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ISOPARAMETRIC FUNCTIONS IN S^{4n+3}

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Abstract. In this article, we consider a homogeneous function of degree four in quaternionic vector spaces and S^{4n+3} which is invariant under S^3 and $U(n+1)$-action. We show it is an isoparametric function providing isoparametric hypersurfaces in S^{4n+3} with $g = 4$ distinct principal curvatures and isoparametric hypersurfaces in quaternionic projective spaces with $g = 5$. This extends study of Nomizu on isoparametric function on complex vector spaces and complex projective spaces.

1. Introduction

A hypersurface M^n embedded in $\mathbb{R}^{n+1}$ or a unit sphere S^{n+1} is said to be *isoparametric* if it has constant principal curvatures. Isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ must have at most two distinct principal curvatures so that the classification consists of an open subset of a hyperplane and a hypersphere or a spherical cylinder $S^k \times \mathbb{R}^{n-k}$. On the other hand, isoparametric hypersurfaces in spheres are rather complicated. In 1938-1940, É. Cartan published a series of four remarkable papers [2, 3, 4, 5] about isoparametric hypersurfaces in spheres which also classified isoparametric hypersurfaces in spheres with $g = 1, 2$ or 3 distinct principal curvatures. More than thirty years later, Münzner showed that isoparametric hypersurfaces in sphere can have only $g = 1, 2, 3, 4$ or 6 distinct principal curvatures in [10, 11]. After Münzner's great achievements, many mathematicians strived to classify cases $g = 4$ and 6. Even though much progress has been made, they are still open.

The concept of isoparametric hypersurfaces in general manifolds is not completely determined. When we consider a compact hypersurface M in a compact symmetric space $\tilde{M}$, we call it isoparametric if all nearby parallel hypersurfaces of M have constant mean curvatures. Wang [15], Kimura [9], Park [13] and Xiao [16] have

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studied isoparametric hypersurfaces in $\mathbb{C}P^n$. Here the isoparametric hypersurfaces in S^{2n+1} and $\mathbb{C}P^n$ are related via the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$. Moreover, Park [13] also considered isoparametric hypersurfaces in S^{4n+3} and $\mathbb{H}P^n$ via the Hopf fibration $S^{4n+3} \to \mathbb{H}P^n$. Therefrom, we are interested in the isoparametric hypersurfaces in S^{4n+3} which are invariant under $Sp(1) = S^3$ action of Hopf fibration. In particular, we consider the work of Nomizu [12] which provided an example of $g = 4$ case in $S^{2n+1} \subset \mathbb{C}^{n+1}$ and extend his study to $S^{4n+3} \subset \mathbb{H}^{n+1}$. We construct an isoparametric function on $S^{4n+3} \subset \mathbb{H}^{n+1}$ which is homogeneous of degree four and invariant under S^3 and $U(n+1)$-action. By using this function, we obtain a homogeneous isoparametric family of hypersurfaces that presents one example of $g = 4$ case as an extension of the result of Nomizu [12]. Here, we obtain a family of isoparametric hypersurfaces in S^{4n+3} of four distinct principal curvatures with multiplicities 2, 2, $2n-1$, $2n-1$. Moreover by applying [13] the family also induces a family of isoparametric hypersurfaces in $\mathbb{H}P^n$ having $g = 5$ with multiplicities 3, 2, 2, $2n-4$, $2n-4$ via Hopf fibration on S^{4n+3} to $\mathbb{H}P^n$, and a family of isoparametric hypersurfaces in $\mathbb{C}P^{2n+1}$ having $g = 5$ with multiplicities 1, 2, 2, $2n-2$, $2n-2$ via Hopf fibration on S^{4n+3} to $\mathbb{C}P^{2n+1}$.

2. PRELIMINARIES

In this section, following [7], [12] and [14], we recall the definition of isoparametric function and isoparametric family in a real space form $\tilde{M}_c^{n+1}$ which is an $(n+1)$-dimensional, simply connected, complete Riemannian manifolds with constant sectional curvature $c\,(=1,0,-1)$. Here $\tilde{M}_1^{n+1}$ is a unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, $\tilde{M}_0^{n+1}$ is $\mathbb{R}^{n+1}$, and $\tilde{M}_{-1}^{n+1}$ is the hyperbolic space H^{n+1}.

Isoparametric family, parallel hypersurfaces, and constant principal curvatures. A non-constant real-valued function F defined on a connected open subset of $\tilde{M}_c^{n+1}$ is called an *isoparametric function* if it satisfies a system of differential equations

$$|\operatorname{grad} F|^2 = T \circ F, \ \Delta F = S \circ F,$$

for some smooth function T and S where ΔF is the Laplacian of F. Moreover, the collection of an 1-parameter hypersurface of $\tilde{M}_c^{n+1}$ which is equal to level sets $F^{-1}(t)$ is called an *isoparametric family* of $\tilde{M}_c^{n+1}$.

