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APPROXIMATE CONVEXITY WITH RESPECT TO INTEGRAL ARITHMETIC MEAN

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Abstract. Let $(\Omega, \mathcal{S}, \mu)$ be a probabilistic measure space, $\varepsilon \in \mathbb{R}$, $\delta \geq 0$, $p > 0$ be given numbers and let $P \subset \mathbb{R}$ be an open interval. We consider a class of functions $f : P \to \mathbb{R}$, satisfying the inequality

$$f(EX) \leq E(f \circ X) + \varepsilon E(|X - EX|^p) + \delta$$

for each $\mathcal{S}$-measurable simple function $X : \Omega \to P$.

We show that if additionally the set of values of μ is equal to $[0, 1]$ then $f : P \to \mathbb{R}$ satisfies the above condition if and only if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+\varepsilon\,[(1-t)^p t + t^p(1-t)]\,|x-y|^p+\delta$$

for $x, y \in P$, $t \in [0, 1]$.

We also prove some basic properties of such functions, e.g. the existence of subdifferentials, Hermite-Hadamard inequality.

1. Introduction

Let I be an interval in $\mathbb{R}$. A function $M : I \times I \to \mathbb{R}$ is called a mean in I if $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$ for $x, y \in I$. Suppose two means are given, M, N on real intervals I and J, respectively. A function $f : I \to J$ is called (M, N)-convex if

$$f(M(x, y)) \leq N(f(x), f(y)) \qquad \text{for } x, y \in I.$$

There are many papers where convexity with respect to means are considered, see e.g. [1], [2], [5], [6], [16]. Some modifications of the concept of (M, N)-convex function have also been studied, mainly when M is arithmetic and N is either an arithmetic or maximum mean, in which some positive term, depending on $x - y$, was added or subtracted to the right hand side. It leads in the first case to "approximate convexity" [3], [10], [14] and in the second one to "strong convexity" [7], [9], [11], [12], [15] with respect to means.

The main idea of this paper is to replace the previous means by the integral mean and to consider strong and approximate convexity with respect to this

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mean. Let μ be a probabilistic measure on a σ-algebra $\mathcal{S}$ of subsets of Ω and let $P \subset \mathbb{R}$ be an interval. It is known [4], [13] that if a function $f : P \to \mathbb{R}$ is convex, whereas $X : \Omega \to P$ is μ-integrable, then the following Jensen integral inequality holds: $f(EX) \leq E(f \circ X)$. We will deal with reversed theorem in some more general settings, assuming that a function $f : P \to \mathbb{R}$ satisfies the Jensen inequality with some bound which depends on the expression $\varepsilon E(|X - EX|^p) + \delta$, with $\varepsilon \in \mathbb{R}$, $p > 0$, $\delta > 0$:

$$f(EX) \leq E(f \circ X) + \varepsilon E(|X - EX|^p) + \delta$$

for each $\mathcal{S}$- measurable simple function $X : \Omega \to P$.

We say then that f is approximately or strongly convex depending on whether the sign of ε is positive or negative.

We show that if a set of values of μ is $[0, 1]$ then we obtain the following characterization: $f : P \to \mathbb{R}$ satisfies the above condition if and only if the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon\left[(1-t)^p t + t^p(1-t)\right]|x-y|^p + \delta$$

for $x, y \in P$, $t \in [0, 1]$.

Such functions in the case $p = 1$, $\varepsilon > 0$ were considered by Zs. Páles [10]. He proved that in this case f is of the form: $f = g + l + h$, where g is convex, h is bounded by $\frac{\delta}{2}$ and l is Lipschitz with constant $\frac{\varepsilon}{2}$. In the case $\varepsilon < 0$, $\delta = 0$, $p = 2$ our concept coincides with the notion of strongly convex functions [7], [11].

Approximately convex functions and strongly convex functions have tended to be considered separately in mathematical literature. In our paper we try to use some common approach and obtain results concerning this class independently whether $\varepsilon \geq 0$ or $\varepsilon < 0$. We prove also some basic properties of the above mentioned functions, e.g. existence of subdifferentials, the Jensen and Hermite-Hadamard inequalities.

