Bull. Korean Math. Soc. **51** (2014), No. 6, pp. 1773–1789
http://dx.doi.org/10.4134/BKMS.2014.51.6.1773

NOTES ON NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS SHARING A SET WITH THEIR DERIVATIVES

Xiao-Min Li, Hong-Xun Yi, and Kai-Mei Wang

Abstract. We study the normality of families of meromorphic functions sharing a set consisting of two or three distinct finite values to improve and extend Theorem 1 in Liu-Pang [15] and Theorem 1.1 in Liu-Chang [16]. Examples are provided to show that the results in this paper, in a sense, are the best possible.

1. Introduction and main results

Let D be a domain on the complex plane $\mathbb{C}$, and let F be a family of meromorphic functions D. The family F is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset F$ contains either a subsequence that converges to a meromorphic function uniformly on each compact subset of D, or a subsequence which converges to ∞ uniformly on each compact subset of D, see, e.g., Hayman [11], Schiff [23] and Yang [26].

Let f and g be two nonconstant meromorphic functions in a domain $D \subset \mathbb{C}$, and let S be a subset of distinct elements in the extended plane. Next we define $E_f(S) =: \bigcup_{a\in S}\{z : z \in D,\, f(z) = a\}$, where each a-point of f with multiplicity m is repeated m times in $E_f(S)$. Similarly we define

$$\overline{E}_f(S) =: \bigcup_{a\in S}\{z : z \in D,\, f(z) = a\},$$

where each a-point in $\overline{E}_f(\{a\})$ is counted only once. We say that f and g share the set S CM in D, provided $E_f(S) = E_g(S)$. We say that f and g share the set S IM in D, provided $\overline{E}_f(S) = \overline{E}_g(S)$ (see [10]). We say that f and g share the value a CM in D if $E_f(\{a\}) = E_f(\{a\})$. Similarly we say that f and g share the value a IM in D if $\overline{E}_f(\{a\}) = \overline{E}_f(\{a\})$. We recall the following result due to Mues and Steinmetz [18]:

Received November 16, 2013; Revised June 7, 2014.
2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. meromorphic functions, shared values, shared sets, normal families.
This work is supported by the NSFC (No. 11171184), the NSFC (No. 11461042), the NSF of Shandong Province, China (No. Z2008A01), and the NSF of Shandong Province, China (No. ZR2009AM008).

Theorem A ([18, Satz 2])**.** *Let f be a nonconstant meromorphic function and let a_1, a_2 and a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3 IM, then $f = f'$.*

Schwick [24] discovered a connection between normality criteria and shared values and proved the following result:

Theorem B ([24, Theorem 2])**.** *Let F be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be distinct complex numbers. If f and f' share a_1, a_2 and a_3 IM in D for each $f \in F$, then F is normal in D.*

Pang and Zalcman proved the following result to improve Theorem B:

Theorem C ([21, Theorem 2])**.** *Let F be a family of meromorphic functions in a domain D and let a, b be two nonzero distinct complex numbers. If f and f' share a and b IM in D for each $f \in F$, then F is normal in D.*

Frank-Schwick [8] generalized Theorem A as follows:

Theorem D. *Let f be a nonconstant meromorphic function, let k be a positive integer, and let a_1, a_2, and a_3 be three distinct complex numbers. If f and $f^{(k)}$ share a_1, a_2, a_3 CM, then $f = f^{(k)}$.*

Regarding Theorems B and D, one may ask, what can be said about the conclusion of Theorem B, if f' is replaced with $f^{(k)}$ for $k \geq 2$. Frank and Schwick [9] observed that Theorem B does not admit the obvious extension obtained by replacing f' as $f^{(k)}$. In this direction, Chen and Fang [4] proved the following results:

Theorem E ([4, Theorem 1])**.** *Let F be a family of meromorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in F$, f and $f^{(k)}$ share a and b IM in D, and the zeros of $f - c$ are of multiplicity $\geq k + 1$, then F is normal in D.*

Theorem F ([4, Theorem 2])**.** *Let F be a family of holomorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in F$, f and $f^{(k)}$ share a and b IM in D, and the zeros of $f - c$ are of multiplicity $\geq k$, then F is normal in D.*

We recall the following example, which shows that some assumption on the zeros of $f - c$ is required for Theorems E and F to hold:

Example A ([4])**.** Let $F = \{f_n(z) : f_n(z) = n(e^z - e^{\lambda z}),\ n = 1, 2, 3, \ldots, \}$, where $\lambda^k = 1$ and $\lambda \neq 1$, $k \geq 2$ is a positive integer. Then we can find that F is a family of holomorphic functions in the domain $D = \{z : |z| < 1\}$. Obviously, for each $f \in F$, we have $f = f^{(k)}$ and that f and $f^{(k)}$ share any complex number b in D. But F is not normal in D.

Regarding Theorems B, C, E and F, one may ask, what can be said about the conclusions of Theorems B, C, E and F, if, for each $f \in F$, f and f' or f and

$f^{(k)}$ share the set $\{a_1, a_2, a_3\}$, where F is a family of meromorphic functions in a domain D, $k \geq 2$ is a positive integer, and a_1, a_2, a_3 are three distinct finite complex values in the complex plane? In this direction, Fang [6] and Liu-Pang [15], respectively proved the following results:

Theorem G ([6, Corollary 1]). *Let F be a family of holomorphic functions in a domain D, and let a_1, a_2 and a_3 be distinct complex numbers in the complex plane. If f and f' share $\{a_1, a_2, a_3\}$ IM in D for each $f \in F$, then F is normal in D.*

Theorem H ([15, Theorem 1]). *Let F be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be distinct complex numbers in the complex plane. If f and f' share $\{a_1, a_2, a_3\}$ IM in D for each $f \in F$, then F is normal in D.*

Next we denote by S_1 and S_2 two nonempty sets consisting of finitely many distinct finite values in the complex plane, denote by $|S_1|$ and $|S_2|$ the numbers of the elements in S_1 and S_2, respectively. Recently Liu-Chang [16] proved the following results to extend Theorems G and H:

Theorem K ([16, Theorem 1.1]). *Let F be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be distinct complex numbers in the complex plane. Suppose that $f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$. If one of the assumptions* (a) $|S_1| \geq 5$; (b) $|S_1| \geq 3$, $|S_2| \geq 3$; *and* (c) $|S_2| \geq 10$ *holds, then F is normal in D.*

Regarding Theorem K, one may ask:

Question 1.1. What can be said about the conclusions of Theorem K, if the assumption "$f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$" is replaced with "$f(z) \in S_1$, $z \in D$ if and only if $f^{(k)}(z) \in S_2$, $z \in D$, where $k \geq 2$ is a positive integer"?