For each connected oriented hypersurface M^n embedded in $\tilde{M}_c^{n+1}$ with a unit normal vector field ξ on it, we define a map

$$\begin{array}{rccc} \phi : & M^n \times \mathbb{R} & \rightarrow & \tilde{M}_c^{n+1} \\ & (X,t) & \mapsto & \phi(X,t) \end{array}$$

where $\phi(X,t)$ is a point in $\tilde{M}_c^{n+1}$ reached after moving the point X of M^n by t along the normal geodesic $\alpha(s)$ in $\tilde{M}_c^{n+1}$ with $\alpha(0) = X$ and $\alpha'(0) = \xi_X$ (unit normal vector of ξ at X). For each fixed $t \in \mathbb{R}$, the image $\phi(M^n, t)$ of M^n is called *parallel hypersurface.*

A connected hypersurface M^n in the space form $\tilde{M}_c^{n+1}$ is said to have *constant principal curvatures* if there are distinct constants $\lambda_1, \ldots, \lambda_g$ representing principal curvatures given by a field of unit normal vector ξ at every point. Here, the multiplicity m_i of λ_i is same throughout M^n and $\sum_{i=1}^{g} m_i = n$. If the oriented hypersurface M^n has constant principal curvatures, one can show that each parallel hypersurface M_t^n also has constant principal curvatures. In particular, it is known that the level hypersurfaces of an isoparametric function F form a family of parallel hypersurfaces with constant principal curvatures. Conversely, for each connected hypersurface M^n of $\tilde{M}_c^{n+1}$ with constant principal curvatures, we can construct an isoparametric function F so that each $\phi(M^n, t)$ is contained in a level set of F which turns out a level set itself. In conclusion, an *isoparametric hypersurface* of the isoparametric family is defined as a hypersurface with constant principal curvatures.

Cartan's work on isoparametric hypersurfaces. Cartan considered an isoparametric hypersurface M^n of $\tilde{M}_c^{n+1}$ with g distinct principal curvatures $\lambda_1, \ldots, \lambda_g$, having respective multiplicities $m_1, \ldots, m_g$. For $g > 1$, Cartan showed an important equation known as Cartan's identity([4])

$$\sum_{j \neq i} m_j \frac{c + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0$$

for each i, $1 \leq i \leq g$. From this, he was able to determine all isoparametric hypersurfaces in the cases $c = 0$ and $c = -1$.

For $\tilde{M}_1^{n+1} = S^{n+1}$, Cartan ([3]) provided examples of isoparametric hypersurfaces with $g = 1, 2, 3$ or 4 distinct principal curvatures. Moreover, he classified isoparametric hypersurfaces with $g \leq 3$ as follows.

(i) ($g = 1$) The isoparametric hypersurface M^n with $g = 1$ is totally umbilic, thus M^n is an open subset of a great or small hypersphere in S^{n+1}.

(ii) ($g = 2$) The isoparametric hypersurface M^n with $g = 2$ is a standard product of two spheres with radius r_1 and r_2 in the unit sphere $S^{n+1}(1)$ in $\mathbb{R}^{n+2}$, namely,

$$M^n = S^p(r_1) \times S^q(r_2) \subset S^{n+1}(1) \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+2},$$

where $r_1^2 + r_2^2 = 1$ and $n = p + q$.

(iii) ($g = 3$) The isoparametric hypersurface M^n with $g = 3$ must have the principal curvatures with the same multiplicity $m = 1, 2, 4$ or 8, and it is a tube of constant radius over a standard embedding of a projective plane $\mathbb{F}P^2$ into S^{3m+1}, where $\mathbb{F}$ is the division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, for $m = 1, 2, 4$ and 8 respectively. Therein, it was showed that any isoparametric family with g distinct principal curvatures of the same multiplicity can be defined by a function F on $\mathbb{R}^{n+2}$ satisfying the equation

$$F|_{S^{n+1}} = \cos gt.$$

Note the function F is a harmonic homogeneous polynomial of degree g on $\mathbb{R}^{n+2}$ with

$$\left|\operatorname{grad}^E F\right|^2 = g^2 r^{2g-2},$$

where the $\operatorname{grad}^E F$ is the Euclidean gradient and $r(X) = |X|$.

