

## OPTION PRICING UNDER STOCHASTIC VOLATILITY MODEL WITH JUMPS IN BOTH THE STOCK PRICE AND THE VARIANCE PROCESSES

JU HONG KIM

ABSTRACT. Yan & Hanson [8] and Makate & Sattayatham [6] extended Bates' model to the stochastic volatility model with jumps in both the stock price and the variance processes. As the solution processes of finding the characteristic function, they sought such a function  $f$  satisfying

$$f(\ell, \nu, t; k, T) = \exp(g(\tau) + \nu h(\tau) + ix\ell).$$

We add the term of order  $\nu^{1/2}$  to the exponent in the above equation and seek the explicit solution of  $f$ .

### 1. INTRODUCTION

The Heston model [5] is the following risk-neutral stock price processes

$$(1.1a) \quad dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^S,$$

$$(1.1b) \quad d\nu_t = \kappa(\theta - \nu_t) dt + \sigma_v \sqrt{\nu_t} dW_t^\nu,$$

where  $S_t$  is a stock process,  $r$  is the riskless rate of return,  $\nu_t$  is the volatility of asset returns,  $\kappa > 0$  is a mean-reverting rate,  $\theta$  is the long term variance,  $\sigma > 0$  is the volatility of volatility, and  $W_t^S$  and  $W_t^\nu$  are two correlated Brownian motions under the risk-neutral measure with constant correlation coefficient  $\rho$ .

The Bates [1] extended the Heston model (1.1) to include jumps in the stock price process. The model has the following dynamics which define the evolution of  $S_t$  satisfying

$$(1.2a) \quad dS_t = (r - \lambda^S m) S_t dt + \sqrt{\nu_t} S_t dW_t^S + S_{t-} Y_t dN_t^S,$$

$$(1.2b) \quad d\nu_t = \kappa(\theta - \nu_t) dt + \sigma_v \sqrt{\nu_t} dW_t^\nu,$$

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where the volatility process  $\nu_t$  is the same as one in the Heston model and the driving Brownian motions in the two processes have an instantaneous correlation coefficient  $\rho$ , the process  $N_t^S$  represents a Poisson process under the risk-neutral measure, with jump intensity  $\lambda$ . The Poisson process is independent of the two Brownian motions in the stock price and the variance processes. The percentage jump size of the stock price is denoted by the random variable  $Y_t$  with log-normal distribution.

Eraker et al. [3] extended Bates model to a stochastic volatility model with contemporaneous jumps in the stock price and its volatility

$$\begin{aligned} dS_t &= (r - \lambda^S m)S_t dt + \sqrt{\nu_t} S_t dW_t^S + S_{t-} Y_t dN_t^S, \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^\nu + Z_t dN_t^\nu. \end{aligned}$$

Eraker et al. tested their model with empirical data and showed that the models with jumps performed better than those without jumps in volatility. Makate and Sattayatham [6] provide a formal 'closed-form solution' of the stochastic-volatility jump-diffusion model.

Heston's [5] 'closed-form solution' for risk-neutral pricing of European options is given by first converting the problem into characteristic functions, then using the Fourier inversion formula for probability distribution functions to find a more numerically robust form which everyone won't call it closed. To solve for the characteristic function  $f_j$  explicitly, Yan & Hanson [8] and Makate & Sattayatham [6] conjecture that its solution is given by

$$(1.3) \quad f_j(\ell, \nu, t; x, t + \tau) = \exp(g_j(\tau) + \nu h_j(\tau) + ix\ell + \beta_j(\tau))$$

where  $\beta_1(\tau) = 0$  and  $\beta_2(\tau) = r\tau$ . In this paper, we add the term of order  $\nu^{1/2}$  to the exponent in (1.3) for the exploit of nonlinearity and seek the explicit solution of  $f_j$ .

This paper is structured as follows. The introduction is given in Section 1. The stochastic-volatility jump-diffusion model is explained in detail in Section 2. The formulation for European call option pricing is given in Section 3.