Question 1.2 ([16]). Can we find an empty set S_2 satisfying $|S_2| < 10$ such that the conclusion of Theorem K still holds if any other assumptions of Theorem K are not changed?

We will prove the following results to deal with Questions 1.1 and 1.2:

Theorem 1.1. *Let F be a family of meromorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Suppose that $f(z) \in S_1$, $z \in D$ if and only if $f^{(k)}(z) \in S_2$, $z \in D$. If, for each $f \in F$, every zero of $f - a_1$ and $f - a_2$ is of multiplicity $\geq k$, then F is normal in D.*

Theorem 1.2. *Let F be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, and let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $a_1 \neq a_2$, $b_2/b_1 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ and $b_1/b_2 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$, where $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the set of positive integers and the set of negative integers, respectively. Suppose* *that $f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$. If, for each $f \in F$, every pole of f is of multiplicity ≥ 2, then F is normal in D.*

From Theorem 1.2 we can get the following result:

Corollary 1.1. *Let F be a family of holomorphic functions in a domain $D \subset \mathbb{C}$, and let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $a_1 \neq a_2$, $b_2/b_1 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ and $b_1/b_2 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$, where $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the set of positive integers and the set of negative integers, respectively. Suppose that $f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$. Then F is normal in D.*

The following example shows that the number 3 of elements of S in Theorems G and H is best possible, and shows that the assumption "$b_2/b_1 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ and $b_1/b_2 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$" in Theorem 1.2 and Corollary 1.1 is necessary.

Example B ([7])**.** Let $S = \{1, -1\}$. Set $F = \{f_n(z) : n = 2, 3, 4, \ldots\}$, where

$$f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz}, \quad D = \{z : |z| < 1\}.$$

Then, for each $f_n \in F$, we have $n^2[f_n^2(z) - 1] = [f_n'(z)]^2 - 1$. Thus f_n and f_n' share S CM in D, but F is not normal in D.

We also prove the following result to deal with Questions 1.1 and 1.2:

Theorem 1.3. *Let F be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, and let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2, b_3\}$, where $a_1, a_2, b_1, b_2, b_3 \in \mathbb{C}$ such that $a_1 \neq a_2$ and that b_1, b_2, b_3 are distinct. Suppose that $f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$. If, for each $f \in F$, every pole of f is of multiplicity ≥ 2, then F is normal in D.*

From Theorem 1.3 we can get the following result:

Corollary 1.2. *Let F be a family of holomorphic functions in a domain $D \subset \mathbb{C}$, and let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2, b_3\}$, where $a_1, a_2, b_1, b_2, b_3 \in \mathbb{C}$ such that $a_1 \neq a_2$ and that b_1, b_2, b_3 are distinct. Suppose that $f(z) \in S_1$, $z \in D$ if and only if $f'(z) \in S_2$, $z \in D$. Then F is normal in D.*

2. Some lemmas

In this section, we introduce some important lemmas to prove the main results in this paper. First of all, we introduce the following result due to Pang-Zalcman:

Lemma 2.1 (Pang-Zalcman Lemma, [19] and [22, Lemma 2])**.** *Let F be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in F$. Then, if F is not normal, there exist, for each*

$-1 < \alpha \le k$, *we have*: (a) *a number* $0 < r < 1$; (b) *points* z_n, $|z_n| < r$; (c) *functions* $f_n \in F$, *and* (d) *positive numbers* $\rho_n \to 0$ *such that*

$$\frac{f(z_n + \rho_n \zeta)}{\rho^{\alpha}} =: g_n(\zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g *is a nonconstant meromorphic function on* $\mathbb{C}$ *such that* $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$.

Remark 2.1. Suppose additionally in Lemma 2.1 that F is a family of zero-free meromorphic functions in the domain D. Then the real number α in Lemma 2.1 can be such that $-1 < \alpha < \infty$.

Lemma 2.2 ([3, Lemma 1])**.** *Let* f *be a meromorphic function on* $\mathbb{C}$. *If* f *has bounded spherical derivative on* $\mathbb{C}$, f *is of order at most* 2. *If, in addition,* f *is entire, then the order of* f *is at most* 1.

Lemma 2.3 (Valiron-Mokhonko lemma, [17])**.** *Let* f *be a nonconstant meromorphic function, and let* $F = \frac{\sum_{k=0}^{p} a_k f^k}{\sum_{j=0}^{q} b_j f^j}$ *be an irreducible rational function in* f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$, *where* $a_p \ne 0$ *and* $b_q \ne 0$. *Then* $T(r, F) = d\,T(r, f) + O(1)$, *where* $d = \max\{p, q\}$.

Next we use the notion of a totally ramified value of a meromorphic function: We say that a value $a \in \mathbb{C} \cup \{\infty\}$ is a totally ramified value of a meromorphic function f if all a-points of f are multiple. A classical result of Nevanlinna says that a nonconstant function meromorphic in the plane can have at most 4 totally ramified values, and that a nonconstant entire function can have at most 2 finite totally ramified values (see [1]).

Lemma 2.4 ([1, Lemma 5])**.** *Let* f *be a nonconstant entire function of order at most* 1 *for which* 1 *and* -1 *are totally ramified. Then* $f(z) = \cos(az + b)$, *where* $a, b \in \mathbb{C}$ *are constants and* $a \ne 0$.

Lemma 2.5 ([25, Theorem 1.10])**.** *Suppose that* f *is a nonconstant rational function. Then* f *has only one deficient value in the extended complex plane.*

We need the following result in Langley [14]:

Lemma 2.6 ([14, Theorem 1.2])**.** *Suppose that* f *is meromorphic of finite order in the complex plane, and that* $f^{(k)}$ *has finitely many zeros, for some* $k \ge 2$. *Then* f *has finitely many poles.*

Lemma 2.7 ([25, Theorem 1.5])**.** *Suppose that* f *is a transcendental meromorphic function in the complex plane. Then*

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty.$$

Finally we give the following results due to Chang-Wang [2]:

Lemma 2.8 ([2, Lemma 10]). *Let P be a nonconstant polynomial of degree k, and a and b distinct nonzero finite values. If $P(z) = 0$ if and only if $P'(z)$ is in $\{a, b\}$, then $k \geq 2$ and either $a + (k-1)b = 0$ or $(k-1)a + b = 0$.*