Münzner's work on isoparametric hypersurfaces in spheres. In the papers [10][11], an important generalization of Cartan's work was produced on isoparametric hypersurface in sphere in 1973 by Münzner. Without assuming that the multiplicities are all the same, he proved that the possibilities for the number g are $1, 2, 3, 4$ and 6 ([14]). Moreover he obtained the following.

If M is a connected oriented isoparametric hypersurface embedded in S^{n+1} with g distinct principal curvatures, there exists a homogeneous polynomial F of degree g on $\mathbb{R}^{n+2}$ such that M is an open subset of a level set of the restriction of F to S^{n+1} satisfying the following *Cartan-Münzner differential equations*,

$$\left|\operatorname{grad}^E F\right|^2 = g^2 r^{2g-2} \tag{1}$$

$$\Delta^E F = c r^{g-2}, \tag{2}$$

where $r(X) = |X|$, $c = g^2(m_2 - m_1)/2$, and m_1, m_2 are the two possible distinct multiplicities of the principal curvatures. If all the multiplicities are equal, then $c = 0$ and F is harmonic on $\mathbb{R}^{n+2}$. Therefore this generalizes the work of Cartan on the case $g = 3$ where all multiplicities are equal and F is harmonic. The homogeneous polynomial F satisfying Cartan-Münzer differential equations is called a *Cartan-Münzner polynomial.*

Conversely, the level sets of the restriction $F|_{S^{n+1}}$ of F satisfying (1) and (2) constitute an isoparametric family of hypersurfaces. Münzner also proved that if M

is an isoparametric hypersurfaces with principal curvatures $\cot\theta_i,\ 0 < \theta_1 < \ldots < \theta_g < \pi$, with multiplicities m_i, then

$$\theta_k = \theta_1 + \frac{k-1}{g}\pi,\ 1 \leq k \leq g,$$

and the multiplicities satisfy $m_i = m_{i+2}$ (subscripts mod g). Therefore, if g is odd, then all of the multiplicities must be equal and if g is even, there are at most two distinct multiplicities.

Remark. Moreover, by [1], [10] and [11] we know (1) if $g = 3$, then $m_1 = m_2 = m_3 = 1, 2, 4$ or 8, (2) if $g = 4$, then $m_1 = m_3$, $m_2 = m_4$, and m_1, m_2 are 1 or even, (3) if $g = 6$, then $m_1 = m_2 = ... = m_6 = 1$ or 2.

3. Homogeneous Functions on Quaternionic Vector Spaces

Homogeneous functions on $\mathbb{R}^{n+2}$ and S^{n+1}. In this section, by following [8], we review computation of the gradient and the Laplacian of a homogeneous function F on $\mathbb{R}^{n+2}$ and S^{n+1} the unit sphere in $\mathbb{R}^{n+2}$.

Let $F : \mathbb{R}^{n+2} \to \mathbb{R}$ be a homogeneous function of degree g, that is, $F(tX) = t^g F(X)$ for all nonzero $t \in \mathbb{R}$ and $X \in \mathbb{R}^{n+2}$. Note the homogeneous function satisfies the following equation by Euler's theorem for $X \in \mathbb{R}^{n+2}$

$$\left\langle X, \operatorname{grad}^E F \right\rangle = gF(X). \tag{3}$$

If F is an isoparametric function defined on a Euclidean space $\mathbb{R}^{n+2}$, the restriction of F to the unit sphere S^{n+1} is also isoparametric by the following theorem.

Theorem 1. *For a homogeneous function $F : \mathbb{R}^{n+2} \to \mathbb{R}$ of degree g, we have*

$$\begin{aligned} \left|\operatorname{grad}^S F\right|^2 = {} & \left|\operatorname{grad}^E F\right|^2 - g^2F^2 \\ \Delta^S F = {} & \Delta^E F - g(g-1)F - g(n+1)F. \end{aligned}$$

Here, $\operatorname{grad}^S F$ *is the gradient of the restriction of F to the unit sphere S^{n+1}. Similarly, $\Delta^E F$ and $\Delta^S F$ denote the Laplacian of F on $\mathbb{R}^{n+2}$ and the unit sphere S^{n+1} respectively.*

Proof. (1) Let $X \in S^{n+1}$. Since X is a position vector of the unit sphere S^{n+1}, $\operatorname{grad}^S F$ can be written by

$$\operatorname{grad}^S F = \operatorname{grad}^E F - \left\langle \operatorname{grad}^E F, X \right\rangle X. \tag{4}$$

