following curvature identities hold*

$$\begin{aligned}
&\tilde{R}(X,Y)Z \\
= \;& (2(\xi(\alpha)+\alpha^2+\beta^2)+\frac{1}{2}\tau)(\tilde{g}(Y,Z)X-\tilde{g}(X,Z)Y) \\
&-(\xi(\alpha)+3(\alpha^2+\beta^2)+\frac{1}{2}\tau)((\tilde{g}(Y,Z)\eta(X)\xi-\tilde{g}(X,Z)\eta(Y)\xi \\
&+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y)+(\tilde{\varphi}Z(\beta)-Z(\alpha))(\eta(Y)X-\eta(X)Y) \\
&+(\tilde{\varphi}Y(\beta)-Y(\alpha))(\eta(Z)X-\tilde{g}(X,Z)\xi) \\
&-(\tilde{\varphi}X(\beta)-X(\alpha))(\eta(Z)Y-\tilde{g}(Y,Z)\xi) \\
&+(\tilde{\varphi}\mathrm{grad}\beta+\mathrm{grad}\alpha)(\eta(Y)\tilde{g}(X,Z)-\eta(X)\tilde{g}(Y,Z)).
\end{aligned} \tag{9}$$

$$\begin{aligned}
\tilde{S}(Y,Z) \;=\; & -(\xi(\alpha)+\alpha^2+\beta^2+\frac{1}{2}\tau)\tilde{g}(\tilde{\varphi}Y,\tilde{\varphi}Z) \\
&+\eta(Z)(\tilde{\varphi}Y(\beta)-Y(\alpha)) \\
&+\eta(Y)(\tilde{\varphi}Z(\beta)-Z(\alpha))-2(\alpha^2+\beta^2)\eta(Y)\eta(Z),
\end{aligned} \tag{10}$$

where $\tilde{R}$, $\tilde{S}$ and τ are resp. Riemannian curvature, Ricci tensor and scalar curvature of M.

Proof. Differentiating (5) covariantly and using (4) we find

$$\begin{aligned}
\tilde{\nabla}_X\tilde{\nabla}_Y\xi \;=\; & \alpha(\tilde{\nabla}_XY-\eta(\tilde{\nabla}_XY)\xi)+(\alpha^2-\beta^2)\tilde{g}(\tilde{\varphi}X,\tilde{\varphi}Y)\xi \\
&+\beta\tilde{\varphi}\tilde{\nabla}_XY+X(\alpha)\tilde{\varphi}^2Y \\
&+X(\beta)\tilde{\varphi}Y-(\alpha^2+\beta^2)\eta(Y)\tilde{\varphi}^2X-2\alpha\beta\eta(Y)\tilde{\varphi}X.
\end{aligned}$$

Therefore, for the curvature transformation $\tilde{R}_{XY}=[\tilde{\nabla}_X,\tilde{\nabla}_Y]-\tilde{\nabla}_{[X,Y]}$ we obtain

$$\begin{aligned}
\tilde{R}(X,Y)\xi \;=\; & -\left\{Y(\alpha)+(\alpha^2+\beta^2)\eta(Y)\right\}\tilde{\varphi}^2X \\
&+\left\{X(\alpha)+(\alpha^2+\beta^2)\eta(X)\right\}\tilde{\varphi}^2Y \\
&-\{Y(\beta)+2\alpha\beta\eta(Y)\}\tilde{\varphi}X+\{X(\beta)+2\alpha\beta\eta(X)\}\tilde{\varphi}Y.
\end{aligned} \tag{11}$$

and

$$\tilde{S}(Y,\xi)=-Y(\alpha)+\tilde{\varphi}Y(\beta)-\left\{\xi(\alpha)+2(\alpha^2+\beta^2)\right\}\eta(Y), \tag{12}$$

where $\tilde{R}$ denotes the curvature tensor and $\tilde{S}$ is the Ricci tensor.

