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CERTAIN CLASSES OF ANALYTIC FUNCTIONS AND DISTRIBUTIONS WITH GENERAL EXPONENTIAL GROWTH

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Abstract. Let $\mathcal{K}'_M$ be the generalized tempered distributions of $e^{M(t)}$-growth, where the function $M(t)$ grows faster than any linear functions as $|t| \to \infty$, and let $\mathbf{K}'_{\mathbf{M}}$ be the Fourier transform spaces of $\mathcal{K}'_M$. We obtain the relationship between certain classes of analytic functions in tubes, $\mathcal{K}'_M$ and $\mathbf{K}'_{\mathbf{M}}$.

1. Introduction

In his book [9], V. S. Vladimirov has considered the relationship between the class of analytic functions in tubes $H(A;C)$ and tempered distributions with polynomial growth $\mathcal{S}'$. Later R. D. Carmichael [3] has introduced two different types of classes of analytic functions in tubes $G_p(A;C)$ and $F_p(A;C)$ both of which are extensions of $H(A;C)$ and has obtained the relationship between $G_p(A;C)$ (and $F_p(A;C)$) and tempered distributions with exponential growth of polynomial powers $\mathcal{K}'_p$, $p > 1$, and the Fourier transform spaces $\mathbf{K}'_{\mathbf{p}}$, $p > 1$ of $\mathcal{K}'_p$, $p > 1$.

In this paper, we introduce two different types of classes of analytic functions in tubes $G_M(A;C)$ and $F_M(A;C)$ which are extensions of $G_p(A;C)$ and $F_p(A;C)$, respectively, and obtain the relationship between $G_M(A;C)$ (and $F_M(A;C)$) and tempered distributions with general exponential powers growth $\mathcal{K}'_M$ and the Fourier transform $\mathbf{K}'_{\mathbf{M}}$ of $\mathcal{K}'_M$.

In the main sections, we show that elements of $G_M(A;C)$ and $F_M(A;C)$ can be represented as the Fourier-Laplace transform of distributions $\mathcal{K}'_M$. Also we present representations of $G_M(A;C)$ and $F_M(A;C)$ as elements in $\mathbf{K}'_{\mathbf{M}}$ in terms of Fourier transforms in $\mathbf{K}'_{\mathbf{M}}$ of certain elements in $\mathcal{K}'_M$ and strong boundedness for $G_M(A;C)$ and $F_M(A;C)$ as elements in $\mathbf{K}'_{\mathbf{M}}$. In particular, we show that elements of $F_M(A;C)$ obtain distributional boundary values in $\mathbf{K}'_{\mathbf{M}}$.

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2. Notation and preliminaries

We denote the points of $\mathbb{R}^n$ spaces by $t = (t_1, t_2, \ldots, t_n)$ and $s = (s_1, s_2, \ldots, y_n)$. The letter n always denotes the dimension. In $\mathbb{C}^n$ the points are denoted by $z = x + iy$, $x, y \in \mathbb{R}^n$. We define $\langle t, s\rangle = t_1 s_1 + t_2 s_2 + \cdots + t_n s_n$ and similarly define $\langle t, z\rangle$, $t \in \mathbb{R}^n$, $z \in \mathbb{C}^n$. α denotes n tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers. $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. If $k = (k_1, k_2, \ldots, k_n)$ is an n tuples of integers, $t^k = t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}$, $t \in \mathbb{R}^n$, with similar definition for z^k, $z \in \mathbb{C}^n$. If $a \in R$, then $at = (at_1, at_2, \ldots, at_n)$. We write $D_j = -\frac{1}{2\pi i}\left(\frac{\partial}{\partial t_j}\right)$, $j = 1, 2, \ldots, n$, and $D_t^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and similarly write D_z^α.

Definition 1. A set $C \subset \mathbb{R}^n$ is a cone with vertex at zero if $y \in C$ implies $\lambda y \in C$ for all positive real scalar λ.

Definition 2. Let C be a cone. $C \cap \{y \in \mathbb{R}^n : |y| = 1\}$ is called the projection of C and is denoted $pr(C)$.

Definition 3. If C' and C are cones such that $\mathrm{pr}(\bar{C}') \subset \mathrm{pr}(C)$, then C' is called a compact subcone of C.

For a cone C, $\mathcal{O}(c)$ will denote the convex hull or envelop of C and $T^C = \mathbb{R}^n + iC \subseteq \mathbb{C}^n$ is a tube in $\mathbb{C}^n$.

Definition 4. If C is open, T^C is called a tubular cone. If C is both open and connected, T^C is called a tubular radial domain.

Definition 5. The set $C^* = \{t \in \mathbb{R}^n : \langle t, y\rangle \geq 0,\ y \in C\}$ is the dual cone of the cone C and $C_* = \mathbb{R}^n \setminus C^*$.

Definition 6. The function

$$u_C = \sup_{y \in pr(C)} (-\langle t, y\rangle)$$

is the indicatrix of the cone C.

It follows that $C^* = \{t \in \mathbb{R}^n :\ u_C(t)\ \leq 0\}$. Further $u_C(t) \leq u_{\mathcal{O}(C)}(t)$ and if $t \in C^*$, then $u_C(t) = u_{\mathcal{O}(C)}(t)$ [9, p. 219]. To characterize the nonconvexity of the cone, we have the following; for a cone C, let

$$\rho_C = \sup_{t \in C^*} \frac{u_{\mathcal{O}(C)}}{u_C(t)}.$$

A cone C is convex if and only if $\rho_C = 1$ [9, Sec. 25.1, Lemma 2] and if a cone is open and consists of finite number of components, then $\rho_C < 1$ [9, Sec. 25.1, Lemma 3]. In this paper we shall be considering the case $1 \leq \rho_C < +\infty$ for all cones C.

Now we present four important facts what will be used frequently later.

Lemma 1 ([9, Sec. 25.1])**.** *Let C be a cone. Then*

$$-\langle t, y\rangle \leq |y| u_{\mathcal{O}(C)}, \quad u_{\mathcal{O}(C)} \leq \rho_C u_C(t), \quad t \in C^*, y \in \mathcal{O}(C).$$

Lemma 2 ([9, Eq.(28), p. 241]). *Let C be an open connected cone and let C'_* be a compact subcone of C_*. Then there exist $\xi = \xi(C'_*)$, depending on C'_*, such that*

$$\xi|t| \leq u_C(t) \leq |t|, \quad t \in C'_*.$$

Lemma 3 ([9, Sec. 25.2, Lemma 2]). *Let C be an open cone and C' that is an arbitrary subcone of $\mathcal{O}(C)$. Then there exist a number $\delta = \delta(C')$ and open cone $(C^*)'$ both depending on C' such that $C^* \subset (C^*)'$ and*

$$\langle y, t\rangle \geq \delta|y||t|, \quad y \in C' \subset \mathcal{O}(C), \ t \in (C^*)'.$$

Lemma 4 ([9, Lemma, p. 241]). *Let C'_* be cone that is compact in the cone C_*. For an arbitrary number $\eta \in (0,1)$, there exists a compact subcone $C' = C'(C'_*, \eta)$ depending on C'_* and on η such that for any $t \in C'_*$, there exists a point $y_t^0 \in Pr(C')$ at which*

$$-\langle t, y_t^0\rangle \geq (1-\eta)u_C(t).$$

3. The distribution spaces $\mathcal{K}'_M$ and $\mathbf{K}'_\mathbf{M}$

Let $\mu(\xi)$, $0 \leq \xi \leq \infty$, denote a continuous increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For $t \geq 0$ we define

$$M(t) = \int_0^t \mu(\xi)d\xi.$$

The function $M(t)$ is an increasing, convex, and continuous function with $M(0) = 0$ and $M(\infty) = \infty$. Further we define $M(t)$ for negative t by $M(-t) = M(t)$. Since the derivative $\mu(t)$ of $M(t)$ is unbounded in R, the function $M(t)$ will grow faster than any linear function as $|t| \to \infty$.

The function $M(t)$ can be defined on $\mathbb{R}^n$ by $M(t_1 + t_2 + \cdots + t_n) = M(t_1) + M(t_2) + \cdots + M(t_n)$ for all $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. (Refer to Sec. 4.1 of Chapter 1 in [4] or p. 130 in [5].)

Definition 7. Let $M(x)$ and $\Omega(y)$ be the functions corresponding to $\mu(\xi)$ and $\omega(\eta)$ as above, respectively. Then $M(x)$ and $\Omega(y)$ are called to be dual in the sense of Young if $\mu(\omega(\eta)) = \eta$ and $\omega(\mu(\xi)) = \xi$.

We have two examples of dual functions in the sense of Young as follow;

1. $M(s) = \dfrac{s^p}{p}, \quad \Omega(t) = \dfrac{t^q}{q}, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1, \quad s, t \geq 0.$
2. $M(s) = e^s - s - 1, \ \Omega(t) = (t+1)\log(t+1) - t, \quad s, t \geq 0.$

We list some properties of function $M(x)$, $x \in \mathbb{R}^n$.

Lemma 5. *For $t \geq 0$ we define $M(t) = \int_0^t \mu(\xi)d\xi$, where $\mu(\xi)$ $(0 \leq \xi \leq \infty)$ is a continuous increasing function such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. Then we have that*

$$M(s) + M(t) \leq M(s+t) \text{ for all } st \geq 0.$$

$$M\left(\frac{t}{k}\right) \leq \frac{M(t)}{k} \quad \textit{for all } k > 1.$$

Hence if we let $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be in* $\mathbb{R}^n$*, then*

$$M(x) + M(y) \leq M(x+y) \textit{ for all } x_i y_i \geq 0, \ (i = 1, 2, \ldots, n).$$

$$M\left(\frac{x}{k}\right) \leq \frac{M(x)}{k} \quad \textit{for all } k > 1 \textit{ and } x \in \mathbb{R}^n.$$

We note an important property of dual functions which will be useful later.