Using (3),

$$\begin{aligned}\left|\operatorname{grad}^S F\right|^2 &= \left\langle \operatorname{grad}^E F - \left\langle \operatorname{grad}^E F, X\right\rangle X, \operatorname{grad}^E F - \left\langle \operatorname{grad}^E F, X\right\rangle X\right\rangle \\ &= \left|\operatorname{grad}^S F\right|^2 - 2gF(X)\left\langle \operatorname{grad}^E F, X\right\rangle X + g^2F^2(X)\,|X|^2 \\ &= \left|\operatorname{grad}^S F\right|^2 - g^2F^2(X)\end{aligned}$$

(2) Let ∇^E and ∇^S denote the Levi-Civita connections on $\mathbb{R}^{n+2}$ and S^{n+1} respectively. Then $\Delta^S F$ is the trace of the operator on $T_X S^{n+1}$ given by

$$\begin{array}{ccc} T_X S^{n+1} & \longrightarrow & T_X S^{n+1} \\ V & \longmapsto & \nabla^S_V \operatorname{grad}^S F. \end{array}$$

For an orthonormal basis $\{V_1, \ldots, V_{n+1}\}$ for $T_X S^{n+1}$,

$$\begin{aligned}\Delta^S F &= \sum_{i=1}^{n+1} \left\langle \nabla^S_{V_i} \operatorname{grad}^S F, V_i\right\rangle = \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^S F - \left\langle \nabla^E_{V_i} \operatorname{grad}^S F, X\right\rangle X, V_i\right\rangle \\ &= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^S F, V_i\right\rangle = \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} (\operatorname{grad}^E F - \ gFX), V_i\right\rangle\end{aligned}$$

Here we use $\langle V_i, X\rangle = 0$ and $\operatorname{grad}^S F = \operatorname{grad}^E F - \ gFX$ by (3) and (4). Since X is just an identity map on $\mathbb{R}^{n+2}$, we obtain

$$\begin{aligned}\nabla^E_{V_i}(\operatorname{grad}^E F - \ gFX) &= \nabla^E_{V_i} \operatorname{grad}^E F - \ \nabla^E_{V_i}(gFX) \\ &= \nabla^E_{V_i} \operatorname{grad}^E F - gV_i(F)X - gFV_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i}(\operatorname{grad}^E F - \ gFX), V_i\right\rangle &= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F - gV_i(F)X - gFV_i, V_i\right\rangle \\ &= \sum_{i=1}^{n+1} \left(\left\langle \nabla^E_{V_i} \operatorname{grad}^E F, V_i\right\rangle - gF\right) \\ &= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F, V_i\right\rangle - g(n+1)F.\end{aligned}$$

Thus we have

$$\Delta^S F = \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F, V_i\right\rangle - g(n+1)F.$$

Now we compute the Laplacian $\Delta^E F$ for the orthonormal basis $\{V_1, \dots, V_{n+1}, X\}$ for $T_X\mathbb{R}^{n+2}$ by applying the above identity and Euler theorem.

$$\begin{aligned}
\Delta^E F &= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F,\ V_i \right\rangle + \left\langle \nabla^E_X \operatorname{grad}^E F, X \right\rangle \\
&= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F,\ V_i \right\rangle + \nabla^E_X \left\langle \operatorname{grad}^E F, X \right\rangle - \left\langle \operatorname{grad}^E F, X \right\rangle \\
&= \sum_{i=1}^{n+1} \left\langle \nabla^E_{V_i} \operatorname{grad}^E F,\ V_i \right\rangle + g(g-1)F \\
&= \Delta^S F + g(n+1)F + g(g-1)F
\end{aligned}$$

□

Remark. When the homogeneous polynomial F of degree g satisfying the Cartan-Münzner differential equations, we conclude the followings. Notice that $Im(F|_{S^{n+1}}) \subset [-1,1]$, in fact the image ranges exactly the whole compact connected set $[-1,1]$. We can consider a level set M_c of $F|_{S^{n+1}}$ defined by

$$M_c := \left\{ X \in S^{n+1} \mid F(X) = c \right\} = \left(F|_{S^{n+1}}\right)^{-1}(c),\ c \in [-1,1]\,.$$

Then M_c ($c \in (-1,1)$) is an isoparametric hypersurface, while $M_1 = \left(F|_{S^{n+1}}\right)^{-1}(1)$ and $M_{-1} = \left(F|_{S^{n+1}}\right)^{-1}(-1)$ are focal submanifolds. In other words, we can denote the level set by

$$M_t = \left\{ X \in S^{n+1} | F(X) = \cos gt \right\},\ t \in [0, \frac{\pi}{g}],$$

where M_0 and $M_{\pi/g}$ are two focal submanifolds and for $t \in (0, \frac{\pi}{g})$, M_t is an isoparametric hypersurface.