Lemma 2.9 ([2, Lemma 11]). *Let R be a non-polynomial rational function, and a and b distinct finite values. If $R(z) = 0$ if and only if $R'(z) \in \{a, b\}$, then $ab \neq 0$ and either $R(z) = a(z - z_0) + d/(z - z_0)^n$ with $b = (n + 1)a$ or $R(z) = b(z - z_0) + d/(z - z_0)^n$ with $a = (n + 1)b$, where $d(\neq 0)$ and z_0 are constants and n is a positive integer.*

3. Proof of theorems

Proof of Theorem 1.1. We may assume that $D = \{z : |z| < 1\}$. Suppose that F is not normal in D. Without loss of generality, we assume that F is not normal at $z_0 = 0$. Then, by Lemma 2.1, Remark 2.1 and the assumption that $f(z) \in \{a_1, a_2\}$, $z \in D$ if and only if $f^{(k)}(z) \in \{b_1, b_2\}$, $z \in D$, we can find that there exist points $z_n \to 0$, $|z_n| < 1$, positive numbers $\rho_n \to 0^+$ and a subsequence of functions $f_n \in F$ such that

$$f_n(z_n + \rho_n\zeta) - a_1 =: g_n(\zeta) \to g(\zeta) \tag{3.1}$$

and

$$f_n(z_n + \rho_n\zeta) - a_2 = g_n(\zeta) + a_1 - a_2 \to g(\zeta) + a_1 - a_2 \tag{3.2}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where g is some nonconstant meromorphic function such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA+1$, where $A = |b_1|+|b_2|+1$. Moreover, from Lemma 2.2 we can find $\rho(g) \leq 2$. By Hurwitz's Theorem and the assumption of Theorem 1.1 we can find that every zero of g and $g + a_1 - a_2$ is of multiplicity $\geq k$. Next we prove that 0 is a Picard exceptional value of g and $g + a_1 - a_2$. We consider the following two cases:

Case 1. Suppose that 0 is not a Picard exceptional value of one of g and $g + a_1 - a_2$. Without loss of generality, we suppose that 0 is not a Picard exceptional value of one of g. Then, there exists some point $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) = 0$. Set

$$H_k = \{h_n : n = 1, 2, 3, \ldots\}, \tag{3.3}$$

where $h_n(\zeta) = \rho_n^{-k} g_n(\zeta) = \rho_n^{-k}(f_n(z_n + \rho_n\zeta) - a_1)$. Now we claim that H_k is not normal at ζ_0. Indeed, if H_k is normal at ζ_0, then, for a given sequence of functions $\{h_n\} \subseteq H_k$, there exist a subsequence of $\{h_n\}$ say itself such that

$$h_n(\zeta) \to h(\zeta) \tag{3.4}$$

or possibly

$$h_n(\zeta) \to \infty \tag{3.5}$$

spherical uniformly on $\mathbb{C}$, as $n \to \infty$. Noting that $g \not\equiv 0$, we can find, by Hurwitz's Theorem, that there exist a sequence of points ζ_n such that $g_n(\zeta_n) =$

0 and $\zeta_n \to \zeta_0$ as $n \to \infty$. Therefore

$$h(\zeta_0) = \lim_{n\to\infty} \rho_n^{-k} g_n(\zeta_n) = 0. \tag{3.6}$$

From (3.6) we can find that (3.4) is valid and so (3.5) is invalid. By the property that zeros of a nonconstant analytic function are isolated, we can find that there exists some deleted neighborhood $\triangle'(\zeta_0, \delta(\zeta_0)) = \{\zeta : 0 < |\zeta - \zeta_0| < \delta(\zeta_0)\}$ of ζ_0 such that $g(\zeta) \neq 0, \infty$ for any $\zeta \in \triangle'(\zeta_0, \delta(\zeta_0))$, where $\delta(\zeta_0)$ is some positive number that depends only upon ζ_0. Then, for a given point $\zeta \in \triangle'(\zeta_0, \delta(\zeta_0))$, there exists some positive number $\rho(\zeta)$ that depends only upon ζ such that $|g_n(\zeta)| \geq \rho(\zeta)$ for the large positive integer n. Therefore $h(\zeta) = \lim_{n\to\infty} \rho_n^{-k} g_n(\zeta) = \infty$, and so $h = \infty$, which contradicts the facts $h \not\equiv \infty$. Therefore, H_k is not normal at ζ_0. Combining this with Lemma 2.1, we can find that there exist points ζ_n such that $\zeta_n \to \zeta_0$, positive numbers η_n such that $\eta_n \to 0^+$ and a subsequence of functions $h_n \in H_k$ such that

$$\eta_n^{-k} h_n(\zeta_n + \eta_n \xi) = \frac{g_n(\zeta_n + \eta_n \xi)}{\rho_n^k \eta_n^k} =: G_n(\xi) \to G(\xi) \tag{3.7}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where G is some nonconstant meromorphic function such that $G^{\#}(\xi) \leq G^{\#}(0) = kA + 1$, where $A = |b_1| + |b_2| + 1$. Moreover, by Lemma 2.2 we have $\rho(G) \leq 2$. By (3.1), (3.7), Hurwitz's Theorem and the assumptions of Theorem 1.1 we can find that every zero of G is of multiplicity $\geq k$. Now we prove the following claims:

(i) The number of zeros of G in $\mathbb{C}$ is finite; (ii) $\overline{E}_G(\{0\}) = \overline{E}_{G^{(k)}}(S_2)$.

We prove the claim (i): Let ζ_0 be a zero of g with multiplicity $p \geq 1$. Then, the number of zeros of G in $\mathbb{C}$ is not more than p. On the contrary, suppose that there exist $p+1$ distinct points $\xi_1, \xi_2, \ldots, \xi_p, \xi_{p+1}$ in $\mathbb{C}$ such that $G(\xi_j) = 0$ for $1 \leq j \leq p+1$. Combining this with the fact $G \not\equiv 0$, we can find, by Hurwitz's Theorem, that there exist a sequences of points ξ_{n_j} satisfying $\xi_{n_j} \to \xi_j$ for $1 \leq j \leq p+1$ such that $G_n(\xi_{n_j}) = 0$ for the large positive number n, and so we have, by (3.7), that $g_n(\zeta_n + \eta_n \xi_{n_j}) = 0$. Noting that $\zeta_n + \eta_n \xi_{n_j} \to \zeta_0$ for $1 \leq j \leq p+1$, we can deduce, by Hurwitz's Theorem, that ζ_0 is a zero of g with multiplicity $\geq p+1$, which contradicts the above supposition. This proves the claim (i).