From (11), we obtain

$$\tilde{R}(\xi,Y,Z,\xi)=(\xi(\alpha)+\alpha^2+\beta^2)\tilde{g}(\tilde{\varphi}Y,\tilde{\varphi}Z)-(\xi(\beta)+2\alpha\beta)\tilde{g}(\tilde{\varphi}Y,Z),$$

where $\tilde{R}(X,Y,Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W)$. By Bianchi identity the last equation follows that

$$\tilde{R}(\xi,Y,Z,\xi) = (\xi(\alpha)+\alpha^2+\beta^2)\tilde{g}(\tilde{\varphi}Y,\tilde{\varphi}Z), \tag{13}$$

$$\xi(\beta)+2\alpha\beta = 0. \tag{14}$$

Next, we recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \tilde{g}(X,W)\tilde{S}(Y,Z) - \tilde{g}(X,Z)\tilde{S}(Y,W) \\ &\quad +\tilde{g}(Y,Z)\tilde{S}(X,W) - \tilde{g}(Y,W)\tilde{S}(X,Z) \\ &\quad -\frac{1}{2}\tau\left\{\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W)\right\}, \end{aligned} \tag{15}$$

where τ is the scalar curvature.

In order to compute (10) we will use (12), (13) and (15). Combining (10) with (15), we obtain (9). ∎

Theorem 3.5. *Let $(M,\tilde{\varphi},\xi,\eta,g)$ be a 3-dimensional normal almost paracontact metric manifold. Then*

$(i)dF = (div\xi)\eta\wedge F$, where $div\xi$ is the divergence of ξ defined by $div\xi = trace\left\{X\to\tilde{\nabla}_X\xi\right\}$.

(ii)If M has constant curvature τ, and α, β are constants, then $\tau = -6(\alpha^2+\beta^2)$.

Proof. From (5), one can easily get $\mathrm{div}\xi = 2\alpha$. Taking a local orthonormal $\tilde{\varphi}$-basis $\{E_0=\xi,\ E_1=\tilde{\varphi}E_2,\ E_2=\tilde{\varphi}E_1\}$ such that $\tilde{g}(E_0,E_0) = \tilde{g}(E_1,E_1) = 1$ and $\tilde{g}(E_2,E_2) = -1$, we obtain

$$(\eta\wedge F)(E_0,\ E_1,E_2) = 1 \tag{16}$$

and using $F(X,Y) = \tilde{g}(X,\tilde{\varphi}Y)$, we get

$$\begin{aligned} dF(E_0,\ E_1,E_2) &= (\tilde{\nabla}_{E_0}F)(E_1,E_2) + (\tilde{\nabla}_{E_2}F)(E_0,E_1) \\ &\quad +(\tilde{\nabla}_{E_1}F)(E_2,E_0) \\ &= \tilde{g}(E_1,(\tilde{\nabla}_{E_0}\tilde{\varphi})E_2) + \tilde{g}(E_0,(\tilde{\nabla}_{E_2}\tilde{\varphi})E_1) \\ &\quad +\tilde{g}(E_2,(\tilde{\nabla}_{E_1}\tilde{\varphi})E_0). \end{aligned}$$

Using the fact that (4) in the last relation we have

$$dF(E_0,\ E_1,E_2) = 2\alpha. \tag{17}$$

Now, we want to note that $dF = \sigma\eta\wedge F$ for a certain function σ on M. From (16) and (17), we conclude that $\sigma = div\xi = 2\alpha$ which concludes the proof of (i). The proof of (ii) is a direct consequence of (9) for α =constant and β =constant. ∎

4. 3-dimensional ξ-projectively flat normal almost paracontact metric manifolds

Let M be a n-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be *locally projectively flat*. For $n \geq 3$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\},$$

for $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is *projectively flat* (namely, $P = 0$) if and only if the manifold is of constant curvature ([17], pp. 84-85). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The definition of projectively flat can be adapted to the definition given by [17] for 3-dimensional almost paracontact metric manifolds. M will be called *projectively flat if* P vanishes in the following formula.

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{18}$$

for $X, Y, Z \in T(M)$, where $\tilde{R}$ is the curvature tensor and $\tilde{S}$ is the Ricci tensor. So one can define ξ-projectively flat almost paracontact manifolds analogous to the definition of ξ-conformally flat manifolds.