Quaternionic vector spaces and quaternionic projective spaces. Each element of $\mathbb{H}$ quaternions can be represented as

$$q = a + bi + cj + dk \in \mathbb{H}$$

with $a, b, c, d \in \mathbb{R}$, and the quaternion multiplication is determined by $i^2 = j^2 = k^2 = ijk = -1$. The standard conjugate of q which is denoted by $\bar{q}$ is the quaternion number $\bar{q} := a - bi - cj - dk$. Moreover, we define the norm of q as $|q| := q\bar{q}$. It is well known that $\mathbb{H}$ is one of the composition algebras satisfying $|q_1 q_2| = |q_1|\,|q_2|$ for $q_1, q_2 \in \mathbb{H}$. If $q \in \mathbb{C}$, the complex number, then $\bar{q}$ is the ordinary complex conjugate of q, and if $q \in \mathbb{R}$, $\bar{q} = q$.

On the other hand, if we regard the field of complex number $\mathbb{C}$ spanned by $\{1, k\}$, we can also present the quaternion number q as

$$q = z + wi, \text{ for } z = a + dk, \ w = b + ck, \text{ where } z, w \in \mathbb{C} = span\{1, k\}.$$

From this, define another conjugate $\widetilde{q}$ of q by

$$\widetilde{q} = \widetilde{z + wi} := \bar{z} + wi,$$

where $\bar{z}$ is the conjugate on $\mathbb{C} = span\{1, k\}$. Moreover, for an $n \times m$ matrices $A = (a_{ij}) \in M_{n\times m}(\mathbb{H})$, we define $\widetilde{A}$ by $\widetilde{A} := (\widetilde{a_{ij}})^t = (\widetilde{a_{ji}})$. Then the conjugation gives us the following lemma.

Lemma 2. *For $q_1, q_2 \in \mathbb{H}$,*

(1) $\widetilde{q_1 + q_2} = \widetilde{q_1} + \widetilde{q_2}$, $\widetilde{q_1 q_2} = \widetilde{q_2}\widetilde{q_1}$

(2) $|\widetilde{q_1}| = |q_1|$

(3) $\widetilde{AB} = \widetilde{B}\widetilde{A}$ *for* $A \in M_{n\times m}(\mathbb{H})$, $B \in M_{m\times l}(\mathbb{H})$

(4) *For* $A \in M_{n\times m}(\mathbb{C})$ *with* $\mathbb{C} = span\{1, k\}$, $\widetilde{A} = A^*$ *where* $A^* = \bar{A}^t$

Now we recall the construction of the quaternionic projective space by Hopf fibration. We consider a $4(n+1)$-dimensional quaternionic space over $\mathbb{R}$

$$\mathbb{H}^{n+1f} = \{q = (q_0, \dots q_n) | q_i \in \mathbb{H}, \ i = 0, \dots, n\}$$

which is also a right $\mathbb{H}$-module, that is, for $\lambda \in \mathbb{H}$, $q = (q_0, \dots, q_n) \in \mathbb{H}^{n+1}$,

$$q \cdot \lambda := (q_0 \lambda, \dots q_n \lambda) \in \mathbb{H}^{n+1}.$$

And the unit sphere S^{4n+3} in $\mathbb{H}^{n+1}$ is defined as

$$S^{4n+3} := \left\{ q = (q_0, \dots, q_n) \in \mathbb{H}^{n+1} \mid |q|^2 = \sum_{i=0}^{n} |q_i|^2 = 1 \right\}.$$

The *quaternionic projective n-space* $\mathbb{H}P^n$ is obtained as the quotient of the unit sphere S^{4n+3} by the right $Sp(1)(= S^3)$-action, that is, $\mathbb{H}P^n \cong S^{4n+3}/S^3$. Note that $U(n+1)$ acts on $\mathbb{H}^{n+1}$ and S^{4n+3} by the matrix multiplication

$$\begin{array}{ccc} U(n+1) \times \mathbb{H}^{n+1} & \longrightarrow & \mathbb{H}^{n+1} \\ (A, q) & \longmapsto & A \cdot q \ := Aq \end{array}$$

where q is represented as column matrix. Moreover, $U(n+1)$ also acts on $\mathbb{H}P^n$ by $(A, [q]) \mapsto [Aq]$, where $A \in U(n+1)$, $[q] \in \mathbb{H}P^n$. Here $[q] \in \mathbb{H}P^n$ is related to $q \in \mathbb{H}^{n+1}$ via Hopf fibration.

Homogeneous functions on quaternionic vector spaces. In this subsection we construct a homogeneous function F on $\mathbb{H}^{n+1}$ which is invariant under $Sp(1)$ and $U(n+1)$. Furthermore we will induce $\tilde{F}$ from F which is defined on the $\mathbb{H}P^n$ that also invariant under those.