## 2. STOCHASTIC-VOLATILITY JUMP-DIFFUSION MODEL

We assume that a risk-neutral probability measure  $Q$  exists. We also assume that the asset price  $S_t$  under  $Q$  follows a jump- diffusion process, and the volatility  $\nu_t$  follows a pure mean-reverting and square root diffusion process with jump, *e.g.*, our model is governed by the following dynamics

$$(2.1a) \quad dS_t = (r - \lambda^S m)S_t dt + \sqrt{\nu_t} S_t dW_t^S + S_t Y_t dN_t^S,$$

$$(2.1b) \quad d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^\nu + Z_t dN_t^\nu.$$

where  $S_t, \nu_t, \kappa, \theta, \sigma, W_t^S, W_t^\nu$  are the same ones defined as in Bates model (1.2),  $r$  is a risk-free interest rate,  $N_t^S$  and  $N_t^\nu$  are independent Poisson processes with constant intensities  $\lambda^S$  and  $\lambda^\nu$  respectively.  $Y_t$  is the jump size of the asset price return with density  $\phi_Y(y)$  and  $E[Y_t] = m$ , and  $Z_t$  is the jump size of the volatility with density  $\phi_Z(z)$ . Moreover, we assume that the Poisson processes  $N_t^S$  and  $N_t^\nu$  are independent of standard Brownian motions  $W_t^S$  and  $W_t^\nu$  with  $Corr(dW_t^S, dW_t^\nu) = \rho$ .

### 3. FORMULATION FOR EUROPEAN CALL OPTION PRICING

Let  $C$  denote the price at time  $t$  of a European style call option on  $S_t$  with strike price  $K$  and expiration time  $T$ . The terminal payoff of a European call option on the underlying stock  $S_t$  is

$$\max\{S_T - K, 0\}.$$

Assume that the short-term risk-free interest rate  $r$  is constant over the lifetime of the option. The price of the European call at time  $t$  equals the discounted and conditional expected payoff

$$\begin{aligned} C(S_t, \nu_t, t; K, T) &= e^{-r(T-t)} E_Q \left[ \max(S_T - K, 0) \middle| S_t, \nu_t \right] \\ &= e^{-r(T-t)} \left( \int_K^\infty S_T P_Q(S_T | S_t, \nu_t) dS_T - K \int_K^\infty P_Q(S_T | S_t, \nu_t) dS_T \right) \\ &= S_t \left( \frac{1}{e^{r(T-t)} S_t} \int_K^\infty S_T P_Q(S_T | S_t, \nu_t) dS_T \right) \\ &\quad - K e^{-r(T-t)} \int_K^\infty P_Q(S_T | S_t, \nu_t) dS_T \\ &= S_t \left( \frac{1}{E_Q[S_T | S_t, \nu_t]} \int_K^\infty S_T P_Q(S_T | S_t, \nu_t) dS_T \right) \\ &\quad - K e^{-r(T-t)} \int_K^\infty P_Q(S_T | S_t, \nu_t) dS_T \\ (3.1) \quad &= S_t P_1(S_t, \nu_t, T; K, T) - K P_2(S_t, \nu_t, T; K, T), \end{aligned}$$

where  $E_Q$  is the expectation with respect to the risk-neutral probability measure  $Q$  and  $P_Q(S_T | S_t, \nu_t)$  is the corresponding conditional density function given  $(S_t, \nu_t)$ . Since

$$\int_0^\infty S_T P_Q(S_T | S_t, \nu_t) dS_T = E_Q[S_T | S_t, \nu_t],$$

$$P_1(S_t, \nu_t, T; K, T) = \frac{1}{E_Q[S_T|S_t, \nu_t]} \int_K^\infty S_T P_Q(S_T|S_t, \nu_t) dS_T$$

is a risk-neutral probability such that

$$S_T > K, E_Q[S_T|S_t, \nu_t] = e^{r(T-t)} S_t.$$

$P_2(S_t, \nu_t, T; K, T) = Prob_Q(S_T > K|S_t, \nu_t)$  is the risk-neutral in-the-money probability. Note that the complement of  $P_2$  is a risk-neutral distribution function. It is difficult to find the cumulative distribution function in European option pricing. The main job is to evaluate  $P_1$  and  $P_2$  under the distribution assumptions embedded in the risk-neutral probability measure.

