

Comparing Solution Methods for a Basic RBC Model

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ABSTRACT

This short article compares different solution methods for a basic RBC model (Hansen, 1985). We solve and simulate the model using two main algorithms: the methods of perturbation and projection, respectively. One novelty is that we offer a type of the hybrid method: we compute easily a second-order approximation to decision rules and use that approximation as an initial guess for finding Chebyshev polynomials. We also find that the second-order perturbation method is most competitive in terms of accuracy for standard RBC model.

Keywords: Projectoion Method, Chebyshev, Perturabtion Method, RBC Model Approximation, DSGE Model ap-
Proximation, Computation Methods

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1. INTRODUCTION

One of the most important applications of the stochastic neoclassical growth model over the past 30 years has been to the analysis of business cycles, i.e. the recurrent fluctuations in economic activity. The real business cycle (RBC) approach, pioneered by the seminal work of Kydland and Prescott (1982) and Long and Plosser (1983), employs the neoclassical growth model with recurrent aggregate productivity shocks. As Cooley (1995) puts it in his preface, the RBC approach combines general equilibrium theory with a set of tools for computing equilibria of artificial economies and studying their empirical properties.

Its conceptual simplicity notwithstanding, the RBC theory argues that exogenous technology shocks account for most of business cycles. Thanks to its relative success in matching certain statistical moments of macro aggregates such as employment, consumption, investment and etc. the RBC theory has become the workhorse for the quantitative analysis of business cycles. In particular, the numerical simulation of dynamic general equilibrium models has become a standard procedure for macroeconomic research over the past 30 years. The methodological innovation the RBC theory provided us

with has spurred the rapid development of a variety of computational solution methods for dynamic equilibrium models.

This short article comes as a witness to illustrate the methodological contribution of the RBC theory. We present and compare different solution methods for computing one of basic RBC models, particularly, Hansen (1985) indivisible RBC model. Hansen's model, regarded as one of the most successful RBC models in terms of its conceptual simplicity and empirical performance, is the stochastic neoclassical growth model with indivisible labor choices. Nevertheless, to our best knowledge, little is reported about the relative performance of new solution methods for computing Hansen's business-cycle model.

We solve and simulate the model using four different algorithms: log-linearization, second-order perturbation, the Chebyshev projection method that uses first-order approximation to model solutions as an initial guess, and the Chebyshev projection method that uses second-order approximation to model solutions as an initial guess. One novelty is that we offer a type of the hybrid method: we compute easily a second-order approximation to all relevant optimal policy functions and use it as an initial guess for finding Chebyshev poly-

mials. This second-order approximation is such a good guess that Chebyshev polynomials will converge rapidly. This type of the hybrid method is reminiscent of the so-called value function perturbation proposed by Caldara *et al.* (2012). The value function perturbation uses a second-order approximation to the value function as an initial guess for value function iteration.

We highlight two main results from our exercise. First, the perturbation method delivers a high level of accuracy and is most competitive in terms of computing time, implementation, and accuracy for Hansen's RBC model. This result is in stark comparison with Caldara *et al.* (2012), for instance; they report that the Euler equation errors of the method of perturbation are larger than those of the Chebyshev projection method in computing the stochastic neoclassical growth model with recursive preferences. It seems that the "proper" solution method in terms of accuracy depends on preferences and model parameters such as risk aversion. Second, the Euler equation errors of the perturbation method increase as the value of capital moves far away from its steady-state. In fact, the hybrid method of projection with second-order perturbation is more accurate than the second-order perturbation method over an extreme range of capital.

Our comparison approach is not new. Indeed, Arouba *et al.* (2006) examined the accuracy of a variety of perturbation methods, including log-linearization and second-order perturbation, in computing the stochastic neoclassical growth model with Cobb-Douglas preferences as their benchmark test problem; they show that these classes of perturbations can accumulate large Euler equation errors as solution functions move away from the steady state. They also demonstrate that the method of projection is preferable in terms of accuracy; by construction, it generates solution functions which fit well over a wide range of state space. Caldara *et al.* (2012) compared four solution methods for the stochastic neoclassical growth model with recursive preferences and stochastic volatility: second and third-order perturbation, Chebyshev polynomials, and value function iteration. They examined four methods in terms of computing time, implementation complexity, and accuracy. They concluded that perturbations are competitive in terms of accuracy along with Chebyshev polynomials and value function iteration while being several orders of magnitude faster to run.