Definition 4.1. *A normal almost paracontact metric manifold M is called ξ-projectively flat if the condition $\tilde{P}(X,Y)\xi = 0$ holds on M, where projective curvature tensor $\tilde{P}$ is defined by (18).*

By Δ we denote the Laplace-Beltrami operator of $\tilde{g}$

$$\Delta f = Trace\{X \mapsto \nabla_X gradf\}.$$

Theorem 4.2. *If the 3-dimensional normal almost paracontact metric manifold is ξ-projectively flat then $\Delta\beta = 0$. If additionally β is constant then the manifold is β-para-Sasakian.*

Proof. Setting $Z=\xi$ in (18) and with the use of (11), (12), we obtain

$$\tilde{P}(X,Y)\xi = \frac{1}{2}\left(X(\alpha)Y - Y(\alpha)X\right) + \left(Y(\alpha)\eta(X) - X(\alpha)\eta(Y)\right)\xi + X(\beta)\tilde{\varphi}Y - Y(\beta)\tilde{\varphi}X + 2\alpha\beta(\eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) + \frac{1}{2}\left(\tilde{\varphi}Y(\beta)X - \tilde{\varphi}X(\beta)Y + \xi(\alpha)(\eta(Y)X - \eta(X)Y)\right). \tag{19}$$

From the assumption of the manifold, we have

$$\frac{1}{2}\left(X(\alpha)Y - Y(\alpha)X\right) + \left(Y(\alpha)\eta(X) - X(\alpha)\eta(Y)\right)\xi + X(\beta)\tilde{\varphi}Y - Y(\beta)\tilde{\varphi}X + 2\alpha\beta(\eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) + \frac{1}{2}\left(\tilde{\varphi}Y(\beta)X - \tilde{\varphi}X(\beta)Y + \xi(\alpha)(\eta(Y)X - \eta(X)Y)\right) = 0. \tag{20}$$

Replacing Y by ξ, in the last equation and using (14), it follows that

$$X(\alpha) + \tilde{\varphi}X(\beta) - \xi(\alpha)\eta(X) = 0. \tag{21}$$

Rewriting (21) in the form

$$X(\alpha) + \tilde{g}(grad\beta, \tilde{\varphi}X) - \xi(\alpha)\eta(X) = 0$$

and taking the covariant derivative of the last equation according to Y, we have

$$\tilde{\nabla}_Y X(\alpha) + \tilde{g}(\tilde{\nabla}_Y grad\beta, \tilde{\varphi}X) + \tilde{g}(grad\beta, (\tilde{\nabla}_Y\tilde{\varphi})X) - Y(\xi(\alpha))\eta(X) - \xi(\alpha)(\tilde{\nabla}_Y\eta)X = 0$$

Antisymmetrizing with respect to X, Y, we have

$$\tilde{g}(\tilde{\nabla}_Y grad\beta, \tilde{\varphi}X) - \tilde{g}(\tilde{\nabla}_X grad\beta, \tilde{\varphi}Y) + \{(\tilde{\nabla}_Y\tilde{\varphi})X(\beta) - (\tilde{\nabla}_X\tilde{\varphi})Y(\beta)\} - Y(\xi(\alpha))\eta(X) + X(\xi(\alpha))\eta(Y) + 2\xi(\alpha)d\eta(X,Y) = 0 \tag{22}$$

On the other hand, applying (4) and Corollary 3.2 to (22), (22) returns to

$$\tilde{g}(\tilde{\nabla}_Y grad\beta, \tilde{\varphi}X) - \tilde{g}(\tilde{\nabla}_X grad\beta, \tilde{\varphi}Y) + \{2\alpha\tilde{g}(\tilde{\varphi}Y, X)\xi - \alpha(\eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) - \beta(\eta(X)Y - \eta(Y)X)\}\beta - \{Y\xi(\alpha)\eta(X) - X\xi(\alpha)\eta(Y)\} - 2\beta\xi(\alpha)F(X,Y) = 0 \tag{23}$$