We prove the claim (ii): Let $G(\xi_0) = 0$. Then, by Hurwitz's Theorem and the fact $G \not\equiv 0$ we can find from (3.1) and (3.7) that there exist a sequences of points ξ_n satisfying $\xi_n \to \xi_0$, such that $G_n(\xi_n) = 0$, and so $f_n(z_n + \rho_n(\zeta_n + \eta_n \xi_n)) = a_1$ for the large positive integer n. Combining this with the assumption $\overline{E}_f(S_1) = \overline{E}_{f^{(k)}}(S_2)$, we have $G_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \rho_n(\zeta_n + \eta_n \xi_n)) \in S_2$, and so $G^{(k)}(\xi_0) = \lim_{n\to\infty} G_n^{(k)}(\xi_n) \in S_2$. This implies

$$\overline{E}_G(\{0\}) \subseteq \overline{E}_{G^{(k)}}(S_2). \tag{3.8}$$

Next we prove

$$\overline{E}_{G^{(k)}}(S_2) \subseteq \overline{E}_G(\{0\}). \tag{3.9}$$

Let $G^{(k)}(\xi_0)=s_2$, $s_2\in S_2$. First of all, we prove $G^{(k)}\not\equiv s_2$. On the contrary, we suppose that $G^{(k)}=s_2$. If $s_2=0$, then G is a nonconstant polynomial with multiplicity $\le k-1$, which contradicts the fact that every zero of G is of multiplicity $\ge k$. If $s_2\neq 0$, then G is a polynomial of degree k such that $G(\xi)=\frac{s_2(\xi-\xi_1)^k}{k!}$, where ξ_1 is a complex number. Therefore,

$$G^{\#}(0)\le\begin{cases}\frac{k}{2}, & \text{if } |\xi_1|\ge 1,\\ |s_2|, & \text{if } |\xi_1|<1,\end{cases}$$

which contradicts the fact $G^{\#}(0)=kA+1$ and $A=|b_1|+|b_2|+1$. Hence, by Hurwitz's Theorem and the fact $G^{(k)}\not\equiv s_2$ we can find that there exist a sequence of points ξ_n satisfying $\xi_n\to\xi_0$, such that $G_n^{(k)}(\xi_n)=f_n^{(k)}(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=s_2\in S_2$ for the large positive integer n. Combining this with the assumption $\overline{E}_f(S_1)=\overline{E}_{f^{(k)}}(S_2)$, we have $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))\in S_1$. Hence, there exists a subsequence of $\{f_n\}$, say itself such that $f_n(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=s_1$ for the large positive integer n, where $s_1\in S_1$ is some complex number.

Suppose that $s_1\neq a_1$, then we have from (3.1) and (3.7) that

$$G(\xi_0)=\lim_{n\to\infty}G_n(\xi_n)=\lim_{n\to\infty}\frac{s_1-a_1}{\rho_n^k\eta_n^k}=\infty,$$

which contradicts the fact

$$G^{(k)}(\xi_0)=\lim_{n\to\infty}G_n^{(k)}(\xi_n)=\lim_{n\to\infty}f_n^{(k)}(z_n+\rho_n(\zeta_n+\eta_n\xi_n))=s_2.$$

Suppose that $s_1=a_1$, and so we have from (3.1) and (3.7) that

$$G(\xi_0)=\lim_{n\to\infty}G_n(\xi_n)=0,$$

which implies (3.9). From (3.8) and (3.9) we have the claim (ii). We consider the following two cases:

Subcase 1.1. Suppose that G, and so $G^{(k)}$ is a transcendental meromorphic function. Then, by the second fundamental theorem and the claims (i) and (ii) we have

$$\begin{aligned}T(r,G^{(k)})&\le\overline{N}(r,G)+\sum_{j=1}^{2}\overline{N}\left(r,\frac{1}{G^{(k)}-b_j}\right)+O(\log r)\\&\le\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+O(\log r)\\&\le\overline{N}(r,G)+O(\log r),\end{aligned}$$

this together with the fact $N(r,G)+k\overline{N}(r,G)=N(r,G^{(k)})\le T(r,G^{(k)})$ gives

$$N(r,G)\le(1-k)\overline{N}(r,G)+O(\log r)\le O(\log r),$$

as $r\longrightarrow\infty$. Hence G has finitely many poles in the complex plane. Combining this with the claim (i) and Lemma 1.24 [25] and the fact that G is of finite order, we have

$$N\left(r,\frac{1}{G^{(k)}}\right) \leq N\left(r,\frac{1}{G}\right) + k\overline{N}(r,G) + O(\log r) \leq O(\log r),$$

and so $G^{(k)}$ has finitely many zeros in the complex plane. Therefore

$$G^{(k)} = \frac{P_1}{P_2}e^{\alpha}, \tag{3.10}$$

where P_1 and P_2 are nonzero polynomials, α is a nonconstant polynomial such that its degree satisfies $\deg(\alpha) = 1$ or $\deg(\alpha) = 2$. Noting that one of b_1 and b_2 is a finite nonzero value, say $b_1 \neq 0$, we can get by (3.10), Hayman [11, p. 7] and the second fundamental theorem that

(3.11)

$$\begin{aligned}\frac{|a_0| r^{\deg(\alpha)}}{\pi} &\sim T(r,G^{(k)}) \\ &\leq \overline{N}(r,G^{(k)}) + \overline{N}\left(r,\frac{1}{G^{(k)}}\right) + \overline{N}\left(r,\frac{1}{G^{(k)}-b_1}\right) + O(\log r) \\ &= \overline{N}\left(r,\frac{1}{G^{(k)}-b_1}\right) + O(\log r),\end{aligned}$$

where a_0 is the coefficient of the highest term of the polynomial α. From (3.11) we can find that $G^{(k)} - b_1$ has infinitely many zeros of in the complex plane, which contradicts the above claim (i) and (ii).