We make a change of variable from  $S_t$  to  $L_t = \ln S_t$ . Let  $k = \ln K$ . By the jump-diffusion chain rule,  $\ln S_t$  satisfies the SDE

$$(3.2) \quad d \ln S_t = \left( r - \lambda^S m - \frac{\nu_t}{2} \right) dt + \sqrt{\nu_t} dW_t^S + \ln(1 + Y_t) dN_t^S.$$

The value  $C$  of a European-style option as a function of  $L_t$  becomes

$$\begin{aligned} C(S_t, \nu_t, t; K, T) &= C(e^{\ln S_t}, \nu_t, t; e^{\ln K}, T) \\ &= C(e^{L_t}, \nu_t, t; e^k, T) \\ &:= \tilde{C}(L_t, \nu_t, t; k, T), \end{aligned}$$

that is, we have

$$\tilde{C}(\ell, \nu, t; k, T) = e^{-r(T-t)} E_Q [\max\{e^{L_T} - K, 0\} | L_t = \ell, \nu_t = \nu].$$

The Dynkin's theorem [4] shows a relationship between stochastic differential equations and partial differential equations. If we apply two-dimensional Dynkin's theorem for the price dynamics (3.2) and volatility  $\nu_t$  in (2.1b) to  $\tilde{C}(L_t, \nu_t, t; k, T)$ , then we obtain the following Partial Integro-Differential Equations (PIDE)

$$\begin{aligned} 0 &= \frac{\partial \tilde{C}}{\partial t} + \bar{\mathcal{A}}[\tilde{C}](\ell, \nu_t, t; k, T) \\ &+ \lambda^S \int_{\mathbb{R}} [\tilde{C}(\ell + y, \nu, t; k, T) - \tilde{C}(\ell, \nu, t; k, T)] \phi_Y(y) dy \\ &+ \lambda^\nu \int_{\mathbb{R}} [\tilde{C}(\ell, \nu + z, t; k, T) - \tilde{C}(\ell, \nu, t; k, T)] \phi_Z(z) dz, \end{aligned}$$

where  $\bar{\mathcal{A}}$  is defined as

$$\begin{aligned} \bar{\mathcal{A}}[\tilde{C}](\ell, \nu, t; k, T) &= \left( r - \lambda^S m - \frac{1}{2} \nu \right) \frac{\partial \tilde{C}}{\partial \ell} + \kappa(\theta - \nu) \frac{\partial \tilde{C}}{\partial \nu} + \frac{\nu}{2} \frac{\partial^2 \tilde{C}}{\partial \ell^2} \\ &+ \rho \sigma \nu \frac{\partial^2 \tilde{C}}{\partial \ell \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 \tilde{C}}{\partial \nu^2} - r \tilde{C}. \end{aligned}$$

In the current state variables  $L_t = \ell$  and  $\nu_t = \nu$ , the option value (3.1) becomes

$$(3.3) \quad \tilde{C}(\ell, \nu, t; k, T) = e^\ell \tilde{P}_1(\ell, \nu, t; k, T) - e^{k-r(T-t)} \tilde{P}_2(\ell, \nu, t; k, T),$$

where  $\tilde{P}_j(\ell, \nu, t; k, T) := P_j(e^\ell, \nu, t; e^k, T)$  for  $j = 1, 2$ .