Taking into account $\{e_1, e_2, \xi\}$ an orthonormal $\tilde{\varphi}$-basis where $X = e_2 = \tilde{\varphi}e_1$, $Y = e_1 = \tilde{\varphi}e_2$, we obtain

$$\tilde{g}(\tilde{\nabla}_{e_1} grad\beta, e_1) - \tilde{g}(\tilde{\nabla}_{e_2} grad\beta, e_2) = 2\alpha\xi(\beta) + 2\beta\xi(\alpha). \tag{24}$$

After differentiating (14) covariantly, we can state

$$\tilde{g}(\tilde{\nabla}_\xi grad\beta, \xi) = -2\alpha\xi(\beta) - 2\beta\xi(\alpha). \tag{25}$$

From (24) and (25), a simple computation shows that $\Delta\beta = 0$, where $\Delta = divgrad$. If additionally β is constant from (14), we obtain $\alpha = 0$ which implies that M is a β-para-Sasakian manifold. ∎

5. 3-dimensional $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifolds

The similar definition of projectively flat which was given in [9] can be given for almost paracontact metric manifolds.

Definition 5.1. *A 3-dimensional normal almost paracontact metric manifold satisfying the condition*

$$\tilde{\varphi}^2\tilde{P}(\tilde{\varphi}X, \tilde{\varphi}Y)\tilde{\varphi}Z = 0$$

is called $\tilde{\varphi}$-projectively flat.

Proposition 5.2. *The scalar curvature τ of a 3-dimensional $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifold is $\tau = -6(\xi(\alpha) + \alpha^2 + \beta^2)$.*

Proof. Assume that M is a 3-dimensional $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifold. Note that $\tilde{\varphi}^2\tilde{P}(\tilde{\varphi}X, \tilde{\varphi}Y)\tilde{\varphi}Z = 0$ holds if and only if

$$\tilde{g}(\tilde{P}(\tilde{\varphi}X, \tilde{\varphi}Y)\tilde{\varphi}Z, \tilde{\varphi}W) = 0,$$

for any $X, Y, Z, W \in T(M)$. Using (18), $\tilde{\varphi}$-projectively flat condition returns to

$$\tilde{g}(\tilde{R}(\tilde{\varphi}X, \tilde{\varphi}Y)\tilde{\varphi}Z, \tilde{\varphi}W) = \frac{1}{2}\left\{ \begin{array}{c} \tilde{S}(\tilde{\varphi}Y, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}W) \\ -\tilde{S}(\tilde{\varphi}X, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}Y, \tilde{\varphi}W) \end{array} \right\}. \tag{26}$$

We can suppose that $\{e_1, e_2, \xi\}$ is a local orthonormal basis of vector fields in M. By using the fact that $\{\tilde{\varphi}e_1, \tilde{\varphi}e_2, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (26) and sum up with respect to i, we obtain

$$\begin{aligned} &\sum_{i=1}^{2}\tilde{g}(\tilde{R}(\tilde{\varphi}e_i, \tilde{\varphi}Y)\tilde{\varphi}Z, \tilde{\varphi}e_i) \\ &= \frac{1}{2}\sum_{i=1}^{2}\left\{\tilde{S}(\tilde{\varphi}Y, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}e_i, \tilde{\varphi}e_i) - \tilde{S}(\tilde{\varphi}e_i, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}Y, \tilde{\varphi}e_i)\right\}. \end{aligned} \tag{27}$$

Using (13) and (10) in (27), we get

$$\left(\frac{\tau}{2}+3(\xi(\alpha)+\alpha^2+\beta^2)\right)(\tilde{g}(\tilde{\varphi}Y,\tilde{\varphi}Z))=0.$$

From the last equation we conclude that $\tau=-6(\xi(\alpha)+\alpha^2+\beta^2)$. ∎

Theorem 5.3. *Let $(M,\tilde{\varphi},\xi,\eta,g)$ be a 3-dimensional normal almost paracontact metric manifold, and α, β are constants. The manifold is $\tilde{\varphi}$-projectively flat if and only if it is an Einstein manifold, and in consequence of constant sectional curvature.*