The rest of the paper proceeds as follows. Section 2 presents our test model. Section 3 describes the solution methods used to approximate the decision rules of our test model. Section 4 reports simulation results and Section 5 concludes.

2. BASIC RBC MODEL

We consider Hansen (1985)'s real business cycle (RBC) model. Hansen's RBC model is described by:

- (i) one endogenous state variable: $\{K_t\}$
- (ii) five control variables: $\{C_t, N_t, I_t, Y_t, R_t\}$
- (iii) one exogenous variable: $\{Z_t\}$

Its equilibrium conditions can be specified as follows:

$$C_t + I_t = Y_t \quad (1)$$

$$K_{t+1} = I_t + (1 - \delta)K_t \quad (2)$$

$$Y_t = Z_t K_t^\alpha N_t^{1-\alpha} \quad (3)$$

$$R_t = \alpha \frac{Y_t}{K_t} + 1 - \delta \quad (4)$$

$$1 = \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^\gamma R_{t+1} \right] \quad (5)$$

$$A = C_t^{-\gamma} (1 - \alpha) \frac{Y_t}{N_t} \quad (6)$$

$$\log Z_{t+1} = (1 - \rho) \log \bar{Z} + \rho \log Z_t + \varepsilon_{t+1} \quad (7)$$

K_t , and C_t are capital and consumption, respectively; I_t represents investment, while Y_t , and N_t represent output, and labor, respectively; R_t is defined as the gross rate of return on capital; lastly, Z_t represents exogenous productivity shocks. α stands for capital income share, while δ means the depreciation rate of capital; γ represents the constant coefficient of relative risk aversion; ρ represents the persistency of the productivity process; and ε_{t+1} follows i.i.d. white noise. The calibration of model parameters is summarized as follows.

$$\alpha = 0.36; \delta = 0.025; \gamma = 5; \rho = 0.95; \sigma_\varepsilon = 0.00712.$$

3. SOLUTION METHODS

3.1 Log-Linearization

To find a first-order approximation to the model's equilibrium conditions, we use a suite of Matlab programs provided by Uhlig (1999). Note that the system of log-linearized equations is:

$$\bar{C}\hat{c}_t + \bar{I}\hat{i}_t = \bar{Y}\hat{y}_t \quad (1)$$

$$\bar{K}\hat{k}_{t+1} = \bar{I}\hat{i}_t + (1 - \delta)\bar{K}\hat{k}_t \quad (2)$$

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_t (1 - \alpha)\hat{n}_t \quad (3)$$

$$\bar{R}\hat{r}_t = \alpha \frac{\bar{Y}}{\bar{K}} (\hat{y}_t - \hat{k}_t) \quad (4)$$

$$0 = E_t [-\gamma\hat{c}_{t+1} + \gamma\hat{c}_t + \hat{r}_{t+1}] \quad (5)$$

$$E_t [-\gamma\hat{c}_t + \hat{y}_t - \hat{n}_t] = 0 \quad (6)$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1} \quad (7)$$

Note that hat and bar represent the percent deviation and the steady-state value respectively.

3.2 2nd Order Perturbation

Schmitt-Grohé and Uribe (2004) employ a perturbation approach to find a second-order approximation to equilibrium conditions. Supposing any policy function is expressed as $y_t = g(x_t, \sigma)$, where σ is a perturbation parameter, we can obtain a second-order approximation to the decision rule as follow.