Subcase 1.2. Suppose that G is a nonconstant rational function. We consider the following two subcases:

Subcase 1.2.1. Suppose that G is a nonconstant polynomial. Then, by the claims (i), (ii) and the second fundamental theorem we have

(3.12)

$$T(r,G^{(k)}) \leq \overline{N}(r,G) + \sum_{j=1}^{2}\overline{N}\left(r,\frac{1}{G^{(k)}-b_j}\right) + O(1) \leq \overline{N}\left(r,\frac{1}{G}\right) + O(1).$$

By Lemma 2.3 we have

$$T(r,G^{(k)}) = (\deg(G) - k)\log r + O(1). \tag{3.13}$$

Noting that every zero of G is of multiplicity $\geq k$, we can get from Lemma 2.3 that

$$\overline{N}\left(r,\frac{1}{G}\right) \leq \frac{1}{k}N\left(r,\frac{1}{G}\right) \leq \frac{\deg(G)}{k}\log r + O(1). \tag{3.14}$$

From (3.12)-(3.14) we get

$$(\deg(G) - k)\log r \leq \frac{\deg(G)}{k}\log r + O(1),$$

and so we have

$$(k-1)\deg(G) \leq k^2. \tag{3.15}$$

Suppose that G has only one zero in the complex plane. Then

$$G(\xi) = c_0(\xi - \hat{\xi}_1)^{\deg(G)}. \tag{3.16}$$

Noting that every zero of G is of multiplicity $\geq k$, we can get from (3.16) and the above claim (ii) we can get a contradiction.

Suppose that G has at least two distinct zeros in the complex plane. Then, by the assumption that every zero of G is of multiplicity $\geq k$, we can deduce that $\deg(G) \geq 2k$. This together with (3.15) gives

$$2k(k-1) \leq (k-1)\deg(G) \leq k^2. \tag{3.17}$$

From (3.17) and the assumption $k \geq 2$ we deduce $k = 2$. Combining this with (3.17) and the claims (i) and (ii), we deduce that $\deg(G) = 4$ and that G has and only has two distinct zeros such that every zero of G is of multiplicity 2, and so we have $b_1 \neq 0$ and $b_2 \neq 0$, this together with the assumption $b_1 \neq b_2$ implies that every zero of $(G'' - b_1)(G'' - b_2)$ is of multiplicity 2. Therefore, by Lemma 2.4 we have

$$G''(\zeta) = \frac{a_2 - a_1}{2}[1 + \cos(A_1\zeta + B_1)] = \frac{a_2 - a_1}{2} \cdot \frac{[e^{i(A_1\zeta + B_1)} + 1]^2}{2e^{i(A_1\zeta + B_1)}}, \tag{3.18}$$

where $A_1 \neq 0$ and B_1 are constants. This contradicts the fact that G, and so G'' is a nonconstant polynomial.

Subcase 1.2.2. Suppose that G is a nonconstant rational function that is not a polynomial. Then

$$G(\xi) = c_m\xi^m + c_{m-1}\xi^{m-1} + \cdots + c_1\xi + c_0 + \frac{P_3(\xi)}{P_4(\xi)}, \tag{3.19}$$

where c_m, c_{m-1}, $\ldots$, c_1, c_0 are complex numbers and $c_m \neq 0$, $m \geq 0$ is an integer, P_3 and P_4 are two relatively prime polynomials such that $P_3 \not\equiv 0$ and that P_4 is not a constant, and that $\deg(P_3) < \deg(P_4)$. Set

$$P_4(\xi) = \alpha_l(\xi - \xi_1)^{n_1}(\xi - \xi_2)^{n_2} \cdots (\xi - \xi_t)^{n_t}, \tag{3.20}$$

where $\alpha_l \neq 0$ is a constant, $t \geq 1$ is a positive integer, n_1, n_2, $\ldots$, n_t are t positive integers such that $l = n_1 + n_2 + \cdots + n_t$, and $\xi_1, \xi_2, \ldots, \xi_t$ are t distinct finite complex values. By (3.19) and (3.20) we have

$$N(r, G^{(k)}) = (l + tk)\log r + O(1), \ \ \overline{N}(r, G) = \overline{N}(r, G^{(k)}) = t\log r + O(1), \tag{3.21}$$

By the claims (i), (ii) and the second fundamental theorem we deduce

$$T(r, G^{(k)}) \leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + O(1). \tag{3.22}$$

We discuss as follows:

Suppose that $m \geq k$. Then, by the fact that every zero of G is of multiplicity $\geq k \geq 2$, we can get by (3.19), (3.20) that

$$\begin{aligned}\overline{N}\left(r, \frac{1}{G}\right) &\leq \frac{1}{k} N\left(r, \frac{1}{G}\right) \\ &= \frac{m+l}{k} \log r + O(1) \\ &\leq \frac{m+l}{2} \log r + O(1)\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}T(r, G^{(k)}) &= N\left(r, \frac{1}{G^{(k)}}\right) + O(1) \\ &= [(m-k) + (l+tk)] \log r + O(1) \\ &= [m+l+(t-1)k)] \log r + O(1).\end{aligned} \tag{3.24}$$

From (3.21)-(3.24) we have

$$[m+l+(t-1)k)] \log r \leq \left(\frac{m+l}{2} + t\right) \log r + O(1),$$

and so

$$\frac{m+l}{2} \leq 2-t. \tag{3.25}$$

Noting that m, l and t are positive integers, we can deduce from (3.25) that $m = l = t = 1$. Combining this with (3.20), we can find that (3.19) can be rewritten as

$$G(\xi) = c_1 \xi + c_0 + \frac{c_{-1}}{\xi - \xi_1}, \tag{3.26}$$

where $P_3/\alpha_1 = c_1$ is a nonzero constant. Noting that $k \geq 2$ is a positive integer, we can get from (3.26) that

$$G^{(k)}(\xi) = \frac{c_{-1}(-1)^k k!}{(\xi - \xi_1)^{k+1}}. \tag{3.27}$$

From (3.27) we have

$$\begin{aligned}&(G^{(k)}(\xi) - b_1)(G^{(k)}(\xi) - b_2) \\ &= \frac{[c_{-1}(-1)^k k! - b_1(\xi - \xi_1)^{k+1}][c_{-1}(-1)^k k! - b_2(\xi - \xi_1)^{k+1}]}{(\xi - \xi_1)^{2(k+1)}}.\end{aligned} \tag{3.28}$$

From (3.26), (3.28), the above claim (ii) and $k \geq 2$, we can get a contradiction.

Suppose that $m < k$. Then, from (3.19), (3.20), the left equality of (3.21) and the fact that every zero of G is of multiplicity $\geq k \geq 2$, we have (3.23) and

$$T(r, G^{(k)}) = N(r, G^{(k)}) + O(1) = (l + tk) \log r + O(1). \tag{3.29}$$

By substituting (3.23), (3.29) and the right equality of (3.21) into (3.22) we have

$$(l+tk)\log r \le t\log r + \frac{m+l}{k}\log r + O(1),$$

which implies that $l+tk \le t+\frac{m+l}{k}$. Combining this with the assumption $m<k$, we have $(k-1)l+(k-1)tk \le m < k$, which is impossible.