**Lemma 3.1** ([6]). *The functions  $\tilde{P}_1$  in (3.3) satisfies the following PIDEs*

$$(3.4) \quad \begin{aligned} 0 &= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}[\tilde{P}_1](\ell, \nu, t; k, T) + \nu \frac{\partial \tilde{P}_1}{\partial \ell} + \rho\sigma\nu \frac{\partial \tilde{P}_1}{\partial \nu} + (r - \lambda^S m) \tilde{P}_1 \\ &\quad + \lambda^S \int_{\mathbb{R}} [(e^y - 1) \tilde{P}_1(\ell + y, \nu, t; k, T)] \phi_Y(y) dy \\ &:= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}_1[\tilde{P}_1](\ell, \nu, t; k, T), \end{aligned}$$

with the boundary condition at expiration time  $t = T$

$$\tilde{P}_1(\ell, \nu, T; k, T) = I_{\ell > k}.$$

$\tilde{P}_2$  in (3.3) also satisfies the following PIDEs

$$(3.5) \quad 0 = \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}[\tilde{P}_2](\ell, \nu, t; k, T) + r\tilde{P}_2 := \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}_2[\tilde{P}_2](\ell, \nu, t; k, T),$$

with the boundary condition at expiration time  $t = T$

$$\tilde{P}_2(\ell, \nu, T; k, T) = I_{\ell > k}.$$

$\mathcal{A}_1$  and  $\mathcal{A}_2$  in Lemma 3.1 are respectively defined as

$$\begin{aligned} \mathcal{A}_1[f](\ell, \nu, t; k, T) &= \left( r - \lambda^S m + \frac{1}{2}\nu \right) \frac{\partial f}{\partial \ell} + (\kappa(\theta - \nu) + \rho\sigma\nu) \frac{\partial f}{\partial \nu} + \frac{\nu}{2} \frac{\partial^2 f}{\partial \ell^2} \\ &\quad + \rho\sigma\nu \frac{\partial^2 f}{\partial \ell \partial \nu} + \frac{1}{2}\sigma^2\nu \frac{\partial^2 f}{\partial \nu^2} - \lambda^S m f + \lambda^S \int_{\mathbb{R}} [(e^y - 1)f(\ell + y, \nu, t; k, T)] \phi_Y(y) dy \\ &\quad + \lambda^S \int_{\mathbb{R}} [f(\ell + y, \nu, t; k, T) - f(\ell, \nu, t; k, T)] \phi_Y(y) dy \\ &\quad + \lambda^\nu \int_{\mathbb{R}} [f(\ell, \nu + z, t; k, T) - f(\ell, \nu, t; k, T)] \phi_Z(z) dz, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2[f](\ell, \nu, t; k, T) &= \left( r - \lambda^S m - \frac{1}{2}\nu \right) \frac{\partial f}{\partial \ell} + \kappa(\theta - \nu) \frac{\partial f}{\partial \nu} + \frac{\nu}{2} \frac{\partial^2 f}{\partial \ell^2} \\ &\quad + \rho\sigma\nu \frac{\partial^2 f}{\partial \ell \partial \nu} + \frac{1}{2}\sigma^2\nu \frac{\partial^2 f}{\partial \nu^2} \\ &\quad + \lambda^S \int_{\mathbb{R}} [f(\ell + y, \nu, t; k, T) - f(\ell, \nu, t; k, T)] \phi_Y(y) dy \\ &\quad + \lambda^\nu \int_{\mathbb{R}} [f(\ell, \nu + z, t; k, T) - f(\ell, \nu, t; k, T)] \phi_Z(z) dz. \end{aligned}$$

For  $j = 1, 2$  the characteristic functions for  $\tilde{P}_j(\ell, \nu, t; k, T)$  with respect to the variable  $k$  are defined as

$$f_j(\ell, \nu, t; k, T) := - \int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(\ell, \nu, t; k, T),$$

in which a minus sign is given to account for the negativity of the measure  $d\tilde{P}_j$ . For  $j = 1, 2$ ,  $f_j$  satisfies similar PIDEs as in (3.4) and (3.5)

$$(3.6) \quad \frac{\partial f_j}{\partial t} + \mathcal{A}_j[f_j](\ell, \nu, t; k, T) = 0,$$

with the boundary conditions

$$\begin{aligned} f_j(\ell, \nu, T; k, T) &= - \int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(\ell, \nu, T; k, T) \\ &= - \int_{-\infty}^{\infty} e^{ixk} (-\delta(k - \ell) dk) = e^{ix\ell}, \end{aligned}$$

since  $d\tilde{P}_j(\ell, \nu, T; k, T) = dI_{\ell > k} = dH(\ell - k) = -\delta(k - \ell) dk$ .