Define a function F on $\mathbb{H}^{n+1}$ by

$$\begin{array}{rcccl} F: & \mathbb{H}^{n+1} & \longrightarrow & \mathbb{R} & \\ & q & \longmapsto & F(q) & := |\widetilde{q}q|^2 = \left|\sum_{i=0}^{n} \widetilde{q_i} q_i\right|^2, \end{array}$$

where the column vector $q = (q_0, \ldots, q_n)^t \in \mathbb{H}^{n+1}$ and $\widetilde{q} := (\widetilde{q_0}, \ldots, \widetilde{q_n})$ which is the conjugate transpose of q.

Lemma 3. *For the function F defined on $\mathbb{H}^{n+1}$,*
(1) F is invariant under $Sp(1) = S^3$ and $U(n+1)$.
(2) F is a homogeneous function of degree 4.

Proof (1) Let $\lambda \in S^3$, $A \in U(n+1)$, and $q \in \mathbb{H}^{n+1}$ the column vector. Using lemma2,

$$\begin{aligned} F(A \cdot q \cdot \lambda) = F(Aq\lambda) &= \left|\widetilde{(Aq\lambda)}Aq\lambda\right|^2 \\ &= \left|\widetilde{\lambda}\widetilde{q}\widetilde{A}Aq\lambda\right|^2 = \left|\widetilde{\lambda}\widetilde{q}A^*Aq\lambda\right|^2 \\ &= \left|\widetilde{\lambda}\right|^2 |\widetilde{q}q|^2 |\lambda|^2 \\ &= F(q), \end{aligned}$$

we complete the proof of (1).

(2) For $t \in \mathbb{R}$, $q \in \mathbb{H}^{n+1}$, $\widetilde{tq} = t\widetilde{q}$ and $tq = qt$ obviously. Therefore

$$F(tq) = \left|\widetilde{(tq)}tq\right|^2 = |t\widetilde{q}tq|^2 = \left|t^2\widetilde{q}q\right|^2 = t^4 F(q).$$

□

Remark. Notice that the restriction of F to the unit sphere S^{4n+3} is also a homogeneous function of degree 4 and invariant under the action of $Sp(1) = S^3$ and $U(n+1)$.

Now we induce a homogeneous function $\tilde{F}$ on $\mathbb{H}P^n$ with the following diagram.

$$\begin{tikzcd}
F: & S^{4n+3} \arrow[r] \arrow[d, "S^3"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, equal] \\
\tilde{F}: & \mathbb{H}P^n \arrow[r] & \mathbb{R} \\
& {[q]} \arrow[r, mapsto] & \tilde{F}([z]) := |\widetilde{q}q|^2,
\end{tikzcd}$$

$[q]$ is corresponding to $q \in S^{4n+3}$. Using the same procedure, we get that $\tilde{F}$ is homogeneous with degree 4 and $U(n+1)$-invariant. Moreover, if $q = (q_0, \dots, q_n)^t \in S^{4n+3}$,

$$0 \le |\widetilde{q}q|^2 = \left|\sum_{i=0}^{n} \widetilde{q}_i q_i\right|^2 \le \left(\sum_{i=0}^{n} |\widetilde{q}_i q_i|\right)^2 = \left(\sum_{i=0}^{n} |q_i|^2\right)^2 = 1,$$

thus both of the images of the restriction $F|_{S^{4n+3}}$ and $\tilde{F}$ are the closed unit interval $[0,1]$ in fact exactly $[0,1]$.

The isoparametric function on S^{4n+3}. In this subsection we prove our homogeneous function F on the unit sphere S^{4n+3} is isoparametric on S^{4n+3}.

Theorem 4. *The homogeneous function* $F(q) = |\widetilde{q}q|^2$ *on the unit sphere* $S^{4n+3} = \{q \in \mathbb{H}^{n+1} |\, |q| = 1\}$ *satisfies*

$$\left|\operatorname{grad}^S F\right|^2 = 16F(1-F), \ \Delta^S F = 24 - 12F - 4(n+1)F,$$

therefore F is isoparametric on the sphere S^{4n+3}.