Lemma 6 ([5, Lemma 1.1])**.** *Let* $M(s)$ *and* $\Omega(t)$ *be defined as in Definition 7, where* $s, t \in R$*. Then*

$$st \leq M(s) + \Omega(t) \textit{ for any } s, \ t \geq 0$$

and the equality holds if and only if $t = \mu(s)$ *or* $s = \omega(t)$*.*

Hence if we let $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be in* $\mathbb{R}^n$*, then*

$$\langle x, y\rangle \leq M(x) + \Omega(y) \textit{ for any } x_i, \ y_i \geq 0, \ (i = 1, 2, \ldots, n)$$

and the equality holds if and only if $y_i = \mu(x_i)$ *or* $x_i = \omega(y_i)$*.*

For more details about the function $M(x)$ and $\Omega(y)$, we can refer to [4, Chapter 1].

Using the function $M(t)$, we define the space $\mathcal{K}_M$ as the space of all functions $\varphi(t)$ in C^∞ such that

$$\nu_k(\varphi) = \sup_{t \in \mathbb{R}^n, \ |\alpha| \leq k} e^{M(kt)} |D_t^\alpha \varphi(t)| < \infty, \ k = 1, 2, \ldots,$$

where $D_t^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and $D_j^{\alpha_j} = -\frac{1}{2\pi i} \frac{\partial^{\alpha_j}}{(\partial t_j)^{\alpha_j}}$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The topology in $\mathcal{K}_M$ is defined by the countably family of semi-norms $\{\nu_k\}_{k=1}^\infty$. It follows that the space $\mathcal{K}_M$ becomes a Fréchet space [5] and the identity mapping $\mathcal{D} \hookrightarrow \mathcal{K}_M \hookrightarrow \mathcal{E}$ are continuous when $\mathcal{E}$ denotes the space of all C^∞ functions on $\mathbb{R}^n$ and $\mathcal{D}$ the space of all C^∞ functions with compact support in $\mathbb{R}^n$.

Lemma 7. $\mathcal{K}_M$ *is a Montel space.*

Proof. If B is a bounded set of $\mathcal{K}_M$, then B is a bounded set in C^∞ since the imbedding $\mathcal{K}_M \hookrightarrow C^\infty$ is continuous. Since C^∞ is a Montel space, it suffices to show that B is a relatively compact set in C^∞. Let (ϕ_j) be a sequence of elements of B such that (ϕ_j) converges to ϕ in C^∞. Since B is a bounded set of $\mathcal{K}_M$, for all $k \in N$ and all $\alpha \in N^n$, there exists a constant $C_{k,\alpha}$ such that

$$\sup_{t \in \mathbb{R}^n} |e^{M(kt)} D^\alpha \phi_j(t)| \leq C_{k,\alpha}, \quad \phi_j \in B. \tag{1}$$

The inequality (1) implies that, given $\epsilon > 0$ there is a constant $M > 0$ such that for t with $|t| > M$,

$$|e^{M(kt)} D^\alpha \phi_j(t)| \leq \epsilon, \quad \phi_j \in B. \tag{2}$$

Since $\phi_j \to \phi$ in C^∞, (2) implies that

$$|e^{M(kt)} D^\alpha \phi(t)| \le \epsilon, \quad |t| > M.$$

Hence $\phi \in \mathcal{K}_M$. On the other hand, since $\phi_j \to \phi$ in C^∞, $(D^\alpha \phi_j)$ converges uniformly to $D^\alpha \phi$ on the compact set $\{t \in \mathbb{R}^n : |t| \le M\}$. This implies that given $\epsilon > 0$, we can find an integer j_0 such that

$$e^{M(kt)} |D^\alpha \phi_j(t) - D^\alpha \phi(t)| \le \epsilon$$

for all t with $|t| \le M$ and all $j \ge j_0$. Last three inequalities imply that

$$\sup_{t \in \mathbb{R}^n} e^{M(kt)} |D^\alpha \phi_j(t) - D^\alpha \phi(t)| \le \epsilon$$

for all $j \ge j_0$; therefore $\phi_j \to \phi$ in $\mathcal{K}_M$. The proof is completed. $\square$

We denote by $\mathcal{K}'_M$ the space of all continuous linear functional on $\mathcal{K}_M$. Clearly when $M(t) = \log(1 + |t|)$, $\mathcal{K}'_M$ is the space of Schwartz's tempered distributions. When $M(t) = |t|$, $\mathcal{K}'_M$ is the space of tempered distributions of $\mathcal{K}'_1$ which is introduced and characterized by J. Sevastião E. Silva [8]. When $M(t) = |t|^p,\ p > 1$, $\mathcal{K}'_M$ is the space of tempered distributions of $\mathcal{K}'_p,\ p > 1$, which is introduced and characterized by Sampson and Zielezny [6].

The restriction $\tilde{T} = T|_{\mathcal{D}}$ of a functional $T \in \mathcal{K}'_M$ to $\mathcal{D}$ is a distribution. Since $\mathcal{D}$ is dense in $\mathcal{K}_M$, T is determined by its values on $\mathcal{D}$. We characterize the distributions in $\mathcal{K}'_M$ by their growth at infinity.

Lemma 8 ([5, Theorem 2.3])**.** *A distribution $T \in \mathcal{D}$ is in $\mathcal{K}'_M$ if and only if there exist positive integers k, α and a bounded continuous function $f(t)$ on $\mathbb{R}^n$ such that*

$$T = D^\alpha \left[e^{M(kt)} f(t) \right].$$

Let $\phi(t) \in L^1(\mathbb{R}^n)$. We define the Fourier transform of $\phi(t)$ by

$$\hat{\phi}(x) = \mathcal{F}[\phi(t); x] = \int_{\mathbb{R}^n} \phi(t) e^{2\pi i \langle x, t \rangle} dt$$

and the inverse Fourier transform of $\phi(t)$ by

$$\mathcal{F}^{-1}[\phi(t); x] = \int_{\mathbb{R}^n} \phi(t) e^{-2\pi i \langle x, t \rangle} dt.$$

Now we have a Paley-Wiener type theorem for the space $\mathcal{K}_M$ from [5, Theorem 4.1]; an entire function $F(\zeta)$ is a Fourier transform of a function φ in $\mathcal{K}_M$ if and only if, for every integer $N \ge 0$ and every $\epsilon > 0$, there exists a constant C such that

$$|F(\xi + i\eta)| \le C(1 + |\zeta|)^{-N} e^{\Omega(\epsilon\eta)}, \quad \zeta = \xi + i\eta \in \mathbb{C}^n.$$

Let $\mathbf{K_M}$ be the space of Fourier transform of functions in $\mathcal{K}_M$. We define in $\mathbf{K_M}$ a locally convex topology by means of the seminorms

$$\omega_k(\hat{\varphi}) = \sup_{\zeta = \xi + i\eta} (1 + |\zeta|)^k e^{-\Omega\left(\frac{\eta}{k}\right)} |\hat{\varphi}(\zeta), \ k = 1, 2, \ldots, \ \varphi \in \mathcal{K}_M.$$

Lemma 9 ([5, Coro. 4.2])**.** *The Fourier transform is a topological isomorphism of* $\mathcal{K}_M$ *onto* $\mathbf{K_M}$.

Let $\mathbf{K'_M}$ be the space of continuous linear functional on $\mathcal{K}_M$ which equipped with the topology of uniform convergence on all bounded set in $\mathbf{K_M}$. Each distribution T in $\mathcal{K}'_M$ has a Fourier transform $\hat{T}$ in $\mathbf{K'_M}$ defined by Parseval's formula

$$\langle \hat{T}, \hat{\varphi} \rangle = (2\pi)^n \langle T, \varphi \rangle, \quad \varphi \in \mathcal{K}_M.$$

Moreover, we have:

Lemma 10 ([5, Coro. 4.3])**.** *The Fourier transform is a topological isomorphism of* $\mathcal{K}'_M$ *onto* $\mathbf{K'_M}$.

For further detailed structure theories about $\mathcal{K}'_M$ and $\mathbf{K'_M}$, we can refer to [4] and [5].

4. The analytic spaces $G_M(A;C)$ and $F_M(A;C)$

To find the relations between the increase in certain classes of analytic functions and the properties of their spectral functions, Vladimirov [9, Sec. 26.4] introduced the following class of analytic functions;

Let C be an open cone in $\mathbb{R}^n$ and C' be an arbitrary compact subcone of C. $p \geq 1$ and $A \geq 0$ are real numbers. A function $f(z)$ belongs to the class $\mathbf{H_p(A;C)}$ if $f(z)$ is analytic in the tubular cone $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ and satisfies

$$|f(z)| \leq K(C')(1+|z|)^N (1+|y|)^{-M} e^{A|y|^p}, \quad z = x+iy \in T^C,$$

where $K(C')$ is a constant depending on C', and N and M are nonnegative real numbers which do not depend on C'.

Motivated by the works of Vladimirov, R. D. Carmichael introduced two different types of classes of analytic functions in tubes both of which are more general spaces than the class $H_p(A;C)$ as follow;

Let C be an open cone in $\mathbb{R}^n$ and C' be an arbitrary compact subcone of C. $p \geq 1$ and $A \geq 0$ are real numbers. Let $m > 0$. $T(C';m)$ denotes the set $T(C';m) = \mathbb{R}^n + i(C' \backslash (C' \cap N(0,m)))$ where $N(0,m)$ is a closed ball in $\mathbb{R}^n$ of radius $m > 0$ with center at the origin. A function$f(z)$ belongs to the class $\mathbf{G_p(A;C)}$ if, for each compact subcone $C' \subset C$, there exists a fixed $m = m(C') > 0$ depending on C' such that $f(z)$ is analytic in $T(C';m)$ and satisfies

$$|f(z)| \leq K(C';m)(1+|z|)^N e^{2\pi A|y|^p}, \quad z = x+iy \in T(C';m),$$

where $K(C';m)$ is a constant depending on C' and on m and N is a nonnegative real number which does not depend on C' and on m.