Let's find the characteristic functions  $f_j$  for  $j = 1, 2$ . Let  $\tau = T - t$  be the time to go. We seek the functions  $f_1$  and  $f_2$  satisfying

$$\begin{aligned} f_1(\ell, \nu, t; k, T) &= \exp\left(g_1(\tau) + \nu^{1/2}h_1(\tau) + (\nu^{1/2})^2h_2(\tau) + ix\ell\right), \\ f_2(\ell, \nu, t; k, T) &= \exp\left(g_2(\tau) + \nu^{1/2}h_3(\tau) + (\nu^{1/2})^2h_4(\tau) + ix\ell + r\tau\right), \end{aligned}$$

respectively with the boundary conditions

$$g_i(0) = 0 = h_j(0) \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3, 4.$$

**Lemma 3.2.** *The functions  $\tilde{P}_1$  and  $\tilde{P}_2$  can be computed by the inverse Fourier transforms of the characteristic function, e.g.,*

$$(3.7) \quad \tilde{P}_j(\ell, \nu, t; k, T) = \frac{1}{2} + \frac{1}{\pi} \int_{0^+}^{+\infty} \text{Re} \left[ \frac{e^{-ixk} f_j(\ell, \nu, t; k, T)}{ix} \right] dx,$$

for  $j = 1, 2$ .  $\text{Re}[\cdot]$  denote the real part of the complex number.

The characteristic function  $f_1$  is given by

$$f_1(\ell, \nu, t; k, T) = \exp\left(g_1(\tau) + \nu^{1/2}h_1(\tau) + \nu h_2(\tau) + ix\ell\right).$$

$h_2$  is given by

$$h_2(\tau) = \frac{(\eta_1^2 - \Delta_1^2) (e^{\Delta_1\tau} - 1)}{\sigma^2 [\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau}]} := \sum_{i=1}^{\infty} b_i \tau^i,$$

where  $\eta_1 = \rho\sigma(ix + 1) - \kappa$  and  $\Delta_1 = \sqrt{\eta_1^2 - \sigma^2ix(ix + 1)}$ .  $h_1$  is given by

$$h_1(\tau) = \sum_{i=1}^{\infty} a_i \tau^i, \quad a_1 = \gamma_1(0+), \quad (n + 1)a_{n+1} = \frac{\eta_1}{2}a_n + \frac{\sigma^2}{2} \sum_{\substack{i+j=n \\ 1 \leq i, j \leq n}} a_i b_j,$$

where  $\gamma_1(0+)$  represents a small value factor which appears in the coefficient of  $\nu^{1/2}$  as the one of  $\nu^{3/2}$ .

$$g_1(\tau) = \kappa\theta \int_0^\tau h_2(\tau)d\tau - \lambda^S m\tau + (r - \lambda^S m)ix\tau + \frac{\sigma^2}{8} \int_0^\tau h_1^2(\tau)d\tau \\ + \lambda^S \tau \int_{\mathbb{R}} \left( e^{(ix+1)y} - 1 \right) \phi_Y(y)dy + \lambda^\nu \int_0^\tau \int_{\mathbb{R}} \left( e^{zh_2(\tau)} - 1 \right) \phi_Z(z)dzd\tau,$$

which is equal to the equations as in [6] if the coefficient  $h_1(\tau)$  of order  $\nu^{1/2}$  is zero.