$$\begin{aligned} [g(x, \sigma)]^i &= [g(\bar{x}, 0)]^i \\ &+ [g_x(\bar{x}, 0)]^i_a [(x - \bar{x})]_a + [g_\sigma(\bar{x}, 0)]^i [\sigma] \\ &+ \frac{1}{2} [g_{xx}(\bar{x}, 0)]^i_{ab} [(x - \bar{x})]_a [(x - \bar{x})]_b \\ &+ \frac{1}{2} [g_{x\sigma}(\bar{x}, 0)]^i_a [(x - \bar{x})]_a [\sigma] \\ &+ \frac{1}{2} [g_{\sigma x}(\bar{x}, 0)]^i_a [(x - \bar{x})]_a [\sigma] \\ &+ \frac{1}{2} [g_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma][\sigma] \end{aligned}$$

Here we are using the notation suggested by Schmitt-Grohé and Uribe (2004). For example, $[f_y]_{\alpha}^i$ is the (i, α) element of the derivative of f with respect to y .

3.3 Chebyshev Projection Method

This section explains how to implement our projection technique in a step-by-step fashion and highlights the critical issues for each step. According to Judd (1992, 1998), the basic concept of projection technique is to find an operator N and a function f such that equilibrium is represented by the solution to $N(f) = 0$. $N: B_1 \rightarrow B_2$ is a vector of Euler conditions and $f: D \rightarrow R^1$ is a vector of decision rules, where B_i are function spaces and D is a vector of state spaces. To make the problem tractable, the method of projection focuses on a finite-dimensional subdomain of $\{D_{\min}, D_{\max}\}$ that can be easily represented on a computer. It may be difficult for the computer to compute N , in which case we find a computable operator \tilde{N} , which is “similar” to N . Following Judd (1992), we operationalize this idea using the residual function R . Therefore, we calculate the solutions to $R = \tilde{N}(\tilde{f}(D)) = 0$ instead (tilde means an approximated solution). Our numerical approach consists of 7 steps. Step 1 is a preliminary procedure for setting up an initial guess. Step 2 corresponds to step 1 and step 2 addressed by Judd (1992).¹⁾ Step 3 and step 4 give shape to the orthogonal collocation method, simplifying step 3 of Judd (1992). Steps 5 and 6 are also in accordance with step 4 and step 5 of

1) Judd (1992) describes his procedure for the method of projection as follows: Step 1. Choose bases and inner products; Step 2. Choose a degree of approximation, a computable approximation \tilde{N} , and a collection of functions f ; Step 3. Compute the approximation and residual function; Step 4. Compute the projections; Step 5. Iterate over step 3 and step 4.

Judd (1992) respectively. Step 7 suggests ways to extend and update each subdomain until it covers the whole domain.

(step 1) Start with an initial guess

The first step is to define decision rules for the choice variables C_t, N_t using the method of log-linearization or the method of perturbation.

(step 2) Set up Chebyshev polynomials as bases

$$\begin{aligned} \tilde{C}(K_t, Z_t | a_C) &= \sum_{i=1}^{N_K} \sum_{j=1}^{N_Z} a_{C,i,j} \psi_{i,j}(K_t, Z_t), \\ \tilde{N}(K_t, Z_t | a_N) &= \sum_{i=1}^{N_K} \sum_{j=1}^{N_Z} a_{N,i,j} \psi_{i,j}(K_t, Z_t), \end{aligned}$$

where

$$\psi_{i,j}(K_t, Z_t) = T_{i-1} \left(\frac{2(K_t - K_{\max})}{K_{\max} - K_{\min}} - 1 \right) \cdot T_{j-1} \left(\frac{2(Z_t - Z_{\max})}{Z_{\max} - Z_{\min}} - 1 \right).$$

We have to determine the approximation degree and initial range of the subdomains of state and exogenous variables. For convenience, we set $N_K = N_Z = 3$, $D_{K,0} = (K_{\min}, K_{\max}) = (0.4\bar{K}, 1.6\bar{K})$ and $D_{Z,0} = (Z_{\min}, Z_{\max}) = (0.8\bar{Z}, 1.2\bar{Z})$.