A function $f(z)$ belongs to $\mathbf{F_p(A;C)}$ if, for each compact subcone $C' \subset C$, $f(z)$ is analytic in $T^{C'} = \mathbb{R}^n + iC'$ and satisfies

$$|f(z)| \leq K(C';m)(1+|z|)^N e^{2\pi A|y|^p}, \quad z = x+iy \in T(C';m),$$

where $K(C';m)$ is a constant depending on C' and on m and N is a nonnegative real number which does not depend on C' and on m.

Carmichael studied the relationship between $G_p(A;C)$ (and $F_p(A;C)$), the distributions $\mathcal{K}'_p$, $p \geq 1$, and the Fourier transform $\mathbf{K}'_p$, $p \geq 1$, of $\mathcal{K}'_p$, $p \geq 1$, in [3]. Since $\mathcal{K}'_p \subset \mathcal{K}'_M$, $p \geq 1$, we need more general classes of analytic functions than $G_p(A : C)$ or $F_p(A;C)$ to find the relationship between the classes of analytic functions, $\mathcal{K}'_M$, and $\mathbf{K}'_\mathbf{M}$ as follow;

For $t \geq 0$, let $M(t) = \int_0^t \mu(\xi)d\xi$, where $\mu(\xi)$ $(0 \leq \xi \leq \infty)$ is a continuous increasing function such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. Let C be an open cone in $\mathbb{R}^n$ and C' be an arbitrary compact subcone of C. $A \geq 0$ are real numbers. Let $m > 0$. $T(C';m)$ denote the set $T(C';m) = \mathbb{R}^n + i(C' \backslash (C' \cap N(0,m)))$ where $N(0,m)$ is a closed ball in $\mathbb{R}^n$ of radius $m > 0$ with center at the origin.

A function$f(z)$ belongs to the class $\mathbf{G_M}(\mathbf{A};\mathbf{C})$ if, for each compact subcone $C' \subset C$, there exists a fixed $m = m(C') > 0$ depending on C' such that $f(z)$ is analytic in $T(C';m)$ and satisfies

$$|f(z)| \leq K(C';m)(1+|z|)^N e^{2\pi M(Ay)}, \quad z = x+iy \in T(C';m),$$

where $K(C';m)$ is a constant depending on C' and on m and N is a nonnegative real number which does not depend on C' and on m.

A function $f(z)$ belongs to $\mathbf{F_M}(\mathbf{A};\mathbf{C})$ if, for each compact subcone $C' \subset C$, $f(z)$ is analytic in $T^{C'} = \mathbb{R}^n + iC'$ and satisfies

$$|f(z)| \leq K(C';m)(1+|z|)^N e^{2\pi M(Ay)}, \quad z = x+iy \in T(C';m),$$

where $K(C';m)$ is a constant depending on C' and on m and N is a nonnegative real number which does not depend on C' and on m.

The 2π in the exponential term in the definition of $G_M(A;C)$ and $F_M(A;C)$ simply reflects the way we have defined the Fourier transform in this paper. Obviously we have the following inclusion relation;

$$F_M(A;C) \subset G_M(A;C), \quad G_p(A;C) \subset G_M(A;C), \quad F_p(A;C) \subset F_M(A;C).$$

We need three lemmas which will be useful to obtain main results in the next two sections.

Lemma 11. *For $t \geq 0$, let $\Omega(t) = \int_0^t \omega(\xi)d\xi$, where $\omega(\xi)$ $(0 \leq \xi \leq \infty)$ is a continuous increasing function such that $\omega(0) = 0$ and $\omega(\infty) = \infty$. Let C be an open connected cone and let C'_* be an arbitrary compact subcone of $C_* = \mathbb{R}^n \backslash C^*$. Let γ be an n-tuple of nonnegative integers. Let $n \geq 1$ be an integer and let $R > 0$. Then we have*

$$(1+|t|)^{n+1+|\gamma|} \leq M_1 \exp[2\pi R\Omega(u_C(t))], \tag{3}$$

where $M_1 = M_1(C'_*)$ depends on $C'_* \subset C_*$.

Hence for $A > 0$

$$(1+|t|)^{n+1+|\gamma|} \leq M_2 \exp\left[2\pi R\Omega\left(\frac{u_C(t)}{A}\right)\right], \tag{4}$$

where $M_2 = M_2(C'_*, A)$ *depends on* $C'_* \subset C_*$ *and on* A.

Proof. From Lemma 2, given $C'_* \subset C_*$ there exists $\xi = \xi(C'_*)$, depending on C'_*, such that

$$\xi|t| \le u_C(t) \le |t|, \quad t \in C'_*. \tag{5}$$

Hence, for any $R > 0$,

$$0 < \exp[2\pi R\Omega(\xi t)] \le \exp[2\pi R\Omega(u_C(t))], \quad t \in C'_*.$$

Since the function $\Omega(t)$ in the hypothesis grows faster than any linear function as $|t| \to \infty$ for $t \in \mathbb{R}^n$,

$$(1+|t|)^{-n-1-|\gamma|} \exp[2\pi R\Omega(\xi t)] \to \infty \text{ as } |t| \to \infty$$

for $t \in \mathbb{R}^n$, hence

$$(1+|t|)^{-n-1-|\gamma|} \exp[2\pi R\Omega(u_C(t))] \to \infty \text{ as } |t| \to \infty \tag{6}$$

for $t \in C'_* \subset C_*$. Let $N(0,m)$ be a closed ball of the origin in $\mathbb{R}^n$ of radius $m > 0$. We can find $O_m > 1$, depending on m, such that

$$Q_m(1+|t|)^{-n-1-|\gamma|} \exp[2\pi R\Omega(\xi t)] \ge 1, \quad t \in N(0,m). \tag{7}$$

By Lemma 2 and (7),

$$Q_m(1+|t|)^{-n-1-|\gamma|} \exp[2\pi R\Omega(u_C(t))] \ge 1, \quad t \in N(0,m) \cap C'_*. \tag{8}$$

Thus we have (3) from (6) and (8). Now if $A > 0$, we have from (5) that given $C'_* \subset C_*$, there exists $\xi = \xi(C'_*)$, depending on C'_*, such that

$$\frac{\xi|t|}{A} \le \frac{u_C(t)}{A} \le \frac{|t|}{A}, \quad t \in C'_*. \tag{9}$$

If we replace (5) by (9), we have from the same process as above that

$$\left(1 + \left|\frac{t}{A}\right|\right)^{n+1+|\gamma|} \le M_1 \exp\left[2\pi R\Omega\left(\frac{u_C(t)}{A}\right)\right]. \tag{10}$$

But since $(1+|t|) \le C_1(1+|t|/A)$ when $C_1 = C_1(A)$, depending on A, equals A if $A \ge 1$ and equals 1 if $0 < A < 1$, we have (4) from (10). □

Lemma 12. *For* $t \ge 0$, *let* $\Omega(t) = \int_0^t \omega(\xi)d\xi$, *where* $\omega(\xi)$ $(0 \le \xi \le \infty)$ *is a continuous increasing function such that* $\omega(0) = 0$ *and* $\omega(\infty) = \infty$. *Let* C *be an open connected cone and let* C' *be an arbitrary open compact subcone of* $\mathcal{O}(C)$. *Let* C'_* *be an arbitrary compact subcone of* $C_* = \mathbb{R}^n \backslash C^*$. *Let* $A > 0$. *Let* $g(t)$ *be a continuous function of* $t \in \mathbb{R}^n$ *which satisfies*

$$|g(t)| \le K(C'_*, \eta) \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right], \ t \in C'_* \subset C_*,$$

for any $\eta \in (0,1)$ *with* $1 - 3\eta > 0$, *where* $K(C'_*, \eta)$ *is a constant, depending on* C'_* *and on* η. *Let* $z_0 \in T^{C'} = \mathbb{R}^n + iC'$ *be an arbitrary but fixed point and let*

$z \in N'(z_0, r) \subset T^{C'}$, where $N'(z_0, r)$ is an open neighborhood of z_0 with radius $r > 0$ whose closure is in $T^{C'}$. Then for any n-tuple γ of nonnegative integer,

$$h_{C'_*}^{\gamma;g}(z) = \int_{C'_*} t^\gamma g(t) e^{2\pi i\langle z,t\rangle} dt$$

converges absolutely and uniformly for $z \in N'(z_0, r)$.

Proof. From Lemma 1 and assumption about the estimation of $g(t)$, for $z = x + iy \in N'(z_0, r)$, there exists a real number T with $|y| = |\mathrm{Im}(z)| \le T$ such that for $A > 0$ and any $\eta > 0$ with $1 - 2\eta > 0$

$$\begin{aligned} (11) \qquad |h_{C'_*}^{\gamma;g}(t)| &\le K(C'_*, \eta) \int_{C'_*} |t^\gamma| e^{-2\pi\langle y,t\rangle} \\ &\quad \cdot \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] dt \\ &\le K(C'_*, \eta) \int_{C'_*} \frac{(1+|t|)^{n+1+|\gamma|}}{(1+|t|)^{n+1}} \exp[2\pi T\rho_C u_C(t)] \\ &\quad \cdot \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] dt, \end{aligned}$$

where n is the dimension.