The characteristic function  $f_2$  is given by

$$f_2(\ell, \nu, T; k, T) = \exp \left( g_2(\tau) + \nu^{1/2}h_3(\tau) + \nu h_4(\tau) + ix\ell + r\tau \right).$$

$h_4$  is given by

$$h_4(\tau) = \frac{(\eta_2^2 - \Delta_1^2) (e^{\Delta_2\tau} - 1)}{\sigma^2 [\eta_2 + \Delta_2 - (\eta_2 - \Delta_2)e^{\Delta_2\tau}]} := \sum_{i=1}^{\infty} d_i \tau^i,$$

where  $\eta_2 = \rho\sigma ix - \kappa$  and  $\Delta_2 = \sqrt{\eta_2^2 - \sigma^2ix(ix - 1)}$ .

$h_3$  is given by

$$h_3(\tau) = \sum_{i=1}^{\infty} c_i \tau^i, \quad c_1 = \gamma_2(0+), \quad (n + 1)c_{n+1} = \frac{\eta_2}{2}c_n + \frac{\sigma^2}{2} \sum_{\substack{i+j=n \\ 1 \leq i, j \leq n}} c_i d_j,$$

where  $\gamma_2(0+)$  represents a small value factor which appears in the coefficient of  $\nu^{1/2}$  as the one of  $\nu^{3/2}$ .

$$g_2(\tau) = \kappa\theta \int_0^\tau h_4(\tau)d\tau + (r - \lambda^S m)ix\tau + \frac{\sigma^2}{8} \int_0^\tau h_3^2(\tau)d\tau \\ + \lambda^S \tau \int_{\mathbb{R}} \left( e^{(ix+1)y} - 1 \right) \phi_Y(y)dy + \lambda^\nu \int_0^\tau \int_{\mathbb{R}} \left( e^{zh_2(\tau)} - 1 \right) \phi_Z(z)dzd\tau,$$

which is equal to the equations as in [6] if the coefficient  $h_3(\tau)$  of order  $\nu^{1/2}$  is zero.

**Theorem 3.3.** *The value of a European call option of (3.3) is*

$$\tilde{C}(\ell, \nu, T; k, T) = e^\ell \tilde{P}_1(\ell, \nu, T; k, T) - e^{k-r(T-t)} \tilde{P}_2(\ell, \nu, T; k, T),$$

where  $\tilde{P}_1$  and  $\tilde{P}_2$  are given in Lemma 3.2.

Now we prove Lemma 3.2.

*Proof.* For the derivation of the equation (3.7), refer to the paper [6]. Let us compute PDE (3.6). First let's calculate some differentials regarding to  $f_1$ .

$$\begin{aligned}\frac{\partial f_1}{\partial t} &= (-g_1'(\tau) - \nu^{1/2}h_1'(\tau) - \nu h_2'(\tau))f_1, & \frac{\partial f_1}{\partial \ell} &= ix f_1, \\ \frac{\partial f_1}{\partial \nu} &= \left(\frac{1}{2}\nu^{-1/2}h_1(\tau) + h_2(\tau)\right) f_1, \\ \frac{\partial f_1}{\partial \ell \partial \nu} &= ix \left(\frac{1}{2}\nu^{-1/2}h_1(\tau) + h_2(\tau)\right) f_1. \\ \frac{\partial^2 f_1}{\partial \nu^2} &= \left(h_2(\tau) + \frac{1}{2}\nu^{-1/2}h_1(\tau)\right)^2 f_1 - \left(\frac{1}{4}\nu^{-3/2}h_1(\tau) - \frac{3}{4}(\text{coeff of } \nu^{3/2})\nu^{-1/2}\right) f_1 \\ &= \left(h_2^2(\tau) + \frac{1}{4}\nu^{-1}h_1^2(\tau) + h_1(\tau)h_2(\tau)\nu^{-1/2} + \dots\right) f_1 - \frac{1}{4}\nu^{-3/2}h_1(\tau)f_1.\end{aligned}$$

$$f_1(\ell + y, \nu, t; x, t + \tau) - f_1(\ell, \nu, t; x, t + \tau) = (e^{ixy} - 1)f_1(\ell, \nu, t; x, t + \tau).$$