(step 3) Evaluate residual functions at collocation nodes

Collocation nodes (K_l, Z_m) are:

$$\begin{aligned} K_l &= \left(-\cos \frac{(2l-1)\pi}{2 \cdot N_K} + 1 \right) \cdot \frac{(K_{\max} - K_{\min})}{2} + K_{\min}, & l = 1, \dots, N_K \\ Z_m &= \left(-\cos \frac{(2m-1)\pi}{2 \cdot N_Z} + 1 \right) \cdot \frac{(Z_{\max} - Z_{\min})}{2} + Z_{\min}, & m = 1, \dots, N_Z \end{aligned}$$

Choose Eq. (4), Eq. (5) as residual functions:

$$\begin{aligned} R1(K_t, Z_m, \tilde{C}_t, \tilde{N}_t | a_C, a_N) &= \beta E_t \left[\left(\frac{\tilde{C}_t}{\tilde{C}_{t+1}} \right)^\gamma R(K_t, Z_m, \tilde{C}_t, \tilde{N}_t) \right] - 1 = 0 \\ R2(K_t, Z_m, \tilde{C}_t, \tilde{N}_t | a_C, a_N) &= \tilde{C}_t^{-\gamma} (1 - \alpha) \frac{Y(K_t, Z_m, \tilde{C}_t, \tilde{N}_t)}{N_t} - A = 0 \end{aligned}$$

(step 4) Calculate Chebyshev coefficients

$$\begin{aligned} C(K_t, Z_m) - \tilde{C}(K_t, Z_m | a_C) &= C(K_t, Z_m) - \sum_{i=1}^{N_K} \sum_{j=1}^{N_Z} a_{C,i,j} \psi_{i,j}(K_t, Z_m) = 0 \end{aligned}$$

$$N(K_l, Z_m) - \tilde{N}(K_l, Z_m | a_N) = N(K_l, Z_m) - \sum_{i=1}^{N_K} \sum_{j=1}^{N_Z} a_{N,i,j} \psi_{i,j}(K_l, Z_m) = 0$$

where $C(K_l, Z_m)$ and $N(K_l, Z_m)$ are the initial guesses at the collocation nodes (K_l, Z_m) . We can calculate the Chebyshev coefficients of C and N using the method of OLS.

$$a_C = (\psi' \psi)^{-1} \psi' C \quad \text{and} \quad a_N = (\psi' \psi)^{-1} \psi' N.$$

(step 5) Check equilibrium conditions and update Chebyshev coefficients

We use the residual functions defined in step 2 for an equilibrium condition check and update the Chebyshev coefficients using Newton’s method.

(step 6) Raise approximation orders and iterate

We set an initial approximation with $N_K = N_Z = 3$ and then update this up to the case with $N_K = N_Z = 13$. The calculation from step 2 to step 5 is performed in each iteration.

(step 7) Increase a range of subdomains of exogenous variables

We set the i th subdomain of Z :

$$D_{Z,i} = (0.8\bar{Z} - i\delta, 1.2\bar{Z} + i\delta), \quad \text{where } i = 0, \dots, 4, \delta = 0.1\bar{Z}.$$

After iteration, the whole subdomain of Z becomes $D_{Z,4} = (0.4\bar{Z}, 1.6\bar{Z})$, as increases. Note that the Chebyshev coefficients at the collocation nodes of the i th iteration,

$$P_l^{(i)} = \left(-\cos \frac{(2l-1)}{2 \cdot N_p \cdot \pi} + 1 \right) \cdot \frac{(P_{\max}^{(i)} - P_{\min}^{(i)})}{2} + P_{\min}^{(i)}, \quad \text{are used}$$

for calculating a “new” initial guess for the consecutive $(i+1)$ th approximation problem.

4. NUMERICAL RESULTS

4.1 Decision Rules

Figure 1 and Figure 2 plots the decision rules of the household for consumption and labor. In Figure 1, the interval for capital is $[0.4 \cdot \bar{K}, 2.5 \cdot \bar{K}]$. In Figure 2 the productivity interval is $[0.6 \cdot \bar{Z}, 1.6 \cdot \bar{Z}]$. These intervals are big enough to encompass all the simulations. Since all methods provide nearly indistinguishable answers, we observe exactly the same line around the steady-state. We can check some differences in labor supply between log-linearization and the other methods only when capital or productivity are far from its steady-state level.