By Lemma 2 and Lemma 11 with $R = \eta$, for $t \in C'_* \subset C_*$, there exists a $\xi = \xi(C'_*)$ such that

$$\begin{aligned} (12) \qquad &\exp\left[2\pi T\rho_C u_C(t) - 2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] \cdot (1+|t|)^{n+1+|\gamma|} \\ &\le M(C'_*, A) \exp\left[2\pi T\rho_C u_C(t) - 2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] \\ &\quad \cdot \exp\left[2\pi\eta\Omega\left(\frac{u_C(t)}{A}\right)\right] \\ &\le M(C'_*, A) \exp\left[2\pi T\rho_C|t| - 2\pi(1-3\eta)\Omega\left(\frac{\xi t}{A}\right)\right]. \end{aligned}$$

Now consider the function of x defined by

$$f(x) = 2\pi T\rho_C x - 2\pi(1-3\eta)\Omega\left(\frac{\xi x}{A}\right), \ x > 0.$$

Since $\Omega(x) = \int_0^x \omega(t)dt$, we have $\Omega'(x) = \omega(x)$, hence

$$f'(x) = 2\pi T\rho_C - 2\pi(1-3\eta)\omega\left(\frac{\xi x}{A}\right).$$

Since $\omega(x)$ is a continuous increasing function, $\omega(x)$ has its inverse function $\omega^{-1}(x)$. Hence if we take $\eta \in (0,1)$, $f(x)$ attains its maximum at

$$x = \frac{A}{\xi} \cdot \omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right) > 0.$$

Hence, for $t \in C'_* \subset C_*$, if we take $\eta \in (0,1)$ with $1-3\eta > 0$,

$$\begin{aligned}(13)\qquad &\exp\left[2\pi T\rho_C|t| - 2\pi(1-3\eta)\Omega\left(\frac{\xi t}{A}\right)\right] \\ &\le \exp\left[2\pi T\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right) - 2\pi(1-3\eta)\Omega\left(\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right)\right)\right].\end{aligned}$$

Thus we have from (11), (12), and (13) that

$$\begin{aligned}(14)\qquad |h^{\gamma,g}_{C'_*}(z)| &\le M(C'_*,A)K(C'_*,\eta)\exp\Bigg[2\pi T\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right) \\ &\qquad - 2\pi(1-3\eta)\Omega\left(\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right)\right)\Bigg]\cdot\int_{C'_*}\frac{1}{(1+|t|)^{n+1}}dt \\ &\le K'(C'_*,A,\eta)\exp\Bigg[2\pi T\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right) \\ &\qquad - 2\pi(1-3\eta)\Omega\left(\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right)\right)\Bigg]\end{aligned}$$

for all $z \in N'(z_0,r)$, where $K'(C'_*,A,\eta)$ is a constant depending on fixed C'_*, on fixed $A > 0$, and on fixed $\eta \in (0,1)$ with $1-3\eta > 0$. Since the last term of (14) is independent of $z \in N'(z_0,r)$, the function $h^{\gamma,g}_{C'_*}(z)$ converges absolutely and uniformly for $z \in N'(z_0,r)$. □

Remark. The estimation of inequalities in (14) will be continued under some additional conditions in Theorem 2 of the next section.

Lemma 13. *Let C be an open connected cone and let C' be an arbitrary open compact subcone of $\mathcal{O}(C)$. Let $(C^*)'$ be an open cone as in Lemma 3 and let $C'_* = \mathbb{R}^n\backslash(C^*)' \subset C_*$. Let $z_0 \in T(C';m) = \mathbb{R}^n + i(C'\backslash(C'\cap N(0,m)))$ be arbitrary but fixed and let $z \in N'(z_0,r) \subset T(C';m)$, where $N'(z_0,r)$ is an open neighborhood of z_0 with radius $r > 0$ whose closure is in $T(C';m)$. Let $g(t)$ satisfies*

$$(15)\qquad |g(t)| \le Ke^{k|t|},\ t \in (C^*)'$$

for some constants K and $k \ge 0$. Then for any n-tuple γ of nonnegative integers,

$$h^{\gamma,g}_{(C_*)'}(z) = \int_{(C_*)'} t^\gamma g(t)e^{2\pi i\langle z,t\rangle}dt$$

converges absolutely and uniformly for $z \in N'(z_0,r)$.

Proof. By Lemma 3, there exist a number $\delta = \delta(C')$ and an open cone $(C^*)'$ both depending on C' such that $C^* \subset (C^*)'$ and

$$(16)\qquad \langle y,t\rangle \ge \delta|y||t|,\quad y \in C' \subset \mathcal{O}(C),\ t \in (C^*)'.$$

We choose the real number $m = m(C') > 0$ depending on C' such that

$$(17)\qquad m = \frac{k}{(2\pi\delta)} + 1,$$

where $k \geq 0$ is as in (15). Then if $y \in C'$ with $|y| > m$, $k - 2\pi\delta|y| < -2\pi\delta < 0$. For the chosen $m > 0$ in (17), let z_0 be an arbitrary but fixed point in $T(C'; m)$. Choose $N'(z_0, r)$ whose closure is in $T(C'; m)$. Then we have from (16) and (17) that for $z \in N'(z_0, r)$,

$$\begin{aligned}(18)\qquad &\left|h^{\gamma;g}_{C'_*}(z)\right| \\ &= \left|\int_{(C^*)'} t^\gamma g(t) e^{2\pi i\langle z,t\rangle} dt\right| \\ &\leq K \int_{(C^*)'} |t^\gamma| e^{k|t|} e^{-2\pi\langle y,t\rangle} dt \\ &\leq K \int_{(C^*)'} |t^\gamma| \exp[(k - 2\pi\delta|y|)|t|] dt \leq K \int_{(C^*)'} |t^\gamma| \exp[-2\pi\delta|t|] dt \\ &\leq K Z_n \int_0^\infty s^{|\gamma|+n-1} \exp[-2\pi\delta s] ds = K Z_n (|\gamma| + n - 1)!(2\pi\delta)^{-|\gamma|-n}.\end{aligned}$$

Here we have used [7, Theorem 32, p. 39] in the second to last step in (18) and integration by parts $(|\gamma| + n - 1)$ times in the last step in (18), where K is the constant as in (15) and Z_n is the area of the unit sphere in $\mathbb{R}^n$. Since the last term of (18) is independent of $z \in N'(z_0, r)$, the function $h^{\gamma;g}_{C'_*}(z)$ converges absolutely and uniformly for $z \in N'(z_0, r)$. □

5. The relationship $G_M(A; C)$, $\mathcal{K}'_M$, and $\mathbf{K}'_\mathbf{M}$

In this section, we show that elements of $G_M(A; C)$ can be represented as the Fourier-Laplace transform of distributions $\mathcal{K}'_M$. Also we present representations of $G_M(A; C)$ as elements in $\mathbf{K}'_\mathbf{M}$ in terms of Fourier transforms in $\mathbf{K}'_\mathbf{M}$ of certain elements in $\mathcal{K}'_M$ and strong boundedness for $G_M(A; C)$ as elements in $\mathbf{K}'_\mathbf{M}$.

Theorem 1. *Let $M(x)$ and $\Omega(y)$ be the functions as in Definition 7. For the open connected cone C, let $f(z) \in G_M(A; C)$. For any compact subcone $C' \subset C$, let $m = m(C')$ be a fixed real number which depends on C' as in the definition of $G_M(A; C)$. Then there exist a unique element $V = D^\alpha_t(g(t)) \in \mathcal{K}'_M$, where α is an n-tuple of nonnegative integers and $g(t)$ is a continuous function of $t \in \mathbb{R}^n$ such that the following are hold.*

(I) *For $A \geq 0$*

$$(19)\qquad f(z) = z^\alpha \mathcal{F}[e^{-2\pi\langle y,t\rangle} g(t); x], \quad z = x + iy \in T(C'; m),$$

where the Fourier transform is taken in the L^2 sense.

(II) *For $A \geq 0$, $g(t)$ satisfies*

$$(20)\qquad |g(t)| \leq K(C', m) \exp[2\pi(M(Ay) + |y||t|)], \quad t \in \mathbb{R}^n,$$

where $C' \subset C$ is arbitrary and $K(C', m)$ depends on C' and on m. Inequality (20) is independent of $y \in (C' \backslash (C' \cap N(0, m)))$ and $supp(g) = supp(V) \subseteq \{t : u_C(t) \leq A\}$.

(III) *For $A > 0$ and any compact subcone of $C'_* \subset C_* = \mathbb{R}^n \backslash C^*$, $g(t)$ satisfies*

$$|g(t)| \leq M(C'_*, \eta) \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right], \ t \in C'_*, \tag{21}$$

where any $\eta \in (0,1)$ is such that $1-2\eta > 0$ and $M(C'_*, \eta)$ is a constant depending on C'_* and on η.

(IV) *Let $A \geq 0$. If $g(t)$ satisfies that $|g(t)| \leq Ke^{k|t|}$ for any $t \in (C^*)'$ and for some constants K and $k > 0$, then*

$$f(z) = \langle V, e^{2\pi i \langle z,t \rangle} \rangle, \quad z = x+iy \in T(C';m). \tag{22}$$

(V) *For $A \geq 0$,*

$$f(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle} V_t], \quad z = x+iy \in T(C';m), \tag{23}$$

where the equality in (23) *holds in $\mathbf{K}'_{\mathbf{M}}$.*

(VI)

$$\{f(z) : y = \mathrm{Im}(z) \in (C' \backslash (C' \cap N(0,m))), |y| \leq Q_m\} \tag{24}$$

is strongly bounded in $\mathbf{K}'_{\mathbf{M}}$, where $Q_m > m > 0$.