2. (ε, δ, p)-convex functions

In the following we assume that $(\Omega, \mathcal{S}, \mu)$ is a probabilistic measure space, P is an open interval in $\mathbb{R}$ and $\varepsilon \in \mathbb{R}$, $\delta \geq 0$, $p > 0$ are fixed numbers.

Proposition 2.1. *If a function $f : P \to \mathbb{R}$ satisfies the following inequality*

$$f(EX) \leq \int_\Omega (f \circ X) d\mu + \varepsilon \int_\Omega |X - EX|^p d\mu + \delta \tag{1}$$

for each $\mathcal{S}$-measurable simple function X defined on Ω with values in P, where $EX := \int_\Omega X d\mu$,
then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon\left[(1-t)^p t + t^p(1-t)\right]|x-y|^p + \delta \tag{2}$$

for all $x, y \in P$ and for $t \in [0, 1]$ such that $t = \mu(S)$, where $S \in \mathcal{S}$.

Proof. Let $S \in \mathcal{S}$, $x, y \in P$ and $X = x\chi_S + y\chi_{\Omega\setminus S}$. Denoting $t := \mu(S)$ we obtain:

$$\begin{aligned}
&f \circ X = f(x)\chi_S + f(y)\chi_{\Omega\setminus S},\\
&EX = tx + (1-t)y,\\
&E(f \circ X) = tf(x) + (1-t)f(y),\\
&X - EX = (1-t)(x-y)\chi_S + t(y-x)\chi_{\Omega\setminus S},\\
&E|X - EX|^p = ((1-t)^p t + t^p(1-t))|x-y|^p.
\end{aligned}$$

Putting these formulas into (1) we obtain the desired inequality. □

Obviously if the range of μ is $[0,1]$, the last inequality holds for all $t \in [0,1]$. We further show in Theorem 2.2 that for such a measure, conditions (1) and (2) are equivalent.

Definition 2.1. Let D be a convex subset of normed space. A function $f : D \to \mathbb{R}$ satisfying

$$(3)\quad f(tx+(1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon\left[(1-t)^p t + t^p(1-t)\right]|x-y|^p + \delta$$

for all $x, y \in D$, $t \in [0,1]$, we call (ε, δ, p)-convex.

Example 2.1. Let D be a nonempty convex subset of a normed space X, let $p \in (0,1]$, $\varepsilon > 0$, $\delta \ge 0$ and let functions $g, h, k : D \to \mathbb{R}$ be such that g is convex, h-Hölder with a constant ε and power p, k-bounded by $\frac{\delta}{2}$. Then the function $f := g + h + k$ is (ε, δ, p)-convex. Indeed, for $x, y \in D$ and $t \in [0,1]$ we have

$$\begin{aligned}
f(tx+(1-t)y) &\le tg(x) + (1-t)g(y) + h(tx+(1-t)y) + k(tx+(1-t)y)\\
&= tf(x) + (1-t)f(y) - th(x) - tk(x) - (1-t)h(y)\\
&\quad - (1-t)k(y) + h(tx+(1-t)y) + k(tx+(1-t)y)\\
&= tf(x) + (1-t)f(y) + t[h(tx+(1-t)y) - h(x)]\\
&\quad + (1-t)[h(tx+(1-t)y) - h(y)]\\
&\quad + t[k(tx+(1-t)y) - k(x)]\\
&\quad + (1-t)[k(tx+(1-t)y) - k(y)]\\
&\le tf(x) + (1-t)f(y) + \varepsilon[(1-t)^p t + t^p(1-t)]\|x-y\|^p + \delta.
\end{aligned}$$

The above example shows also that (ε, δ, p)-convex function need not be continuous.

Proposition 2.2. *If $f : P \to \mathbb{R}$ is (ε, δ, p)-convex, then f is locally bounded. If f is $(\varepsilon, 0, p)$-convex, then f is continuous.*

Proof. Let $a < b$, $a, b \in P$. For arbitrary $a \le x \le b$ we have $x = ta + (1-t)b$ for some $t \in [0,1]$. By (3) we obtain

$$f(x) \le tM + (1-t)M + 2|\varepsilon||b-a|^p + \delta,$$

where $M = \max\{f(a), f(b), 0\}$. Hence f is bounded from above on $[a, b]$ and consequently locally bounded from above on P.