Case 2. Suppose that 0 is a Picard exceptional value of g and $g+a_1-a_2$. Then, from Lemma 2.5 we can see that g is a transcendental meromorphic function. From (3.1) and (3.2) we have

$$[\rho_n^k(f_n^{(k)}(z_n+\rho_n\zeta)-b_1)][\rho_n^k(f_n^{(k)}(z_n+\rho_n\zeta)-b_2)] \to [g^{(k)}(\zeta)]^2 \tag{3.30}$$

spherical uniformly on compact subsets of $\mathbb{C}$. By the supposition that 0 is a Picard exceptional value of g we have $g^{(k)} \not\equiv 0$. Noting that $f(z)\in S_1, z\in D$ if and only if $f^{(k)}(z)\in S_2, z\in D$, from (3.1), (3.2), (3.28), Hurwitz's Theorem and the supposition that 0 is a Picard exceptional value of g and $g+a_1-a_2$ we can deduce $g^{(k)} \neq 0$. Combining this with $\rho(g)\le 2$ and Lemma 2.6 we can find that g has finitely many poles in the complex plane. Therefore, by the second fundamental theorem we have

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g+a_1-a_2}\right) + O(\log r) \\ &\le O(\log r). \end{aligned} \tag{3.31}$$

From (3.31) and Lemma 2.7 we can see that g is a rational function, which is impossible. Theorem 1.1 is thus completely proved. □

Proof of Theorem 1.2. We may assume that $D=\{z:|z|<1\}$. Suppose that F is not normal in D. Without loss of generality, we assume that F is not normal at $z_0=0$. Then, by Lemma 2.1, Remark 2.1 and the assumption $\overline{E}_f(S_1)=\overline{E}_{f'}(S_2)$ we can find that there exist points $z_n\to 0$, $|z_n|<1$, positive numbers ρ_n, $\rho_n\to 0^+$ and a subsequence of functions $f_n\in F$ such that (3.1)-(3.2) hold, where $g(\zeta)$ is some nonconstant meromorphic function such that $g^{\#}(\zeta)\le g^{\#}(0)=kA+1$, where $A=|b_1|+|b_2|+1$. Moreover, from Lemma 2.2 we can find $\rho(g)\le 2$. By the assumptions of Theorem 1.2 we find that every pole of f_n is of multiplicity ≥ 2. Combining this with Hurwitz's Theorem, we can find that every pole of g is of multiplicity ≥ 2. We consider the following two cases:

Case 1. Suppose that 0 is not a Picard exceptional value of one of g and $g+a_1-a_2$. Without loss of generality, we suppose that 0 is not a Picard exceptional value of one of g. Then, there exists some point $\zeta_0\in\mathbb{C}$ such that $g(\zeta_0)=0$. Set

$$H_1=\{\hat{h}_n : n=1,2,3,\ldots\}, \tag{3.32}$$

where $\hat{h}_n(\zeta)=\rho_n^{-1}g_n(\zeta)=\rho_n^{-1}(f_n(z_n+\rho_n\zeta)-a_1)$. In the same manner as in the proof of Theorem 1.1 we can prove that H_1 is not normal at ζ_0. Combining this with Lemma 2.1, we can find that there exist some points ζ_n such that $\zeta_n \to \zeta_0$, some positive numbers η_n such that $\eta_n \to 0^+$ and some subsequence of functions $\hat{h}_n \in H_1$ such that

$$\eta_n^{-1}\hat{h}_n(\zeta_n + \eta_n\xi) = \frac{g_n(\zeta_n + \eta_n\xi)}{\rho_n\eta_n} =: \hat{G}_n(\xi) \to \hat{G}(\xi) \tag{3.33}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where $\hat{G}$ is some nonconstant meromorphic function such that $\hat{G}^{\#}(\xi) \le \hat{G}^{\#}(0) = A+1$, where $A = |b_1|+|b_2|+1$. Noting that every pole of f_n is of multiplicity ≥ 2, we can deduce by (3.1), (3.32), (3.33) and Hurwitz's Theorem that every pole of $\hat{G}$ is of multiplicity ≥ 2. By Lemma 2.2 we have $\rho(\hat{G}) \le 2$. In the same manner as in the proof of Theorem 1.1 we can prove following claims:

(iii) The number of zeros of $\hat{G}$ in $\mathbb{C}$ is finite; (iv) $\overline{E}_{\hat{G}}(\{0\}) = \overline{E}_{\hat{G}'}(S_2)$.

We consider the following two subcases:

Subcase 1.1. Suppose that $\hat{G}$, and so $\hat{G}'$ is a transcendental meromorphic function. Then, by the fact $\rho(\hat{G}) = \rho(\hat{G}') \le 2$, the claims (iii) and (iv), and the second fundamental theorem we have

$$\begin{aligned} T(r,\hat{G}') &\le \overline{N}(r,\hat{G}') + \sum_{j=1}^{2}\overline{N}\left(r,\frac{1}{\hat{G}'-b_j}\right) + O(\log r) \\ &\le \frac{1}{2}N(r,\hat{G}') + \overline{N}\left(r,\frac{1}{\hat{G}}\right) + O(\log r) \\ &\le \frac{1}{2}T(r,\hat{G}') + O(\log r), \end{aligned}$$

which implies that $T(r,\hat{G}') = O(\log r)$. Combining this with Lemma 2.7 we can see that $\hat{G}'$ is a rational function, which is impossible.

Subcase 1.2. Suppose that $\hat{G}$ is a rational function. We consider the following two subcases:

Suppose that $\hat{G}$ is a nonconstant polynomial. Then, by the above claim (iv) and Lemma 2.8 we deduce $\deg(\hat{G}) =: l \ge 2$ and either $(l-1)b_1 + b_2 = 0$ or $(l-1)b_2 + b_1 = 0$, and so we have $b_2/b_1 \in \mathbb{Z}^-$ or $b_1/b_2 \in \mathbb{Z}^-$, which contradicts the assumptions $b_2/b_1 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ and $b_1/b_2 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ of Theorem 1.2. Next we suppose that $\hat{G}$ is a nonconstant rational function. Then, by the above claim (iv) and Lemma 2.9 we can see that $b_1b_2 \ne 0$ and either $\hat{G}(\xi) = b_1(\xi-\xi_0) + d/(\xi-\xi_0)^n$ with $b_2 = (n+1)b_1$ or $\hat{G}(\xi) = b_2(\xi-\xi_0) + d/(\xi-\xi_0)^n$ with $b_1 = (n+1)b_2$, where $d \ne 0$ and ξ_0 are constants, $n \ge 1$ is a positive integer. Combining this with $b_1b_2 \ne 0$, we have $b_2/b_1 \in \mathbb{Z}^+$ or $b_1/b_2 \in \mathbb{Z}^+$, which contradicts the assumptions $b_2/b_1 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ and $b_1/b_2 \notin \mathbb{Z}^- \cup \mathbb{Z}^+$ of Theorem 1.2.