4.2 Simulations

To characterize the behavior of the model through statistics from simulated paths of the economy, we simulate the model, starting from the deterministic steady state, for 150 periods and 100 times iterations, using the decision rules. Table 1 reports business cycle statistics. We see that nearly all values are the same, a simple consequence of the similarity of the decision rules

4.3 Euler Equation Errors

It is also important to evaluate the accuracy of each of the procedures. Euler equation errors, introduced by Judd (1992), have become a common tool for determining the quality of solution methods. Figure 3 plots the Euler equation errors of each solution method. It deserves several comments. First, the second-order perturba-

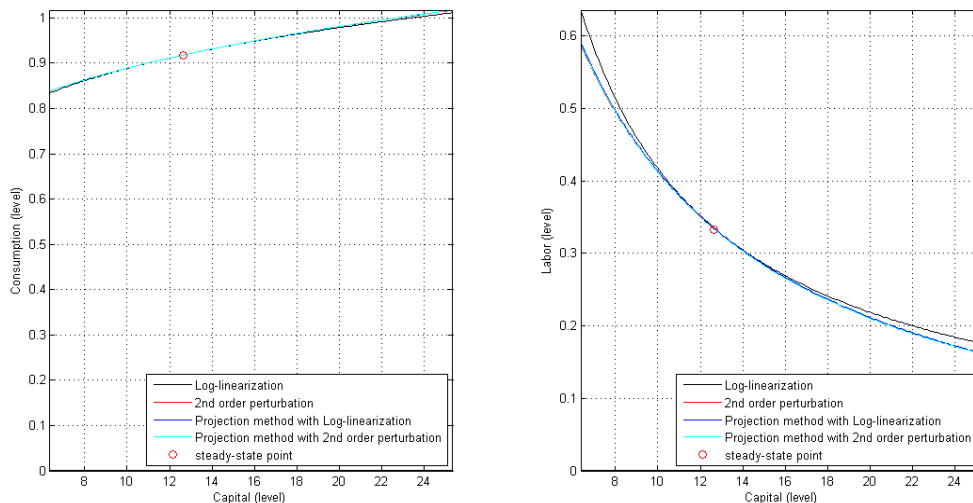


Figure 1. Policy Functions: Consumption and Labor Rates Against Capital (Assuming Productivity at Steady State)

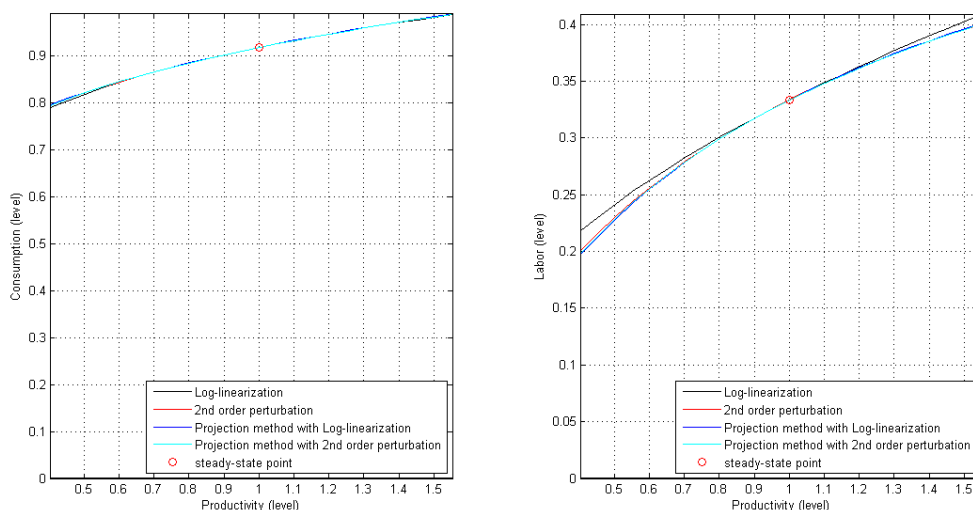


Figure 2. Policy Functions: Consumption and Labor Rates Against Productivity (Assuming Capital at Steady State)