If $\delta = 0$, by [14, Thr. 2.2], f is locally uniformly continuous. □

Example 2.1 shows also that for $\varepsilon > 0$, $p \in (0, 1)$ an $(\varepsilon, 0, p)$-convex function need not be differentiable at any point, because the famous Weierstrass continuous nowhere differentiable function satisfies the Hölder condition with each power from $(0, 1)$.

Proposition 2.3. *If a function $f : D \to \mathbb{R}$ defined on an open convex subset D in $\mathbb{R}^n$ is $(\varepsilon, 0, p)$-convex and locally Lebesgue integrable on D (i.e., Lebesgue integrable on each compact subset of D) or continuous and (ε, δ, p)-convex, then f can be uniformly approximated on compact subsets of its domain by smooth $(\varepsilon, 0, p)$-convex functions.*

Proof. Let ϕ be a mollifier on $\mathbb{R}^n$, i.e., a smooth, nonnegative function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi(x)dx = 1 \quad \text{and } supp\phi \subset B(0, 1).$$

Let $\phi_h(x) = \frac{1}{h^n}\phi(\frac{x}{h})$ for $x \in \mathbb{R}^n$, $h > 0$.

It is well known that the convolution $(f * \phi_h)(x) := \int_{B(0,h)} f(x - y)\phi_h(y)dy$ for $x \in D_h := \{q \in \mathbb{R}^n : \ dist(q, \partial D) > h\}$ of a locally Lebesgue integrable function f with mollifier, is of class $C^\infty(D_h)$, and that by continuity of f (which in the case $\delta = 0$ follows by Proposition 2.2), $(f * \phi_h)(x) \rightrightarrows f(x)$ as $h \to 0$ for each compact $K \subset D$. We have only to show that $f * \phi_h$ is (ε, δ, p)-convex.

For $x_1, x_2 \in D_h$, $t \in [0, 1]$ by (ε, δ, p)-convexity of f we have

$$\begin{aligned}
&(f * \phi_h)(tx_1 + (1 - t)x_2) \\
&= \int_{B(0,h)} f(t(x_1 - y) + (1 - t)(x_2 - y))\phi_h(y)dy \\
&\leq t \int_{B(0,h)} f(x_1 - y)\phi_h(y)dy + (1 - t) \int_{B(0,h)} f(x_2 - y)\phi_h(y)dy \\
&\quad + \varepsilon[(1 - t)^p t + t^p(1 - t)]\|x_1 - x_2\|^p \int_{\mathbb{R}^n} \phi_h(y)dy + \delta \int_{\mathbb{R}^n} \phi_h(y)dy \\
&= t(f * \phi_h)(x_1) + (f * \phi_h)(x_2) + \varepsilon[(1 - t)^p t + t^p(1 - t)]\|x - y\|^p + \delta.
\end{aligned}$$
□

The following theorem generalizes Theorem 1 from [10].

Theorem 2.1. *Let $f : P \to \mathbb{R}$. The following conditions are equivalent:*

(i) *the function f is (ε, δ, p)-convex;*
(ii) *for all $u, x, y \in P$ such that $x < u < y$,*

$$\frac{f(u) - f(x) - \delta}{u - x} - \varepsilon(u - x)^{p-1} \leq \frac{f(y) - f(u) + \delta}{y - u} + \varepsilon(y - u)^{p-1};$$

(iii) *for all* $u \in P$,

$$\sup_{\{x\in P: x<u\}} \left(\tfrac{f(u)-f(x)-\delta}{u-x} - \varepsilon(u-x)^{p-1}\right) \le \inf_{\{x\in P: u<x\}} \left(\tfrac{f(x)-f(u)+\delta}{x-u} + \varepsilon(x-u)^{p-1}\right);$$

(iv) *there exists a function* $K : P \to \mathbb{R}$ *such that for all* $u, x \in P$,

$$f(u) + K(u)(x-u) - \varepsilon|x-u|^p - \delta \le f(x);$$

(v) *for all* $x_1, \dots, x_n \in P$, $t_1, \dots, t_n \ge 0$, $t_1 + \cdots + t_n = 1$, $u = \sum_{i=1}^n t_i x_i$,

$$f(u) \le \sum_{i=1}^n t_i f(x_i) + \varepsilon \sum_{i=1}^n t_i |x_i - u|^p + \delta;$$