Proof. Let C be an open connected cone and let C' be an arbitrary open compact subcone of C. For any compact subcone $C' \subset C$, let $m = m(C')$ be a fixed real number which depends on C' as in the definition of $G_M(A;C)$ corresponding to $f(z)$. Since $f(z) \in G_M(A;C)$, we can choose an n-tuple α of nonnegative integers which is independent of C' and of m such that for $z = x+iy \in T(C';m)$ and $\epsilon > 0$,

$$|z^{-\alpha} f(z)| \leq K'(C';m)(1+|z|)^{-n-\epsilon} e^{2\pi M(Ay)}, \tag{25}$$

where $K'(C';m)$ is a constant and n is a dimension. Put

$$g(t) = \int_{\mathbb{R}^n} z^{-\alpha} f(z) e^{-2\pi i \langle z,t\rangle} dx, \quad z = x+iy \in T(C';m), \tag{26}$$

which is a continuous function of $t \in \mathbb{R}^n$. By [2, Theorem 1, p. 846] and (25), $g(t)$ is independent of $y = \mathrm{Im}(z) \in (C' \backslash (C' \cap N(0,m)))$.

Proof of (I). We have from (25) that $z^{-\alpha} f(z) \in L_1 \cap L_2$ as a function of $x = Re(z)$ for an arbitrary $y \in (C' \backslash (C' \cap N(0,m)))$. Thus from (26),

$$e^{-2\pi\langle y,t\rangle} g(t) = \mathcal{F}^{-1}[z^{-\alpha} f(z); t], \quad z = x+iy \in T(C';m), \tag{27}$$

where the Fourier transform is taken in the L_2 sense. By the Plancherel theorem, $e^{-2\pi\langle y,t\rangle} g(t) \in L_2$ and

$$z^{-\alpha} f(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle} g(t); x], \quad z = x+iy \in T(C';m), \tag{28}$$

where the Fourier transform is taken in the L_1 or L_2 sense. This complete the proof of (I).

Proof of (II). From (25) and (26),

$$|g(t)| \leq K'(C';m) e^{2\pi M(Ay)} e^{2\pi\langle y,t\rangle} \int_{\mathbb{R}^n} (1+|x|)^{-n-\epsilon} dx \tag{29}$$

$$\leq K''(C';m)\ \exp\,[2\pi(M(Ay)+\langle y,t\rangle)],$$

where $K''(C';m)$ is a constant. Since $g(t)$ is independent of $y=\mathrm{Im}(z)\in(C'\backslash(C'\cap N(0,m)))$, (29) holds independently of $y=\mathrm{Im}(z)\in(C'\backslash(C'\cap N(0,m)))$. From exactly the same process in [1, pp. 846–847], we have that $\mathrm{supp}(g)=\mathrm{supp}(V)\subseteq\{t:u_C(t)\leq A\}$. This complete the proof of (II).

Consider

$$V=D_t^\alpha(g(t)). \tag{30}$$

Since $g(t)$ is a continuous function and satisfies (II), $g(t)\in\mathcal{K}'_M$ by Lemma 8, hence $V=D_t^\alpha(g(t))\in\mathcal{K}'_M$. In fact $V=D_t^\alpha(g(t))\in\mathcal{K}'_1\subset\mathcal{K}'_p,\ p>1$.

Proof of (III). Let C'_* be an arbitrary but fixed compact subcone of C_*. By Lemma 4, for any $\eta\in(0,1)$, there exists a compact subcone $C'=C'(C'_*,\eta)$ of $C\subset\mathcal{O}(C)$, depending on C'_* and on η, such that we can find a point $y_t^0\in Pr(C')$ where

$$-\langle t,y_t^0\rangle\ \geq\ (1-\eta)u_C(t)$$

for any $t\in C'_*$. Put

$$y_t=\frac{1}{A}y_t^0\left|M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right|. \tag{31}$$

Since C' is a cone and $y_t^0\in Pr(C')$, $y_t\in C'\subset C$ for any $t\in C'_*$. Choose a real number $R>0$ such that

$$R>\frac{A(\Omega^{-1}(M(Am)))}{\xi}, \tag{32}$$

where $m=m(C')$ is as in the definition of $G_M(A;C)$ corresponding to $f(z)$ and $\xi=\xi(C'_*)$ is as in Lemma 2. Then for $t\in C'_*$ with $|t|>R>0$, we have from Lemma 2, (31), and (32) that

$$\begin{aligned}|y_t|&=\frac{1}{A}\left|M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right|\geq\frac{1}{A}\left|M^{-1}\left(\Omega\left(\frac{\xi t}{A}\right)\right)\right|\\&\geq\frac{1}{A}\left|M^{-1}\left(\Omega\left(\frac{\xi R}{A}\right)\right)\right|\geq m.\end{aligned} \tag{33}$$

Hence if $t\in C'_*$ with $|t|>R>0$, $y_t\in(C'\backslash(C'\cap N(0,m)))$. We have from (II) that for $t\in C'_*$ with $|t|>R$

$$|g(t)|\leq K(C',m)\exp[2\pi(M(Ay_t)+\langle y_t,t\rangle)]. \tag{34}$$

By Lemma 4 and (31), we have for all $t\in C'_*$ that

$$\begin{aligned}\langle y_t,t\rangle&=\frac{1}{A}\left|M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right|\ \langle y_t^0,t\rangle\\&\leq-(1-\eta)\frac{1}{A}u_C(t)\left|M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right|.\end{aligned} \tag{35}$$

Since $|y_t^0| = 1$, we have from (31) that

$$M(Ay_t) = \Omega\left(\frac{u_C(t)}{A}\right). \tag{36}$$

Applying (35) and (36) to (29), we have for all $t \in C'_*$ with $|t| > R$ that

$$|g(t)| \le K(C', m) \exp\left[2\pi\Omega\left(\frac{u_C(t)}{A}\right) - 2\pi(1-\eta)\frac{1}{A}u_C(t)M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right]. \tag{37}$$

Using the Young's inequality in Lemma 6,

$$\begin{aligned} u_C(t)M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right) &= A\frac{u_C(t)}{A}M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right) \\ &\le A\left(M\left(M^{-1}\left(\Omega\left(\frac{u_C(t)}{A}\right)\right)\right) + \Omega\left(\frac{u_C(t)}{A}\right)\right) \\ &= 2A\Omega\left(\frac{u_C(t)}{A}\right). \end{aligned} \tag{38}$$

Applying (38) to (37), we have for all $t \in C'_*$ with $|t| > R$ that

$$|g(t)| \le K(C', m) \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right]. \tag{39}$$

We find the estimation like (39) for $t \in C'_*$ with $|t| \le R$ for a fixed $R > 0$ of (32). Put

$$y'_t = Qy_t^0 \tag{40}$$

for a $y_t^0 \in Pr(C')$ corresponding to $t \in C'_* \subset C_*$ and a fixed $Q > m > 0$.

Then since $y_t^0 \in (C' \backslash (C' \cap N(0, m)))$ and the estimation (29) of (II) holds for $t \in C'_* \subset C_*$ independently of $y \in (C' \backslash (C' \cap N(0, m)))$, we have from (29), Lemma 4, and the fact that $|y_t^0| = 1$ that for $t \in C'_*$,

$$|g(t)| \le K(C', m)\ \exp[2\pi M(AQ)] \cdot \exp[-2\pi Q(1-\eta)u_C(t)]. \tag{41}$$

Since $\eta \in (0, 1)$ and $u_C(t) > 0$ for $t \in C'_* \subset C_*$, we have that

$$\exp[-2\pi Q(1-\eta)u_C(t)] \le 1,\ \ t \in C'_* \subset C_*. \tag{42}$$

From (41) and Lemma 2, if we take $\eta \in (0, 1)$ with $1 - 2\eta > 0$, we have for $t \in C'_*$ with $|t| \le R$ that

$$\begin{aligned} g(t) &\le K(C', m)\ \exp[2\pi M(AQ)] \cdot \exp[-2\pi Q(1-\eta)u_C(t)] \\ &\le K(C', m)\ \exp[2\pi M(AQ)] \\ &= K(C', m)\ \exp[2\pi M(AQ)] \\ &\quad \cdot \exp\left[2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] \cdot \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right] \\ &\le K(C', m)\ \exp[2\pi M(AQ)] \end{aligned} \tag{43}$$

$$\cdot \exp\left[2\pi(1-2\eta)\Omega\left(\frac{R}{A}\right)\right]\cdot \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right]$$
$$= K(C',m)C_{A,Q}(\eta)\ \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right],$$

where the constant $K(C',m)$ depends on C' and on m and the constant $C_{A,Q}(\eta)$ depends on η for two fixed constants A and Q. Since $C' = C'(C'_*, \eta)$ depends on C'_* and on η and $m = m(C')$ depends on C', $K(C',m)C_{A,Q}(\eta)$ depends on C'_* and on η for two fixed constants A and Q. Thus we can find a constant $M(C'_*, \eta)$, depending on $C'_* \subset C_*$ and on η, such that if t is an element of $C'_* \subset C_*$, then (21) holds for any $\eta \in (0,1)$ with $1-2\eta > 0$. This completes the proof of (III).