Table 1. Business cycle statistics

	k	c	n	y	i	r
	Mean					
Log-linear	12.9716	0.9233	0.3282	1.2505	0.3272	1.0097
2nd order perturbation	12.9806	0.9233	0.3282	1.2508	0.3275	1.0097
Projection with log-linear	12.9916	0.9239	0.3285	1.2518	0.3279	1.0097
Projection with 2nd order perturbation	12.9948	0.9234	0.3283	1.2514	0.328	1.0097
	Variance					
Log-linear	0.0018	0.0012	0.0039	0.0095	0.0338	0.0003
2nd order perturbation	0.0018	0.0012	0.0039	0.0094	0.0337	0.0003
Projection with log-linear	0.0019	0.0012	0.0045	0.0099	0.0355	0.0004
Projection with 2nd order perturbation	0.0019	0.0012	0.0045	0.0098	0.0354	0.0004

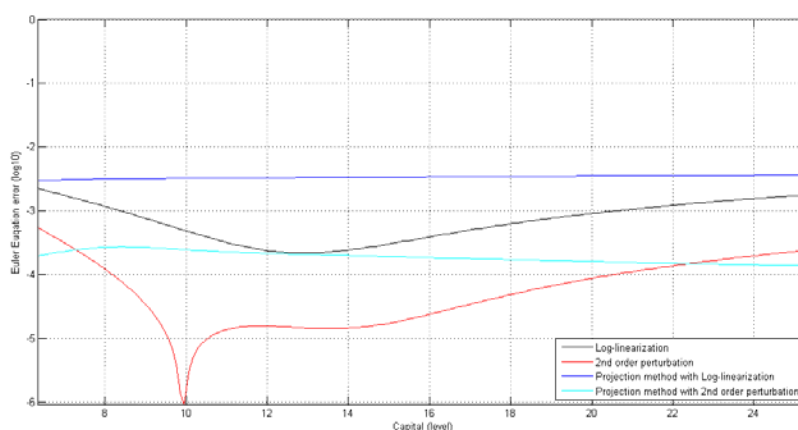


Figure 3. Hansen Model: Consumption and Labor Rates Against Productivity (Assuming Capital at Steady State)

tion method displays a better level of accuracy as compared to other methods. This result is in stark comparison with Caldara *et al.* (2012); they report that the Euler equation errors of the method of perturbation are larger than those of the Chebyshev projection method in com-

puting the stochastic neoclassical growth model with recursive preferences. It seems that the “proper” solution method in terms of accuracy depends on preferences and model parameters such as risk aversion. Second, the projection method which used second-order perturbation

for calculating an initial guess is more accurate than the projection one which used log-linearization.

Third, the Euler equation errors of the perturbation method increase as the value of capital moves far away from steady-state. In fact, Figure 3 shows that the hybrid method of projection with second-order perturbation is more accurate than the second-order perturbation method over an extreme range of capital.

4.4 Computing Times

We briefly summarize computing time on Table 2. Log-linearization method and perturbation method take only 0.3 and 1.1 seconds respectively. Calculation time for Cheyshev projection methods depends on approximation order (N_K, N_Z).

Table 2. Computing Times

Solution Methods	Elapsed Time
Log-linearization	0.27 seconds
2nd order perturbation	1.08 seconds
Projection with Log-linearization (7 order)	17.13 seconds
Projection with 2nd order perturbation (7 order)	14.41 seconds
Projection with Log-linearization (10 order)	17.13 seconds
Projection with 2nd order perturbation (10 order)	63.71 seconds
Projection with Log-linearization (13 order)	225.96 seconds
Projection with 2nd order perturbation (13 order)	171.46 seconds

5. CONCLUSIONS

The purpose of this paper is to provide comparison results of different solution methods for the standard RBC model (Hansen, 1985). Four approaches are taken into account: log-linearization, second-order perturbation, and a type of the hybrid methods which use as an initial guess first-order and second-order approximations, respectively. Two main results merit a mention. First, the perturbation method delivers a high level of accuracy and is a most reliable tool for the standard RBC model. Second, the hybrid method of projection with second-order perturba-

tion is more accurate than the second-order perturbation method over an extreme range of capital.

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