(vi) *for all* $u, x, y \in P$ *such that* $x < u < y$,

$$f(u) \le f(x) + \frac{f(y)-f(x)}{y-x}(u-x) + \varepsilon \frac{(u-x)(y-u)}{y-x}\left[(u-x)^{p-1} + (y-u)^{p-1}\right] + \delta.$$

Proof. (i) $\Rightarrow$ (ii) For $x < u < y$ we have:

$$\begin{aligned} f(u) \le\ & \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y) \\ & + \varepsilon \left[\left(\frac{u-x}{y-x}\right)^p \frac{y-u}{y-x} + \left(\frac{y-u}{y-x}\right)^p \frac{u-x}{y-x}\right](y-x)^p \\ & + \delta \frac{y-u}{y-x} + \delta \frac{u-x}{y-x}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \frac{y-u}{y-x}(f(u)-f(x)-\delta) \le\ & \frac{u-x}{y-x}(f(y)-f(u)-\delta) \\ & + \varepsilon \left[\left(\frac{u-x}{y-x}\right)^p \frac{y-u}{y-x} + \left(\frac{y-u}{y-x}\right)^p \frac{u-x}{y-x}\right](y-x)^p. \end{aligned}$$

Dividing both sides of the above inequality by $(y-u)(u-x)$ we get

$$\begin{aligned} \frac{1}{y-x}\frac{f(u)-f(x)-\delta}{u-x} \le\ & \frac{1}{y-x}\frac{f(y)-f(u)+\delta}{y-u} \\ & + \varepsilon \left[\frac{(u-x)^{p-1}}{(y-x)^{p+1}} + \frac{(y-u)^{p-1}}{(y-x)^{p+1}}\right](y-x)^p \end{aligned}$$

and then, by multiplying the above inequality by $(y-x)$ we obtain

$$\frac{f(u)-f(x)-\delta}{u-x} - \varepsilon(u-x)^{p-1} \le \frac{f(y)-f(u)+\delta}{y-u} + \varepsilon(y-u)^{p-1}.$$

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) We put

$$L(u) := \sup_{\{x \in P : x < u\}} \left(\tfrac{f(u)-f(x)-\delta}{u-x} - \varepsilon(u-x)^{p-1} \right) \qquad \text{for } u \in P,$$
$$R(u) := \inf_{\{x \in P : u < x\}} \left(\tfrac{f(x)-f(u)+\delta}{x-u} + \varepsilon(x-u)^{p-1} \right) \qquad \text{for } u \in P.$$

By (iii) we have

$$-\infty < L(u) \le R(u) < \infty \qquad \text{for } u \in P.$$

To get (iv) it suffices to take arbitrary $K : P \to \mathbb{R}$ such that

$$K(u) \in [L(u), R(u)] \qquad \text{for } u \in P.$$

(iv) $\Rightarrow$ (v) For each u of the form $u = \sum_{i=1}^{n} t_i x_i$, where $x_1, \ldots, x_n \in P$, $t_1, \ldots, t_n \ge 0$, $t_1 + \cdots + t_n = 1$, by (iv) we have

$$\begin{aligned} f(u) &= \sum_{i=1}^{n} t_i [f(u) + K(u)(x_i - u)] \\ &\le \sum_{i=1}^{n} [t_i f(x_i) + t_i \varepsilon |x_i - u|^p + t_i \delta] \\ &= \sum_{i=1}^{n} t_i f(x_i) + \varepsilon \sum_{i=1}^{n} t_i |x_i - u|^p + \delta. \end{aligned}$$

(v) $\Rightarrow$ (i) Putting in (iv) $x_1 = x$, $x_2 = y$, $t_1 = t$, $t_2 = 1 - t$, we obtain (i).