Proof of (IV). Firstly, in order to show that the Fourier transform in (19) can be taken in the L_1 sense, we will show that $(e^{-2\pi\langle y,t\rangle}g(t)) \in L^P,\ 1 \le p < \infty$, for $A > 0$ and $y \in (C'\backslash(C' \cap N(0,m)))$. Let $A > 0$ and let p be arbitrary with $1 \le p < \infty$. If we let y be arbitrary but fixed in $(C'\backslash(C' \cap N(0,m)))$, then $y \in C'$ with $|y| > m > 0$. Also if we choose a positive real number ζ such that $0 < m/|y| < \zeta < 1$, then $\zeta y \in C'$ and $|\zeta y| > m$, hence $\zeta y \in (C'\backslash(C' \cap N(0,m)))$. Then we have from (29) that

$$\text{(44)}\quad \begin{aligned}|e^{-2\pi\langle y,t\rangle}g(t)|^p &\le K(C',m)e^{-2\pi p\langle y,t\rangle}\exp[2\pi p(M(A\zeta y) + \langle \zeta y, t\rangle)]\\ &\le K(C',m)\exp[2\pi p(M(A\zeta y)]\cdot\exp[-2\pi p((1-\zeta)\langle y,t\rangle)]\end{aligned}$$

for all $t \in \mathbb{R}^n$. We have from the fact that $(1-\zeta) > 0$, Lemma 3, (40), and [7, p. 39, Theorem 3.2] that

$$\text{(45)}\quad \begin{aligned}&\int_{(C^*)'} |e^{-2\pi\langle y,t\rangle}g(t)|^p dt\\ &\le K^p(C',m)\exp[2\pi pM(A\zeta y)]\int_{(C^*)'}\exp[-2\pi p\delta(1-\zeta)|y||t|]dt\\ &\le K^p(C',m)Z_n\exp[2\pi pM(A\zeta y)]\int_0^\infty s^{n-1}\exp[-2\pi p\delta(1-\zeta)|y|s]ds\\ &= K^p(C',m)Z_n\exp[2\pi pM(A\zeta y)](n-1)!(2\pi p\delta(1-\zeta)|y|)^{-n}.\end{aligned}$$

Here we have used the same techniques as in (18) in the last two steps in (45), where Z_n is the area of the unit sphere in $\mathbb{R}^n$.

Put $C'_* = \mathbb{R}^n\backslash(C^*)'$. Since $C^* \subset (C^*)'$ and $(C^*)'$ is an open cone, C'_* is a compact subcone of C_* and (III) holds for C'_*. Then we have from (14) that

$$\text{(46)}\quad \begin{aligned}&\int_{C'_*} |e^{-2\pi\langle y,t\rangle}g(t)|^p dt\\ &\le K'(C'_*, A, \eta)\exp\Bigg[2\pi pT\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right)\\ &\quad - 2\pi p(1-\eta)\Omega\left(\omega^{-1}\left(\frac{T\rho_C}{1-3\eta}\right)\right)\Bigg],\end{aligned}$$

where $K'(C'_*, A, \eta)$ is a constant depending on C'_*, on a fixed $\eta \in (0,1)$, and on a fixed $A > 0$. Here $\xi = \xi(C'_*)$ is the number in Lemma 2.

The open cone $(C^*)'$ in (45) is fixed depending on the compact subcone $C' \subset \mathcal{O}(C)$. Then the compact subcone $C'_* \subset C_*$ in (46) was defined by $C'_* = \mathbb{R}^n \backslash (C^*)'$. Since $(C^*)' \cup C'_* = \mathbb{R}^n$ and $(C^*)' \cap C'_* = \emptyset$, we have from (45) and (46) that $(e^{-2\pi\langle y,t\rangle} g(t)) \in L^P$, $1 \le p < \infty$, for $A > 0$ and $y \in (C' \backslash (C' \cap N(0,m)))$.

Now if $A = 0$, then $g(t)$ satisfies (29) and supp $(g) \subseteq C^*$. The open cone $(C^*)'$ for which Lemma 3 holds contains C^*, hence we have from (45) that $(e^{-2\pi\langle y,t\rangle} g(t)) \in L^P$, $1 \le p < \infty$, for $y \in (C' \backslash (C' \cap N(0,m)))$.

Thus for either of the cases $A > 0$ or $A = 0$, the Fourier transform in (15) can be taken in the L_1 sense.

Secondly, we show that $\langle V, e^{2\pi\langle z,t\rangle}\rangle$ is well defined on $T(C';m)$. We consider

$$\int_{\mathbb{R}^n} g(t) e^{2\pi i\langle z,t\rangle} dt, \quad z \in T(C';m). \tag{47}$$

Since $(C^*)' \cap C'_* = \emptyset$ and $(C^*)' \cup C'_* = \mathbb{R}^n$, (47) can be rewrite as

$$\begin{aligned}\int_{\mathbb{R}^n} g(t) e^{2\pi i\langle z,t\rangle} dt &= \int_{C'_*} g(t) e^{2\pi i\langle z,t\rangle} dt + \int_{(C^*)'} g(t) e^{2\pi i\langle z,t\rangle} dt \\ &= h^{0,g}_{C'_*} + h^{0,g}_{(C^*)'}.\end{aligned} \tag{48}$$

Here $h^{0,g}_{C'_*}$ and $h^{0,g}_{(C^*)'}$ are the functions corresponding to $(\gamma, g) = (0, g)$ in Lemma 12 and Lemma 13, respectively. Since $T(C';m) \subset T^{C'}$, $h^{0,g}_{C'_*}$ converges absolutely and uniformly on $T^{C'}$ by Lemma 12 and $h^{0,g}_{(C^*)'}$ converges absolutely and uniformly on $T(C';m)$ by Lemma 13, $\langle V, e^{2\pi\langle z,t\rangle}\rangle$ is well-defined on $T(C';m)$.

Hence since the Fourier transform in (19) can be taken in the L_1 sense and $\langle V, e^{2\pi i\langle z,t\rangle}\rangle$ is well-defined on $T(C';m)$, if we use differentiation in the distributional sense, then we have that

$$\begin{aligned}\langle V, e^{2\pi i\langle z,t\rangle}\rangle &= (-1)^{|\alpha|} \langle g(t), D^\alpha_t (e^{2\pi i\langle z,t\rangle})\rangle \\ &= z^\alpha \int_{\mathbb{R}^n} g(t) e^{2\pi i\langle z,t\rangle} dt = z^\alpha \mathcal{F}[e^{-2\pi\langle y,t\rangle} g(t); x]\end{aligned} \tag{49}$$

for $z \in T(C';m)$ and the Fourier transform is taken in either the L^1 or L^2 sense. From (19) and (49) we have (22). This completes the proof of (IV).

Proof of (V). The proof of (V) follows from only replacing $\mathcal{K}_r, \mathbf{K_r}, \mathcal{K}'_r$, and $\mathbf{K'_r}$ in proving (7.3) of [3, pp. 1056–1057] by $\mathcal{K}_M$, $\mathbf{K_M}$, $\mathcal{K}'_M$, and $\mathbf{K'_M}$, respectively.

Proof of (VI). Firstly, we show that $\{e^{-2\pi\langle y,t\rangle} V_t : y \in (C' \backslash (C' \cap N(0,m))), |y| \le Q_m\}$, $Q_m > M > 0$, is strongly bounded set in $\mathcal{K}'_M$. Let Φ be an arbitrary bounded set in $\mathcal{K}_M$ and let $\phi \in \Phi$. Since $(e^{-2\pi\langle y,t\rangle} V_t) \in \mathcal{K}'_M$ for any $y \in \mathbb{R}^n$, we have from Lemma 8 and general Leibnitz rule that for some n-tuple

α of nonnegative integers, some integer $k \geq 0$, and some continuous function f on $\mathbb{R}^n$ bounded by $M > 0$,

$$\begin{aligned}(50)\qquad &\langle e^{-2\pi\langle y,t\rangle}V_t\ ,\ \phi(t)\rangle \\ &= \langle D_t^\alpha(\exp[M(kt)f(t)])\ ,\ e^{-2\pi\langle y,t\rangle}\phi(t)\rangle \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} e^{M(kt)}f(t)\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\left(\frac{1}{i}\right)^{|\beta|} y^\beta e^{-2\pi\langle y,t\rangle}D_t^\gamma(\phi(t))dt \\ &= (-1)^{|\alpha|}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\left(\frac{1}{i}\right)^{|\beta|} y^\beta I_y(\gamma),\end{aligned}$$

where

$$(51)\qquad I_y(\gamma) = \int_{\mathbb{R}^n} e^{M(kt)}f(t)e^{-2\pi\langle y,t\rangle}D_t^\gamma(\phi(t))dt.$$

Let $Q_m > 0$ be an arbitrary but fixed real number. For $y \in \mathbb{R}^n$ with $|y| \leq Q_m$, if we choose $r \geq \max\{|\alpha|, 2k + 2\pi Q_m\}$, we have from Lemma 5 and the fact that $\phi \in \mathcal{K}_M$ that

$$\begin{aligned}(52)\qquad |I_y(\gamma)| &\leq M\int_{\mathbb{R}^n} e^{M(kt)}e^{2\pi|y||t|}\,|D_t^\gamma(\phi(t))|\,dt \\ &\leq M\int_{\mathbb{R}^n} e^{M(kt)}e^{2\pi Q_m|t|}\,|D_t^\gamma(\phi(t))|\,dt \\ &\leq M\int_{\mathbb{R}^n} e^{-M(kt)}e^{M(2kt)}e^{M(2\pi Q_m t)}\,|D_t^\gamma(\phi(t))|\,dt \\ &\leq M\int_{\mathbb{R}^n} e^{M((2k+2\pi Q_m)t)}\,|D_t^\gamma(\phi(t))|\,e^{-M(kt)}dt \\ &\leq M\|\phi\|_{\mathcal{K}_M}\int_{\mathbb{R}^n} e^{-M(kt)}dt,\end{aligned}$$

where M is such that $\sup_{t\in\mathbb{R}^n}|f(t)| \leq M$. Since Φ is a bounded set in $\mathcal{K}_M$, there exist a constant W_γ, depending only γ, such that $\|\phi\|_{\mathcal{K}_M} \leq W_\gamma$ for all $\phi \in \Phi$. Hence for each γ with $\beta + \gamma = \alpha$,

$$(53)\qquad |I_y(\gamma)| \leq MW_\gamma\int_{\mathbb{R}^n} e^{-M(kt)}dt = W'_\gamma$$

for all $\phi \in \Phi$. Thus we have from (50) and (53) that

$$(54)\qquad \left|\langle e^{-2\pi\langle y,t\rangle}V_t\ ,\ \phi(t)\rangle\right| \leq W'_\gamma\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}Q_m^{|\beta|},\quad \phi \in \Phi.$$