Proof. The column vector $q = (q_0, \dots, q_n)^t \in \mathbb{H}^{n+1}$ can be denoted as

$$q = z + wi, \ z, w \in \mathbb{C}^{n+1}$$

where $\mathbb{C} = span\,\{1, k\}$. And we can write

$$F(q) = |\widetilde{q}q|^2 = \left(|z|^2 - |w|^2\right)^2 + 4\,|z^* w|^2\,,$$

where the standard real inner product in $\mathbb{C}^{n+1}$ is given by

$$\langle x, y\rangle := \operatorname{Re}\left(x^* y\right), \ x, y \in \mathbb{C}^{n+1}.$$

Here we consider $x, y \in \mathbb{C}^{n+1}$ as vectors in $\mathbb{R}^{2n+2}$. We also denote $\operatorname{Im}\left(x^* y\right)$ as $-\omega\,(z, w)$ which is a skew symmetric form on $\mathbb{R}^{2n+2}$, and we write

$$\begin{aligned} z^* w &= \left(\langle a, c\rangle + \langle b, d\rangle\right) - \left(\langle b, c\rangle - \langle a, d\rangle\right) k \\ &= Re\left(z^* w\right) + Im\left(z^* w\right) k \\ &= \langle z, w\rangle - \omega\,(z, w)\,k, \ \text{where } z = a + dk, w = c + dk, \ a, b, c, d \in \mathbb{R}^{n+1}. \end{aligned}$$

Then

$$\frac{\partial F}{\partial a_i} = 2\left(|a|^2 + |b|^2 - |c|^2 - |d|^2\right) 2a_i + 8\,\langle z, w\rangle\, c_i + 8\omega\,(z, w)\,(-d_i)$$

$$\begin{aligned}
\frac{\partial F}{\partial b_i} &= 2\left(|a|^2+|b|^2-|c|^2-|d|^2\right)2b_i+8\left\langle z,w\right\rangle d_i+8\omega\left(z,w\right)c_i\\
\frac{\partial F}{\partial c_i} &= 2\left(|a|^2+|b|^2-|c|^2-|d|^2\right)2c_i+8\left\langle z,w\right\rangle a_i+8\omega\left(z,w\right)b_i\\
\frac{\partial F}{\partial d_i} &= 2\left(|a|^2+|b|^2-|c|^2-|d|^2\right)2d_i+8\left\langle z,w\right\rangle b_i+8\omega\left(z,w\right)\left(-a_i\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 F}{\partial a_i^2} &= 4\left(|a|^2+|b|^2-|c|^2-|d|^2\right)+8\left(a_i^2+c_i^2+d_i^2\right)\\
\frac{\partial^2 F}{\partial b_i^2} &= 4\left(|a|^2+|b|^2-|c|^2-|d|^2\right)+8\left(b_i^2+c_i^2+d_i^2\right)\\
\frac{\partial^2 F}{\partial c_i^2} &= 4\left(|a|^2+|b|^2-|c|^2-|d|^2\right)+8\left(a_i^2+b_i^2+c_i^2\right)\\
\frac{\partial^2 F}{\partial d_i^2} &= 4\left(|a|^2+|b|^2-|c|^2-|d|^2\right)+8\left(a_i^2+b_i^2+d_i^2\right).
\end{aligned}$$

Therefore, we obtain

$$\left|\operatorname{grad}^E F\right|^2 = 16\left|q\right|^2 F,\ \Delta^E F = 24\left|q\right|^2.$$

Using Theorem 1, we get

$$\begin{aligned}
\left|\operatorname{grad}^S F\right|^2 &= \left|\operatorname{grad}^E F\right|^2 - 4^2F^2 = 16F\left(1-F\right),\\
\Delta^S F &= \Delta^E F - 4\left(4-1\right)F - 4\left(n+1\right)F = 24 - 12F - 4\left(n+1\right)F
\end{aligned}$$

since F is homogeneous of degree 4. ☐

Remark. Theorem 4 extends Nomizu's work ([12]) on construction of isoparametric function on $S^{2n+1} = \left\{z \in \mathbb{C}^{n+1} \middle| \, |z| = 1\right\}$ by

$$h(z) = \left|z^t z\right|^2 = \left(|x|^2-|y|^2\right)^2 + 4\left\langle x,y\right\rangle^2,\ \text{for } z = x+iy\,,\ x,y \in \mathbb{R}^{n+1}.$$

The function $h\left(z\right)$ is invariant under the actions of $U\left(1\right) = S^1$ and $O\left(n+1\right)$. Moreover, it is an isoparametric function indeed since it satisfies

$$\left|\operatorname{grad}^S h\right|^2 = 16h\left(1-h\right),\ \Delta^S h = 16 - 12h - 4\left(n+1\right)h,$$

by correcting the computation of the Laplacian of h in [12]. In [12], he showed that h induces isoparametric hypersurfaces in S^{2n+1} of $g = 4$ with multiplicities 1, 1, $n-1$, $n-1$.

By above Theorem4, the homogeneous function does not satisfy Cartan-Münzner differential equations. In the following theorem, we show that the isoparametric function gives a family of isoparametric hypersurfaces with $g = 4$.