(ii) $\Leftrightarrow$ (vi) Let $x < u < y$. It is easy to see that inequalities below are equivalent.

$$\frac{f(u) - f(x) - \delta}{u - x} - \varepsilon(u-x)^{p-1} \le \frac{f(y) - f(u) + \delta}{y - u} + \varepsilon(y-u)^{p-1},$$

$$\frac{y-x}{(u-x)(y-u)} f(u) \le \frac{f(x)+\delta}{u-x} + \frac{f(y)+\delta}{y-u} + \varepsilon\left[(u-x)^{p-1} + (y-u)^{p-1}\right],$$

$$\begin{aligned} f(u) \le{} & (f(x)+\delta)\frac{y-u}{y-x} + (f(y)+\delta)\frac{u-x}{y-x} \\ & + \varepsilon \frac{(u-x)(y-u)}{y-x}\left[(u-x)^{p-1} + (y-u)^{p-1}\right], \end{aligned}$$

$$\begin{aligned} f(u) \le{} & f(x) + \frac{f(y)-f(x)}{y-x}(u-x) \\ & + \varepsilon \frac{(u-x)(y-u)}{y-x}\left[(u-x)^{p-1} + (y-u)^{p-1}\right] + \delta. \end{aligned}$$

□

Corollary 2.1. *Let $f : P \to \mathbb{R}$ be $(\varepsilon, 0, p)$-convex. If f is differentiable at a point $u \in P$, then*

$$f(u) + f'(u)(x-u) - \varepsilon|x-u|^p \le f(x) \qquad \textit{for } x \in P.$$

Proof. Let $L(u), R(u)$ be such as in the proof of the part (iii) $\Rightarrow$ (iv) of Theorem 2.1. Then by Theorem 2.1(iii) we have $L(u) = R(u) = f'(u)$, which gives the assertion. $\square$

As a consequence of Theorem 2.1 we obtain the Jensen integral inequality for (ε, δ, p)-convex functions.

Theorem 2.2. *Assume that $\mu(\mathcal{S}) = [0,1]$. If a function $f : P \to \mathbb{R}$ is (ε, δ, p)-convex, a function $X : \Omega \to P$ is μ-integrable and moreover functions $|X|^p$, $f \circ X$ are μ-integrable, then*

$$f(EX) \leq E(f \circ X) + \varepsilon E|X - EX|^p + \delta.$$

Proof. Assume that $f : P \to \mathbb{R}$ is (ε, δ, p)-convex. Let $u = EX$. By Theorem 2.1(iv) there exists a function $K : P \to \mathbb{R}$ such that

$$f(u) + K(u)(s-u) - \varepsilon|s-u|^p - \delta \leq f(s) \qquad \text{for } s \in P.$$

Substituting $s = X(t)$, $t \in \Omega$, we get

$$f(u) + K(u)(X(t)-u) - \varepsilon|X(t)-u|^p - \delta \leq f(X(t)) \qquad \text{for } t \in \Omega.$$

Integrating this inequality with respect to t we get the assertion. $\square$

The assumption that $\mu(\mathcal{S}) = [0,1]$ seems to be essential in Theorem 2.2. Let $x_1 \neq x_2$, $x, y \in \Omega$, and let, for $i = 1, 2$, $\mu_i(A) = 1$ if $x_i \in A$ and 0 if $x_i \notin A$. Then $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ is a probabilistic measure. For an arbitrary function $f : P \to \mathbb{R}$ the assumptions are equivalent to the condition:

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2} + \frac{\varepsilon}{2^p}|x-y|^p + \delta \qquad \text{for } x, y \in P,$$

whereas the assertion -to the condition

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon\left[(1-t)^p t + t^p(1-t)\right]|x-y|^p + \delta$$

for $x, y \in P,\ t \in [0,1]$.

Theorem 2.3. *Let $\varepsilon > 0$, $p \in (0,1]$ or $\varepsilon < 0$, $p \geq 1$. A function $f : P \to \mathbb{R}$ is (ε, δ, p)-convex if and only if there exist functions $g, h : P \to \mathbb{R}$ such that g is $(\varepsilon, 0, p)$-convex,* $\sup_{x \in P}|h(x)| \leq \frac{\delta}{2}$ *and $f = g + h$.*