Proof. Let $(M,\tilde{\varphi},\xi,\eta,g)$ be a 3-dimensional $\tilde{\varphi}$-projectively flat normal almost paracontact metric manifold. From Proposition 5.2, we know $\tau=-6(\xi(\alpha)+\alpha^2+\beta^2)$. Hence, by (10), the manifold is an Einstein manifold for $\alpha,\beta=$const. Conversely, assume that $(M,\tilde{\varphi},\xi,\eta,g)$ is an Einstein manifold for $\alpha,\beta=$const. From (27), we get the manifold is $\tilde{\varphi}$-projectively flat. Note that arbitrary 3-dimensional pseudo-Riemannian Einstein manifold has constant sectional curvature. ∎

6. Example

Now, we will give an example of a 3-dimensional normal almost paracontact metric manifold.

Example 6.1. *We consider the 3-dimensional manifold*

$$M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$$

and the vector fields

$$X=\frac{\partial}{\partial x},\quad \tilde{\varphi}X=\frac{\partial}{\partial y},\quad \xi=(x+2y)\frac{\partial}{\partial x}+(2x+y)\frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

The 1-form $\eta=dz$ defines an almost paracontact structure on M with characteristic vector field $\xi=(x+2y)\frac{\partial}{\partial x}+(2x+y)\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$. Let $\tilde{g}$, $\tilde{\varphi}$ be the pseudo-Riemannian metric and the $(1,1)$-tensor field given by

$$\tilde{g} = \begin{pmatrix} 1 & 0 & -\frac{1}{2}(x+2y) \\ 0 & -1 & \frac{1}{2}(2x+y) \\ -\frac{1}{2}(x+2y) & \frac{1}{2}(2x+y) & 1-(2x+y)^2+(x+2y)^2 \end{pmatrix},$$

$$\tilde{\varphi} = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix},$$

with respect to the basis $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}$.

Using (5) we have

$$\begin{array}{ccc} \tilde{\nabla}_X X = -\xi, & \tilde{\nabla}_{\tilde{\varphi}X} X = 0, & \tilde{\nabla}_\xi X = -2\tilde{\varphi}X, \\ \tilde{\nabla}_X \tilde{\varphi}X = 0, & \tilde{\nabla}_{\tilde{\varphi}X} \tilde{\varphi}X = \xi, & \tilde{\nabla}_\xi \tilde{\varphi}X = -2X, \\ \tilde{\nabla}_X \xi = X, & \tilde{\nabla}_{\tilde{\varphi}X} \xi = \tilde{\varphi}X, & \tilde{\nabla}_\xi \xi = 0. \end{array}$$

for $\alpha = 1$ and $\beta = 0$. Hence the manifold is a para-Kenmotsu manifold. One can easily compute,

$$\begin{array}{ccc} \tilde{R}(X,\tilde{\varphi}X)\xi = 0, & \tilde{R}(\tilde{\varphi}X,\xi)\xi = -\tilde{\varphi}X, & \tilde{R}(X,\xi)\xi = -X, \\ \tilde{R}(X,\tilde{\varphi}X)\tilde{\varphi}X = X, & \tilde{R}(\tilde{\varphi}X,\xi)\tilde{\varphi}X = -\xi, & \tilde{R}(X,\xi)\tilde{\varphi}X = 0, \\ \tilde{R}(X,\tilde{\varphi}X)X = \tilde{\varphi}X, & \tilde{R}(\tilde{\varphi}X,\xi)X = 0, & \tilde{R}(X,\xi)X = \xi. \end{array} \tag{28}$$

We have constant scalar curvature as follows,

$$\tau = S(X,X) - S(\tilde{\varphi}X,\tilde{\varphi}X) + S(\xi,\xi) = -6.$$

From (10), we conclude that M is an Einstein manifold. Moreover, M is $\tilde{\varphi}$-projectively flat for $\alpha = 1$, $\beta = 0$ ($\tau = -6 = -6(\alpha^2 + \beta^2)$).

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