We use the series expansion, which is valid only when  $|z| < \nu$

$$(\nu + z)^{1/2} = \nu^{1/2} + \frac{1}{2}\nu^{-1/2}z - \frac{1}{8}\nu^{-3/2}z^2 + \dots$$

in the following equation.

$$\begin{aligned}& f_1(\ell, \nu + z, t; x, t + \tau) - f_1(\ell, \nu, t; x, t + \tau) \\ &= \left[\exp\left(zh_2(\tau) + \{(\nu + z)^{1/2} - \nu^{1/2}\}h_1(\tau)\right) - 1\right] f_1 \\ &= \left[e^{zh_2(\tau)} \left\{1 + \left(\frac{1}{2}\nu^{-1/2}z - \frac{1}{8}\nu^{-3/2}z^2 + \frac{1}{16}\nu^{-5/2}z^3 + \dots\right)h_1(\tau) \right. \right. \\ &\quad \left. \left. + \frac{1}{2}\left(\frac{1}{4}\nu^{-1}z^2 + \dots\right)h_1^2(\tau)\right\} - 1\right] f_1, \\ &= \left[e^{zh_2(\tau)} \left(1 + \frac{z}{2}\nu^{-1/2}h_1(\tau) + \frac{z^2}{8}\nu^{-1}h_1^2(\tau) + \dots\right) - 1\right] f_1 \\ &= \left(e^{zh_2(\tau)} - 1\right) f_1 + \frac{z}{2}h_1(\tau)\nu^{-1/2}e^{zh_2(\tau)} f_1 + \frac{z^2}{8}h_1^2(\tau)\nu^{-1}e^{zh_2(\tau)} f_1 + \dots.\end{aligned}$$

If we substitute the above differentials and equations into the equation (3.6), then we have

$$\begin{aligned}0 &= -g_1'(\tau) - \nu h_2'(\tau) - \nu^{1/2}h_1'(\tau) + (r - \lambda^S m)ix + \frac{1}{2}ix\nu - \lambda^S m \\ &\quad + \kappa\theta \left(h_2(\tau) + \frac{1}{2}\nu^{-1/2}h_1(\tau) + \frac{3}{2}(\text{coeff of } \nu^{3/2})\nu^{1/2}\right) - \frac{1}{2}x^2\nu\end{aligned}$$



$$\begin{aligned}
 & +\nu\left(\rho\sigma(ix+1)-\kappa\right)\left(h_2(\tau)+\frac{1}{2}\nu^{-1/2}h_1(\tau)+\frac{3}{2}(\text{coeff of } \nu^{3/2})\nu^{1/2}\right) \\
 & +\frac{1}{2}\sigma^2\nu\left(h_2^2(\tau)+\frac{1}{4}\nu^{-1}h_1^2(\tau)+h_1(\tau)h_2(\tau)\nu^{-1/2}-\frac{1}{4}\nu^{-3/2}h_1(\tau)+\dots\right) \\
 & +\lambda^S\int_{\mathbb{R}}(e^{(ix+1)y}-1)\phi_Y(y)dy \\
 & +\lambda^\nu\int_{\mathbb{R}}\left[\left(e^{zh_2(\tau)}-1\right)+\frac{z}{2}h_1(\tau)\nu^{-1/2}e^{zh_2(\tau)}+\dots\right]\phi_Z(z)dz.
 \end{aligned}$$

The coefficients of  $\nu$  are

$$h_2'(\tau) = \frac{1}{2}ix + (\rho\sigma(ix+1) - \kappa)h_2(\tau) - \frac{1}{2}x^2 + \frac{1}{2}\sigma^2h_2^2(\tau).$$