Proof. ($\Leftarrow$) Assume that $g : P \to \mathbb{R}$ is $(\varepsilon, 0, p)$-convex, $h : P \to \mathbb{R}$ bounded by $\frac{\delta}{2}$ and let $f = g + h$. Then we have for $x, y \in P$, $t \in [0,1]$

$$\begin{aligned} f(tx + (1-t)y) &\leq tg(x) + (1-t)g(y) + \varepsilon[t^p(1-t) + (1-t)^p t]|x-y|^p \\ &\quad + h(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) + \varepsilon[t^p(1-t) + (1-t)^p t]|x-y|^p \\ &\quad + t[h(tx + (1-t)y) - h(x)] \\ &\quad + (1-t)[h(tx + (1-t)y) - h(y)] \\ &\leq tf(x) + (1-t)f(y) + \varepsilon[t^p(1-t) + (1-t)^p t]|x-y|^p + \delta. \end{aligned}$$

($\Rightarrow$) By Theorem 2.1(iv) there exists a function $K : P \to \mathbb{R}$ such that

$$f(u) + K(u)(x-u) - \varepsilon|x-u|^p - \delta \le f(x) \qquad \text{for } x, u \in P.$$

Let

$$g(x) = \sup_{u \in P} \left[f(u) + K(u)(x-u) - \varepsilon|x-u|^p - \frac{\delta}{2} \right] \qquad \text{for } x \in P.$$

Then for every $x \in P$, $d > 0$ there is a $u \in P$ such that

$$g(x) - d < f(u) + K(u)(x-u) - \varepsilon|x-u|^p - \frac{\delta}{2} \le f(x) + \frac{\delta}{2}.$$

Hence $g(x) \le f(x) + \frac{\delta}{2}$. By definition of g also $g(x) \ge f(x) - \frac{\delta}{2}$ for $x \in P$. Therefore $|g(x) - f(x)| \le \frac{\delta}{2}$ for $x \in P$. We show that g is $(\varepsilon, 0, p)$-convex. For $x, y \in P$, $t \in [0,1]$, $d > 0$ we have for some $u \in P$

$$\begin{aligned}
&g(tx + (1-t)y) - d \\
&< f(u) + K(u)[tx + (1-t)y - u] - \varepsilon|tx + (1-t)y - u|^p - \frac{\delta}{2} \\
&= t\left[f(u) + K(u)(x-u) - \varepsilon|x-u|^p - \frac{\delta}{2} \right] \\
&\quad + (1-t)\left[f(u) + K(u)(y-u) - \varepsilon|y-u|^p - \frac{\delta}{2} \right] \\
&\quad + \varepsilon t\left[|x-u|^p - |tx + (1-t)y - u|^p\right] \\
&\quad + \varepsilon(1-t)[|y-u|^p - |tx + (1-t)y - u|^p \\
&\le tg(x) + (1-t)g(y) + \varepsilon[t^p(1-t) + (1-t)^p t]|x-y|^p.
\end{aligned}$$

Taking $h := f - g$, we obtained a required decomposition. □

3. Hermite-Hadamard inequality

As we know [4] if a function $f : P \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

for $a < b$, $a, b \in P$. The above inequality is called the Hermite-Hadamard inequality. We give now analogue of those inequalities for (ε, δ, p)-convex functions.

Theorem 3.1. *Let a locally Lebesgue integrable (i.e., Lebesgue integrable on compact subsets of P) function $f : P \to \mathbb{R}$ be (ε, δ, p)-convex. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx + \frac{\varepsilon}{2^p(p+1)}(b-a)^p + \delta, \tag{4}$$

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} + \frac{2\varepsilon}{(p+1)(p+2)}(b-a)^p + \delta \tag{5}$$

for $a < b$, $a, b \in P$.

Proof. Let $a < b$. By Theorem 2.1(vi) we have

$$f(x) \le f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \varepsilon \frac{(x - a)(b - x)}{b - a}\left[(x - a)^{p-1} + (b - x)^{p-1}\right] + \delta$$

for $a \le x \le b$.

Integrating this inequality, from a to b, we get

$$\int_a^b f(x)dx \le (b - a)\frac{f(a) + f(b)}{2} +$$
$$\frac{\varepsilon}{b - a}\left[\int_a^b (b - x)(x - a)^p dx + \int_a^b (x - a)(b - x)^p dx\right] + \delta(b - a).$$

By substitution $y = \frac{x-a}{b-a}$ – in the first and $y = \frac{b-x}{b-a}$ – in the second integral we obtain that each of these integrals is equal $(b - a)^{p+2}\int_0^1 (y^p - y^{p+1})dy = (b - a)^{p+2}(\frac{1}{p+1} - \frac{1}{p+2}) = \frac{1}{(p+1)(p+2)}(b - a)^{p+2}$. Dividing both sides of the last inequality by $b - a$, we obtain (5).