Theorem 5. *For the above isoparametric function F, the level sets M_t forms the isoparametric family of hypersurfaces in S^{4n+3} with four distinct principal curvatures with multiplicities* 2, 2, $2n-1$, $2n-1$.

Proof. By Münzner [10], the hypersurface M_t in the sphere can have only $g = 1, 2, 3, 4$ or 6 distinct principal curvatures. We consider the preimage of F at zero which is a focal set in S^{4n+3}. For the preimage of F at zero, we have

$$0 = |\widetilde{q}q|^2 = \left|\left(\bar{z}^t + w^t i\right)(z + wi)\right|^2 = \left||z|^2 - |w|^2 + 2z^* wi\right|^2,$$

and that we get $|z| = |w|$ and $z^* w = 0$. Since $|q|^2 = |z|^2 + |w|^2 = 1$, $F^{-1}(0)$ consists of ordered orthogonal complex vectors in $\mathbb{C}^{n+1}$ of length $1/2$. In other words, $F^{-1}(0)$ in S^{4n+3} is equivalently $\frac{U(n+1)}{U(n-1)}$ a complex Stiefel manifold of all orthogonal pairs of vectors in $\mathbb{C}^{n+1}$ which has dimension $4n$.

Therefore each isoparametric hypersurface of dimension $4n + 2$ has one principal curvature with multiplicity 2. By dimension counting according to Remark 2, we conclude $g = 4$. In particular, the four distinct principal curvature of isoparametric hypersurfaces in consideration have multiplicities 2, 2, $2n - 1$, $2n - 1$. ☐

Remark.

1. In [13], Park showed that the possible g in S^{4n+3} are only 2 and 4. Since we show that a focal set $F^{-1}(0)$ is not a sphere, we exclude the case $g = 2$ so that we conclude $g = 4$.

2. The preimage $F^{-1}(1)$ which is the other focal set and the isoparametric hypersurface $F^{-1}(t)$, $t \in (0, 1)$ are homogeneous spaces. Identifying these spaces is an interesting question related to Veronese imbedding of complex projective spaces. We will explain it in another paper.

A complex projective space $\mathbb{C}P^n$ is obtained from the Hopf fibration π of the unit sphere S^{2n+1} by the unit sphere S^1. The isoparametric hypersurface M in $\mathbb{C}P^n$ has constant principal curvatures if and only if M is homogeneous([9]). Moreover, a hypersurface M in $\mathbb{C}P^n$ is isoparametric if and only if its inverse image $\pi^{-1}(M)$ under the well known Hopf map is isoparametric in S^{2n+1}([15]). Similar study on quaternionic projective space $\mathbb{H}P^n$ such as [13] is also very interesting. By applying the study in [13], we also obtain the following corollary.

Corollary 6. *By Hopf fibration on* S^{4n+3} *to* $\mathbb{H}P^n$*, the isoparametric hypersurfaces in* S^{4n+3} *given by* F *induce a family of isoparametric hypersurfaces in* $\mathbb{H}P^n$ *having* $g = 5$ *with multiplicities* 3, 2, 2, $2n-4$, $2n-4$. *By Hopf fibration on* S^{4n+3} *to* $\mathbb{C}P^{2n+1}$*, the isoparametric hypersurfaces in* S^{4n+3} *given by* F *give a family of isoparametric hypersurfaces in* $\mathbb{C}P^{2n+1}$ *having* $g = 5$ *with multiplicities* 1, 2, 2, $2n-2$, $2n-2$.

Proof. According to the study in [13] , we can easily conclude F induces a family of isoparametric hypersurfaces in $\mathbb{H}P^n$ having $g = 5$ with multiplicities 3, 2, 2, $2n-4$, $2n-4$ because F gives family of isoparametric hypersurfaces in S^{4n+3} with $g = 4$ of multiplicities 2, 2, $2n-1$, $2n-1$. Here, we observe S^3-action on S^{4n+3} is nontrivial for the principal distributions with the multiplicities with $2n-1$ and trivial for the principal distributions with the multiplicities with 2. Since the $S^1(\subset S^3)$-action of Hopf fibration also has the similar properties, we get F induces a family of isoparametric hypersurfaces in $\mathbb{C}P^{2n+1}$ having $g = 5$ with multiplicities 1, 2, 2, $2n-2$, $2n-2$. □

Remark. With similar argument, we also know that the homogeneous function S^{2n+1} of Nomizu in Remark in the Therorem 4 produces a family of isoparametric hypersurfaces in $\mathbb{C}P^n$ having $g = 5$ with multiplicities 1, 1, 1, $n-2$, $n-2$.

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