The solution of  $h_2(\tau)$  is given by

$$h_2(\tau) = \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1\tau} - 1)}{\sigma^2[\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau}]},$$

where  $\eta_1 = \rho\sigma(ix+1) - \kappa$  and  $\Delta_1 = \sqrt{\eta_1^2 - \sigma ix(ix+1)}$  (See [6] for detail). The coefficients of  $\nu^{1/2}$  are

$$(3.8) \quad h_1'(\tau) = \frac{1}{2}\eta_1h_1(\tau) + \frac{1}{2}\sigma^2h_1(\tau)h_2(\tau) + \gamma_1(0+),$$

where we denote  $\gamma_1(0+)$  a small value factor which appears in the coefficient of  $\nu^{1/2}$  as the one of  $\nu^{3/2}$ . We seek  $h_1(\tau)$  as series solution such as

$$(3.9) \quad h_1(\tau) = \sum_{i=1}^{\infty} a_i\tau^i.$$

$h_2$  can be written as

$$\begin{aligned}
 (3.10) \quad h_2(\tau) &= \frac{A_1(e^{\Delta_1\tau} - 1)}{B_1 - e^{\Delta_1\tau}} = \sum_{i=1}^{\infty} b_i\tau^i, \\
 b_1 &= \frac{A_1\Delta_1}{B_1 - 1}, b_2 = \frac{A_1\Delta_1^2}{B_1 - 1} \left( \frac{1}{2} + \frac{1}{B_1 - 1} \right), \dots,
 \end{aligned}$$

where  $A_1 = \sigma^{-2}(\eta_1 + \Delta_1)$ ,  $B_1 = (\eta_1 + \Delta_1)/(\eta_1 - \Delta_1)$ . Substituting (3.9) and (3.10) into (3.8), we obtain

$$\begin{aligned}
a_1 &= \gamma_1(0+), \\
2a_2 &= \frac{\eta_1}{2}a_1, \\
3a_3 &= \frac{\eta_1}{2}a_2 + \frac{\sigma^2}{2}a_1b_1, \\
4a_4 &= \frac{\eta_1}{2}a_3 + \frac{\sigma^2}{2}(a_1b_2 + a_2b_1), \\
&\dots \\
(n+1)a_{n+1} &= \frac{\eta_1}{2}a_n + \frac{\sigma^2}{2} \sum_{\substack{i+j=n \\ 1 \leq i, j \leq n}} a_i b_j,
\end{aligned}$$

which can be solved in turn.

The constant terms are

$$\begin{aligned}
g'_1(\tau) &= \kappa\theta h_2(\tau) - \lambda^S m + (r - \lambda^S m)ix + \frac{\sigma^2}{8}h_1^2(\tau) \\
(3.11) \quad &+ \lambda^S \int_{\mathbb{R}} \left( e^{(ix+1)y} - 1 \right) \phi_Y(y) dy + \lambda^\nu \int_{\mathbb{R}} \left( e^{zh_2(\tau)} - 1 \right) \phi_Z(z) dz.
\end{aligned}$$

By integrating (3.11) from 0 to  $\tau$ , we obtain

$$\begin{aligned}
g_1(\tau) &= \kappa\theta \int_0^\tau h_2(\tau) d\tau - \lambda^S m\tau + (r - \lambda^S m)ix\tau + \frac{\sigma^2}{8} \int_0^\tau h_1^2(\tau) d\tau \\
&+ \lambda^S \tau \int_{\mathbb{R}} \left( e^{(ix+1)y} - 1 \right) \phi_Y(y) dy + \lambda^\nu \int_0^\tau \int_{\mathbb{R}} \left( e^{zh_2(\tau)} - 1 \right) \phi_Z(z) dz d\tau.
\end{aligned}$$

Similarly, we can compute  $h_3$ ,  $h_4$  and  $g_2$ . □

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DEPARTMENT OF MATHEMATICS, SUNGSHIN WOMEN'S UNIVERSITY, SEOUL 136-742, REPUBLIC OF KOREA  
*Email address:* [jhkkim@sungshin.ac.kr](mailto:jhkkim@sungshin.ac.kr)