We show (4). We have

$$\frac{1}{b - a}\int_a^b f(x)dx = \frac{1}{b - a}\int_a^{\frac{a+b}{2}} f(x)dx + \frac{1}{b - a}\int_{\frac{a+b}{2}}^b f(x)dx$$
$$= \int_0^1 \frac{1}{2}\left[f\left(\frac{a + b - t(b - a)}{2}\right) + f\left(\frac{a + b + t(b - a)}{2}\right)\right] dt.$$

Next by (1), with $t = \frac{1}{2}$, we get

$$\frac{1}{b - a}\int_a^b f(x)dx \ge \int_0^1 \left[f\left(\frac{a + b}{2}\right) - \frac{\varepsilon}{2^p}t^p(b - a)^p - \delta\right] dt$$
$$= f\left(\frac{a + b}{2}\right) - \frac{\varepsilon}{2^p(p + 1)}(b - a)^p - \delta.$$

□

It is easy to verify that if a function $f : D \to \mathbb{R}$ is convex, then for each $x, y \in D$ the function $g : [0, 1] \to \mathbb{R}$, $g(t) = f(tx + (1 - t)y)$, for $t \in [0, 1]$, is convex. We give below an analogue of this fact for (ε, δ, p)-convex functions.

Proposition 3.1. *Let D be a convex subset of a normed space. If $f : D \to \mathbb{R}$ is (ε, δ, p)-convex, then for fixed $x, y \in D$, the function $g : [0, 1] \to \mathbb{R}$, $g(t) = f(tx + (1 - t)y)$, for $t \in [0, 1]$, is $(\varepsilon', \delta, p)$-convex with $\varepsilon' = \varepsilon\|x - y\|^p$.*

Proof. Let $s, t \in [0, 1]$, $\alpha, \beta \ge 0$, $\alpha + \beta = 1$. Then

$$\begin{aligned} g(\alpha t + \beta s) &= f((\alpha t + \beta s)x + (\alpha + \beta - \alpha t - \beta s)y) \\ &= f(\alpha(tx + (1 - t)y) + \beta(sx + (1 - s)y)) \\ &\le \alpha f(tx + (1 - t)y) + \beta f(sx + (1 - s)y) \\ &\quad + \varepsilon(\alpha^p\beta + \alpha\beta^p)|t - s|^p\|x - y\|^p + \delta \end{aligned}$$

$$= \alpha g(t) + \beta g(s) + \varepsilon|t-s|^p(\alpha^p\beta + \alpha\beta^p)\|x-y\|^p + \delta.$$

Hence g is $(\varepsilon', \delta, p)$-convex. □

By Theorem 3.1 and the above proposition we obtain the Hermite-Hadamard inequality for (ε, δ, p)-convex functions in normed spaces.

Corollary 3.1. *Let D be a convex subset of a normed space. Let $f : D \to \mathbb{R}$ be (ε, δ, p)-convex function such that for every $x, y \in D$ the function*

$$[0,1] \ni t \mapsto f(tx + (1-t)y) \in \mathbb{R}$$

is Lebesgue integrable. Then

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f(tx+(1-t)y)dt + \frac{\varepsilon\|x-y\|^p}{2^p(p+1)} + \delta, \tag{6}$$

$$\int_0^1 f(tx+(1-t)y)dt \le \frac{f(x)+f(y)}{2} + \frac{2\varepsilon\|x-y\|^p}{(p+1)(p+2)} + \delta \tag{7}$$

for $x, y \in D$.

Proof. Let $x, y \in D$, and let $g(t) = f(tx+(1-t)y)$ for $t \in [0,1]$. By Proposition 3.1 the function g satisfies (3), with $\varepsilon\|x-y\|^p$ instead of ε. Making use of Theorem 3.1 for g and $a = 0$, $b = 1$, we obtain immediately (6) and (7). □

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