Here the bound in (54) is independent of $\phi \in \Phi$. Hence $\{e^{-2\pi\langle y,t\rangle}V_t : y \in (C'\backslash(C' \cap N(0,m))),\ |y| \leq Q_m\}$, $Q_m > M > 0$, is a bounded set in complex plane. Since Φ be an arbitrary bounded set in $\mathcal{K}_M$, $\{e^{-2\pi\langle y,t\rangle}V_t : y \in (C'\backslash(C' \cap$

$N(0,m))), |y| \le Q_m\}, \ Q_m > M > 0$, is strongly bounded set in $\mathcal{K}'_M$. Thus we have from Lemma 10 and (23) that

$$\begin{aligned}&\{f(z) : y = \operatorname{Im}(z) \in (C' \backslash (C' \cap N(0,m))), \ |y| \le Q_m\}\\ &= \{\mathcal{F}[e^{-2\pi\langle y,t\rangle} V_t] : y = \operatorname{Im}(z) \in (C' \backslash (C' \cap N(0,m))), \ |y| \le Q_m\}\end{aligned}$$

is strongly bounded set in $\mathbf{K'_M}$. This completes the proof of (VI). □

We consider the converse of Theorem 1. We note that the inequality (20) in Theorem 1 can be rewrite as

$$|g(t)| \le K_g e^{k_g|t|}, \quad t \in \mathbb{R}^n, \tag{55}$$

for some two positive constants K_g and k_g both of which are depend on g. We will use the inequality (55) instead of the inequality (20) in the next theorem.

Theorem 2. *Let C be an open connected cone in $\mathbb{R}^n$ and let C'_* be an arbitrary compact subcone of $C_* = \mathbb{R}^n \backslash C^*$. Let $A > 0$ be such that $A/\xi \le 1$, where $\xi = \xi(C'_*)$ is a constant, depending on C'_*, as in Lemma 2. Let V be a finite sum*

$$V = \sum_\alpha D_t^\alpha(g_\alpha(t)), \tag{56}$$

where each g_α are continuous function of $t \in \mathbb{R}^n$. Assume that for each n-tuple of nonnegative integers α, $g_\alpha(t)$ satisfies

$$|g_\alpha(t)| \le K_\alpha e^{k_\alpha|t|}, \quad t \in \mathbb{R}^n, \tag{57}$$

where some two positive constants K_α and k_α both of which are depend on g_α. Also assume that each g_α satisfies

$$|g_\alpha(t)| \le M(C'_*, \eta) \exp\left[-2\pi(1-2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right], \ t \in C'_* \subset C_*, \tag{58}$$

for any $\eta \in (0,1)$ with $1 - 3\eta > 0$, where $M(C'_*, \eta)$ is a constant depending on C'_* and on η. Then $V \in \mathcal{K}'_1 \subset \mathcal{K}'_M$. Furthermore the function

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$$

and any derivative of $f(z)$ belong to $G_M\left(\frac{\rho_C}{1-3\eta}; \mathcal{O}(C)\right)$.

Proof. Since $g_\alpha(t)$ is continuous and $g_\alpha(t) \in \mathcal{K}'_1 \subset \mathcal{K}'_M$, $V \in \mathcal{K}'_1 \subset \mathcal{K}'_M$. Using the differentiation in the distribution sense, we write $f(z)$ as

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle = \sum_\alpha z^\alpha \int_{\mathbb{R}^n} g_\alpha(t) e^{2\pi i\langle z,t\rangle} dt. \tag{59}$$

To show the existence and analyticity of $f(z)$ for a certain z, consider

$$h_\alpha(z) = \int_{\mathbb{R}^n} g_\alpha(t) e^{2\pi i\langle z,t\rangle} dt. \tag{60}$$

Since $(C^*)' \cap C'_* = \emptyset$ and $(C^*)' \cup C'_* = \mathbb{R}^n$, (60) can be rewritten by

$$\begin{aligned} h_\alpha(z) &= \int_{\mathbb{R}^n} g_\alpha(t) e^{2\pi i \langle z,t \rangle} dt \\ &= \int_{C'_*} g_\alpha(t) e^{2\pi i \langle z,t \rangle} dt + \int_{(C^*)'} g_\alpha(t) e^{2\pi i \langle z,t \rangle} dt \\ &= h^{0,\alpha}_{C'_*}(z) + h^{0,\alpha}_{(C^*)'}(z), \end{aligned} \tag{61}$$

where $h^{0,\alpha}_{C'_*}(z)$ and $h^{0,\alpha}_{(C^*)'}(z)$ are the functions corresponding to $(\gamma, g) = (0, g_\alpha)$ in Lemma 12 and Lemma 13, respectively.

Also we have from (59), the generalized Leibnitz rule, and the fact that $T(C'; m) \subset T^{C'}$ that

(62)

$$\begin{aligned} D^\gamma_z(f(z)) &= \sum_\alpha \sum_{\beta+\mu=\gamma} \frac{\gamma!}{\beta!\mu!} D^\beta_z(z^\alpha) \left[D^\mu_z(\left(h^{0,\alpha}_{C'_*} \right)(z)) + D^\mu_z(\left(h^{0,\alpha}_{(C^*)'} \right)(z)) \right] \\ &= \sum_\alpha \sum_{\beta+\mu=\gamma} \frac{\gamma!}{\beta!\mu!} D^\beta_z(z^\alpha)(-1)^{|\mu|} \left[h^{\gamma,\alpha}_{C'_*}(z) + h^{\gamma,\alpha}_{(C^*)'}(z) \right], \ z \in T(C'; m), \end{aligned}$$

where γ, β and μ are n-tuples of nonnegative integers. Here $h^{\gamma,\alpha}_{C'_*}(z)$ and $h^{\gamma,\alpha}_{(C^*)'}(z)$ are the functions corresponding to $(\gamma, g) = (\gamma, g_\alpha)$ in Lemma 12 and Lemma 13, respectively.

Let C' be an arbitrary compact subcone of $\mathcal{O}(C)$. Choose $m_\alpha = m_\alpha(C')$, depending on α and on C', such that

$$m_\alpha = (k_\alpha/(2\pi\delta)) + 1, \tag{63}$$

where k_α is as in (57) and δ is as in Lemma 3. For $m_\alpha > 0$ in (63), let z_0 be an arbitrary but fixed point in $T(C'; m_\alpha) = \mathbb{R}^n + i(C' \backslash (C' \cap N(0, m_\alpha)))$. If we choose an open neighborhood $N'(z_0, r)$ of z_0 with radius $r > 0$ whose closure is contained in $T(C'; m_\alpha) \subset T^{C'}$, $h^{\gamma,\alpha}_{C'_*}(z)$ and $h^{\gamma,\alpha}_{(C^*)'}(z)$ converge absolutely and uniformly for $z \in N'(z_0, r)$ from Lemma 12 and Lemma 13, respectively. Since z is an arbitrary point in $T(C'; m_\alpha)$, we have from (62) that $f(z)$ and its derivative is analytic in $T(C'; m_\alpha)$.

We put

$$m = \max_\alpha \{m_\alpha\}, \tag{64}$$

where m_α is as in (63) for each α. Since $T(C'; m) \subset T(C'; m_\alpha)$, $h_\alpha(z)$ in (61) is analytic in $T(C'; m)$ for each α, hence $\sum_\alpha z^\alpha h_\alpha(z)$ is also analytic in $T(C'; m)$. Thus $f(z) = \langle V, e^{2\pi i \langle z,t \rangle} \rangle$ and any derivative of $f(z)$ are analytic in $T(C'; m), C' \subset \mathcal{O}(C)$, for the fixed m in (64).

Now we will obtain a growth of $f(z)$ and any derivative of $f(z)$ like the inequality in the definition of $G_M(A; C)$ for any compact subcone $C' \subset \mathcal{O}(C)$ and the corresponding $m > 0$ taken in (64).

In order to estimate integral representations of $f(z)$ and any derivative of $f(z)$ of the form (59) on C'_*, we will continue the estimation of inequalities in (14) under the additional condition of A and ξ in this theorem.

Let $A > 0$ be such that $A/\xi \le 1$, where $\xi = \xi(C'_*)$ is a constant, depending on C'_*, as in Lemma 2 and let $z \in N'(z_0, r) \subset T^{C'}$, where $N'(z_0, r)$ is an open neighborhood of z_0 with radius $r > 0$ whose closure is in $T^{C'}$.