Case 2. Suppose that 0 is a Picard exceptional value of one of g and $g + a_1 - a_2$. Then, by Lemma 2.5 we can deduce that g is a transcendental meromorphic function. From the fact $\rho(g) \leq 2$, the fact that every pole of g is of multiplicity ≥ 2 and the second fundamental theorem, we have

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g+a_1-a_2}\right) + O(\log r) \\ &\leq \frac{1}{2}N(r,g) + O(\log r) \\ &\leq \frac{1}{2}T(r,g) + O(\log r), \end{aligned}$$

i.e., $T(r,g) = O(\log r)$. Combining this with Lemma 2.7 we can see that g is a rational function, which is impossible. This proves Theorem 1.2. □

Proof of Theorem 1.3. We may assume that $D = \{z : |z| < 1\}$. Suppose that F is not normal in D. Without loss of generality, we assume that F is not normal at $z_0 = 0$. Then, by Lemma 2.1, Remark 2.1 and the assumption $\overline{E}_f(S_1) = \overline{E}_{f'}(S_2)$ we can find that there exist points $z_n \to 0$, $|z_n| < 1$, positive numbers ρ_n, $\rho_n \to 0^+$ and a subsequence of functions $f_n \in F$ such that (3.1) and (3.2) hold, where g is a nonconstant meromorphic function such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$, where $A = |b_1| + |b_2| + |b_3| + 1$. Moreover, from Lemma 2.2 we can find $\rho(g) \leq 2$. We consider the following two cases:

Case 1. Suppose that 0 is not a Picard exceptional value of one of g and $g + a_1 - a_2$. Without loss of generality, we suppose that 0 is not a Picard exceptional value of one of g. Then, there exists some point $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) = 0$. Set

$$H_2 = \{\tilde{h}_n : n = 1, 2, 3, \ldots\}, \tag{3.34}$$

where $\tilde{h}_n(\zeta) = \rho_n^{-1} g_n(\zeta) = \rho_n^{-1}(f_n(z_n + \rho_n\zeta) - a_1)$. In the same manner as in the proof of Theorem 1.1 we can prove that H_2 is not normal at ζ_0. Combining this with Lemma 2.1, we can find that there exist some sequence of points ζ_n such that $\zeta_n \to \zeta_0$, some sequence of positive numbers η_n such that $\eta_n \to 0^+$ and some subsequence of functions $\tilde{h}_n \in H_2$ such that

$$\eta_n^{-1}\tilde{h}_n(\zeta_n + \eta_n\xi) = \frac{g_n(\zeta_n + \eta_n\xi)}{\rho_n\eta_n} =: \tilde{G}_n(\xi) \to \tilde{G}(\xi) \tag{3.35}$$

spherical uniformly on compact subsets of $\mathbb{C}$, where $\tilde{G}$ is some nonconstant meromorphic function such that $\tilde{G}^{\#}(\xi) \leq \tilde{G}^{\#}(0) = A + 1$, where $A = |b_1| + |b_2| + |b_3| + 1$. By Lemma 2.2 we have $\rho(\tilde{G}) \leq 2$. In the same manner as in the proof of Theorem 1.1 we can prove following claims:

(v) The number of zeros of $\tilde{G}$ in $\mathbb{C}$ is finite; (vi) $\overline{E}_{\tilde{G}}(\{0\}) = \overline{E}_{\tilde{G}'}(S_2)$.

We consider the following two subcases:

Subcase 1.1. Suppose that $\tilde{G}$, and so $\tilde{G}'$ is a transcendental meromorphic function. Then, by the fact $\rho(\hat{G}) = \rho(\hat{G}') \leq 2$, the claims (v) and (vi), and the second fundamental theorem we have

$$\begin{aligned}2T(r,\tilde{G}') &\le \overline{N}(r,\tilde{G}') + \sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{\tilde{G}'-b_j}\right) + O(\log r)\\ &\le \frac{1}{2}N(r,\tilde{G}') + \overline{N}\left(r,\frac{1}{\tilde{G}}\right) + O(\log r)\\ &\le \frac{1}{2}T(r,\tilde{G}') + T\left(r,\frac{1}{\tilde{G}}\right) + O(\log r)\\ &\le \frac{3}{2}T(r,\tilde{G}') + O(\log r),\end{aligned}$$

which implies that $T(r,\tilde{G}') = O(\log r)$. Combining this with Lemma 2.7 we can see that $\tilde{G}'$, and so $\tilde{G}$ is a rational function, which is impossible.

Subcase 1.2. Suppose that $\tilde{G}$ is a rational function. We consider the following two subcases:

Subcase 1.2.1. Suppose that $\tilde{G}$ is a nonconstant polynomial with degree $\tilde{d} \ge 1$. Then, by the claim (vi) we can find that $\tilde{d} \ge 2$. Combining this with Lemma 2.3, the claims (v) and (vi) and the second fundamental theorem we have

$$\begin{aligned}2(\tilde{d}-1)\log r &= 2T(r,\tilde{G}') + O(1)\\ &\le \overline{N}(r,\tilde{G}') + \sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{\tilde{G}'-b_j}\right) + O(1)\\ &\le \overline{N}\left(r,\frac{1}{\tilde{G}}\right) + O(1)\\ &\le \tilde{d}\log r + O(1),\end{aligned}$$

which implies that $\tilde{d} \le 2$. This together with $\tilde{d} \ge 2$ gives $\tilde{d} = 2$. Therefore, $(\tilde{G}'-b_1)(\tilde{G}'-b_2)(\tilde{G}'-b_3)$ has at least three distinct zeros in the complex plane. Combining this with the claim (vi), we can see that $\tilde{G}$ has at least three distinct zeros in the complex plane, which contradicts $\deg(\tilde{G}) = 2$. Next we suppose that $\tilde{G}$ is a non-polynomial rational function. Then

$$\tilde{G}(\xi) = d_p\xi^p + d_{p-1}\xi^{p-1} + \cdots + d_1\xi + d_0 + \frac{P_5(\xi)}{P_6(\xi)}, \tag{3.36}$$

where $d_p, d_{q-1}, \ldots, d_1, d_0$ are complex numbers and $d_p \ne 0$, $p \ge 0$ is an integer, P_5 and P_6 are two relatively prime polynomials such that $P_5 \not\equiv 0$ and that P_6 is not a constant, and that $\deg(P_5) < \deg(P_6)$. Set

$$P_6(\xi) = \beta_q(\xi-\eta_1)^{r_1}(\xi-\eta_2)^{r_2}\cdots(\xi-\eta_q)^{r_q}, \tag{3.37}$$

where $\beta_q \ne 0$ is a complex number, $\eta_1, \eta_2, \ldots, \eta_q$ are q distinct complex numbers, and r_1, r_2, $\ldots$, r_q are positive integers, $q \ge 1$ is a positive integer. From

(3.36), (3.37), Lemma 2.3, the claim (vi) and the second fundamental theorem we deduce

$$\begin{aligned}2(p+\deg(P_6))\log r &\le 2T(r,\tilde{G}')+O(1)\\ &\le \overline{N}(r,\tilde{G}')+\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{\tilde{G}'-b_j}\right)+O(1)\\ &\le q\log r+\overline{N}\left(r,\frac{1}{\tilde{G}}\right)+O(1)\\ &\le (p+q+\deg(P_6))\log r+O(1),\end{aligned}$$

which implies that $p+\deg(P_6)=q$, and so $p=0$ and $r_j=1$ for $1\le j\le q$. Therefore, by (3.36), Lemma 2.3 and the second fundamental theorem we have

$$\begin{aligned}2(q+\deg(P_6)) &= 2T(r,\tilde{G}')+O(1)\\ &\le \overline{N}(r,\tilde{G}')+\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{\tilde{G}'-b_j}\right)+O(1)\\ &\le \overline{N}(r,\tilde{G}')+\overline{N}\left(r,\frac{1}{\tilde{G}}\right)+O(1)\\ &\le (q+\deg(P_6))\log r+O(1),\end{aligned}$$

and so we have $q+\deg(P_6)=0$, which contradicts the supposition $\deg(P_6)\ge q\ge 1$.

Case 2. Suppose that 0 is a Picard exceptional value of one of g and $g+a_1-a_2$. Then, in the same manner as in Case 2 in the proof of Theorem 1.2 we can get a contradiction.

Theorem 1.3 is thus completely proved. □

Acknowledgements. The authors wish to express their thanks to the referee for his/her valuable suggestions and comments.

References

[1] W. Bergweiler and A. Eremenko, *A property of the derivative of an entire function*, Ann. Acad. Sci. Fenn. Math. **37** (2012), no. 1, 301–307.

[2] J. M. Chang and Y. F. Wang, *Shared values, Picard values and normality*, Tohoku Math. J. **63** (2011), no. 2, 149–162.

[3] J. M. Chang and L. Zalcman, *Meromorphic functions that share a set with their derivatives*, J. Math. Anal. Appl. **338** (2008), no. 2, 1020–1028.

[4] H. H. Chen and M. L. Fang, *Shared values and normal families of meromorphic functions*, J. Math. Anal. Appl. **260** (2001), no. 1, 124–132.

[5] H. H. Chen and X. H. Hua, *Normal families concerning shared values*, Israel J. Math. **115** (2000), 355–362.

[6] M. L. Fang, *A note on sharing values and normality*, J. Math. Study **29** (1996), no. 4, 29–32.

[7] M. L. Fang and L. Zalcman, *Normal families and uniqueness theorems for entire functions*, J. Math. Anal. Appl. **280** (2003), no. 2, 273–283.

[8] G. Frank and W. Schwick, *Meromorphic Funktionen, die mit einer Abteilung drei Werte teilen*, Results Math. **22** (1992), no. 3-4, 679–684.
[9] ———, *A counterexample to the generalized Bloch principle*, New Zealand J. Math. **23** (1994), no. 2, 121–123.
[10] F. Gross, *On the distribution of values of meromorphic functions*, Trans. Amer. Math. Soc. **131** (1968), 199–214.
[11] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
[12] ———, *Picard values of meromorphic functions and their derivatives*, Ann. Math. **70** (1959), 9–42.
[13] G. Jank, E. Mues, and L. Volkmann, *Meromorphe Funktionen, die mit ihrer ersten un zweiten Ableitung einen endlichen Wert teilen*, Complex Variables Theory Appl. **6** (1986), no. 1, 51–71.
[14] J. K. Langley, *The second derivative of a meromorphic function of finite order*, Bull. London Math. Soc. **35** (2003), no. 1, 97–108.
[15] X. J. Liu and X. C. Pang, *Shared values and normal families*, Acta. Math. Sinica (Chin, Ser.) **50** (2007), no. 2, 409–412.
[16] X. Y. Liu and J. M. Chang, *Normal families of meromorphic functions and shared sets*, Acta. Math. Sinica (Chin. Ser.) **54** (2011), no. 6, 1049–1056.
[17] A. Z. Mokhonko, *On the Nevanlinna characteristics of some meromorphic functions*, in: Theory of Functions, Functional Analysis and Their Applications, vol. 14, pp. 83–87, Izd-vo Khar'kovsk. Un-ta, 1971.
[18] E. Mues and N. Steinmetz, *Meromorphe funktionen, die mit ihrer ableitung werte teilen*, Manuscripta Math. **29** (1979), no. 2-4, 195–206.
[19] X. C. Pang, *Normality conditions for differential polynomials*, Kexue Tongbao (Chinese) **33** (1988), no. 22, 1690–1693.
[20] ———, *Shared values and normal families*, Analysis **22** (2002), no. 2, 175–182.
[21] X. C. Pang and L. Zalcman, *Normality and shared values*, Ark. Mat. **38** (2000), no. 1, 171–182.
[22] ———, *Normal families and shared values*, Bull. London Math. Soc. **32** (2000), no. 3, 325–331.
[23] J. Schiff, *Normal Families*, Springer-Verlag, New York, 1993.
[24] W. Schwick, *Sharing values and normality*, Arch. Math. (Basel) **59** (1992), no. 1, 50–54.
[25] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[26] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin Heidelberg, 1993.

Xiao-Min Li
Department of Mathematics
Ocean University of China
Qingdao, Shandong 266100, P. R. China
E-mail address: `lixiaomin@ouc.edu.cn`

Hong-Xun Yi
Department of Mathematics
Shandong University
Jinan, Shandong 250100, P. R. China
E-mail address: `hxyi@sdu.edu.cn`

Kai-Mei Wang
Department of Mathematics
Ocean University of China
Qingdao, Shandong 266100, P. R. China
E-mail address: `kidsky2008@163.com`