If we replace T by $|y|$ in (11), (12), and (13), we have from (14) that

$$\begin{aligned}|h^{\gamma,\alpha}_{C'_*}(z)| &= \left|\int_{C'_*} t^\gamma g(t) e^{2\pi i\langle z,t\rangle}dt\right| \\ &\le K'(C'_*, A, \eta)\exp\Bigg[2\pi|y|\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right) \\ &\quad - 2\pi(1-3\eta)\Omega\left(\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right)\right)\Bigg]\end{aligned} \tag{65}$$

for n tuples γ and α of nonnegative integers and all $z \in N'(z_0, r)$, where $K'(C'_*, A, \eta)$ is a constant depending on fixed C'_*, on fixed $A > 0$, and on fixed $\eta \in (0, 1)$ with $1 - 3\eta > 0$. By Lemma 6,

$$\frac{|y|\rho_C}{1-3\eta}\cdot\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right) = M\left(\frac{|y|\rho_C}{1-3\eta}\right) + \Omega\left(\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right)\right). \tag{66}$$

Since $A/\xi \le 1$ and $0 < 1 - 3\eta < 1$, we have from (66) that

$$\begin{aligned}&|y|\rho_C\frac{A}{\xi}\cdot\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right) - (1-3\eta)\Omega\left(\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right)\right) \\ &\le |y|\rho_C\cdot\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right) - (1-3\eta)\Omega\left(\omega^{-1}\left(\frac{|y|\rho_C}{1-3\eta}\right)\right) \\ &= (1-3\eta)M\left(\frac{|y|\rho_C}{1-3\eta}\right) \le M\left(\frac{|y|\rho_C}{1-3\eta}\right).\end{aligned} \tag{67}$$

Applying (67) to (65), since z is an arbitrary point in $T^{C'}$, we have that for n tuples γ and α of nonnegative integers

$$|h^{\gamma,\alpha}_{C'_*}(z)| \le K'(C'_*, A, \eta)\exp\left[M\left(\frac{|y|\rho_C}{1-3\eta}\right)\right], \quad z \in T^{C'}, \tag{68}$$

where $K'(C'_*, A, \eta)$ is a constant depending on fixed C'_*, on fixed $A > 0$ with $A/\xi \le 1$, and on fixed $\eta \in (0, 1)$ with $1 - 3\eta > 0$.

We now consider the integral representations of $f(z)$ and any derivative of $f(z)$ in (59) on $(C^*)'$. Let m_α and m be as in (63) and (64), respectively. Since $m \ge m_\alpha$ for each m_α, $k_\alpha - 2\pi\delta|y| < -2\pi\delta < 0$ when $|y| > m$ and $y \in C'$. Then we have from Lemma 13 that for n tuples γ and α of nonnegative integers,

$$\left|h^{\gamma,\alpha}_{(C^*)'}(z)\right| = \left|\int_{(C^*)'} |t^\gamma| g(t) e^{2\pi i\langle z,t\rangle}dt\right| \tag{69}$$

$$\leq K_\alpha Z_n(|\gamma|+n-1)!(2\pi\delta)^{-|\gamma|-n}$$
$$= Q_\alpha(m, C') < \infty, \quad z \in T(C'; m),$$

where $Q_\alpha(m, C')$ is a constant depending on m chosen in (64) and on $C' \subset \mathcal{O}(C)$ since $\delta = \delta(C')$ depends on C'.

Applying (68) and (69) to (62), we can find a nonnegative real number N which does not depend on m chosen in (64) or on C' such that for n tuples γ and α of nonnegative integers and $z \in T(C'; m)$,

$$\begin{aligned} |D_z^\gamma(f(z))| \leq & \sum_\alpha C_\alpha(1+|z|)^N \\ & \cdot \left[K'(C'_*, A, \eta)\exp\left[M\left(\frac{|y|\rho_C}{1-3\eta}\right)\right] + Q_\alpha(m, C')\right] \\ \leq & K_1(m, C', A, \eta)(1+|z|)^N \exp\left[M\left(\frac{|y|\rho_C}{1-3\eta}\right)\right], \end{aligned} \tag{70}$$

where $K_1(m, C', A, \eta)$ is a constant depending on m chosen in (64), on C', on fixed $A > 0$ with $A/\xi \leq 1$, and on fixed $\eta \in (0,1)$ with $1-3\eta > 0$. Here we note that $C'_* = \mathbb{R}^n \backslash (C^*)'$ depends on $C' \subset \mathcal{O}(C)$. This complete the proof of Theorem 2. □

We can extend the results that are described from last paragraph of [3, p. 1060] to Corollary 7.1 of [3, p. 1061] to the results in the context of spaces $\mathcal{K}'_M$ or spaces $G_M(A; C)$ by the exactly same line there as follow;

(i) Under the hypothesis of Theorem 2, (V) and (VI) are hold for $z \in T(C'; m), C' \subset \mathcal{O}, m = m(C') > 0$.

(ii) Let C be an open connected cone and let C'_* be an arbitrary compact subcone of $C_* = \mathbb{R}^n \backslash C^*$. Let $A > 0$ be such that $A/\xi \leq 1$, where $\xi = \xi(C'_*)$ is a constant, depending on C'_*, as in Lemma 2. If $f(z) \in G_M(A; C)$, then $f(z)$ and any derivative of $f(z)$ can be extended to an element of $G_M\left(\frac{\rho_C}{1-3\eta}; \mathcal{O}(C)\right)$ for a constant $\eta \in (0,1)$ with $1-3\eta > 0$.

6. The relationship between $F_M(A; C)$, $\mathcal{K}'_M$, and $\mathbf{K'_M}$ and distributional boundary values of the spaces $F_M(A; C)$

In this section, we only state without proof the relationship between $F_M(A; C)$, $\mathcal{K}'_M$, and $\mathbf{K'_M}$ since the ideas, methods, and any others needed to obtain the relationship between $F_M(A; C)$, $\mathcal{K}'_M$, and $\mathbf{K'_M}$ are the same as that of obtaining the relationship between $G_M(A; C)$, $\mathcal{K}'_M$ and $\mathbf{K'_M}$ in the previous section.

Exceptionally, we show that the elements of the spaces $F_M(A; C)$ can obtain distributional boundary values in $\mathbf{K'_M}$.

Theorem 3. *Let $M(x)$ and $\Omega(y)$ be the functions as in Definition 7. For the open connected cone C, let $f(z) \in F_M(A; C)$. For any compact subcone $C' \subset C$, let $m = m(C')$ be a fixed real number which depends on C' as in the* *definition of* $F_M(A;C)$. *Then there exist a unique element* $V = D_t^\alpha(g(t)) \in \mathcal{K}'_M$, *where* α *is an* n-*tuple of nonnegative integers and* $g(t)$ *is a continuous function of* $t \in \mathbb{R}^n$ *such that the following are hold.*

(I) *For* $A \geq 0$

$$f(z) = z^\alpha \mathcal{F}[e^{-2\pi\langle y,t\rangle} g(t); x], \quad z = x + iy \in T^{C'},$$

where the Fourier transform is taken in the L^2 *sense.*

(II) *For* $A \geq 0$, $g(t)$ *satisfies*

$$|g(t)| \leq K(C', m) \exp[2\pi(M(Ay) + |y||t|)], \quad t \in \mathbb{R}^n, \tag{71}$$

where $C' \subset C$ *is arbitrary and* $K(C', m)$ *depends on* C' *and on* m. *Inequality* (71) *is independent of* $y \in (C' \backslash (C' \cap N(0, m)))$ *and supp*$(g) =$ *supp*$(V) \subseteq \{t : u_C(t) \leq A\}$.

(III) *For* $A > 0$ *and any compact subcone of* $C'_* \subset C_* = \mathbb{R}^n \backslash C^*$, $g(t)$ *satisfies*

$$|g(t)| \leq M(C'_*, \eta) \exp\left[-2\pi(1 - 2\eta)\Omega\left(\frac{u_C(t)}{A}\right)\right], \ t \in C'_*,$$

for any $\eta \in (0, 1)$ *with* $1 - 2\eta > 0$, *where* $M(C'_*, \eta)$ *is a constant depending on* C'_* *and on* η.

(IV) *For* $A \geq 0$, *if* $g(t)$ *satisfies that* $|g(t)| \leq Ke^{k|t|}$, $t \in (C^*)'$, *for some constant* K *and* $k > 0$, *then*

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle, \quad z = x + iy \in T^{C'}.$$

(V) *For* $A \geq 0$,

$$f(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle} V_t], \quad z = x + iy \in T^{C'}, \tag{72}$$

where the equality in (72) *holds in* $\mathbf{K'_M}$.

(VI)

$$\{f(z) : y = \mathrm{Im}(z) \in (C' \backslash (C' \cap N(0, m))), |y| \leq Q_m\}$$

is strongly bounded in $\mathbf{K'_M}$, *where* $Q_m > m > 0$.

(VII) $f(z) \to \mathcal{F}[V] \in \mathbf{K'_M}$ *in the strong and weak topology of* $\mathbf{K'_M}$ *as* $y = \mathrm{Im}(z) \to 0$, $y \in C' \subset C$, *where this boundary value is obtained independently of how* $y \to 0$ *in* $C' \subset C$.

Proof. It suffices to prove only (VII). Since $V \in \mathcal{K}'_M$, if we replace $\mathcal{K}'_p$ and $e^{k|t|^p}$ in the proof of Lemma 5.9 in [3, pp. 1052–1053] by $\mathcal{K}'_M$ and $e^{M(kt)}$, respectively, we have that

$$\lim_{y \to 0} e^{-2\pi\langle y,t\rangle} V_t = V_t, \quad y \in \mathbb{R}^n, \tag{73}$$

in the weak topology of $\mathcal{K}'_M$. Since $\mathcal{K}_M$ is a Montel space by Lemma 7, we also have the convergence (73) in the strong topology of $\mathcal{K}'_M$. Since the Fourier transform is a topological isomorphism of $\mathcal{K}'_M$ onto $\mathbf{K'_M}$ by Lemma 10, $f(z) \to \mathcal{F}[V] \in \mathbf{K'_M}$ in the strong and weak topology of $\mathbf{K'_M}$ as $y = \mathrm{Im}(z) \to 0, y \in C' \subset C$. Since V is independent of how $y \to 0$ in $C' \subset C$, the boundary value

$\mathcal{F}[V]$ is obtained independently of how $y \to 0$ in $C' \subset C$. This completes the proof of (VII). □

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