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INDEFINITE GENERALIZED SASAKIAN SPACE FORM ADMITTING A GENERIC LIGHTLIKE SUBMANIFOLD

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Abstract. In this paper, we study the geometry of indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a generic lightlike submanifold M subject such that the structure vector field of $\bar{M}(f_1, f_2, f_3)$ is tangent to M. The purpose of this paper is to prove a classification theorem of such an indefinite generalized Sasakian space form.

1. Introduction

In 1985, Oubina [22] introduced the notion of a trans-Sasakian manifold of type (α, β). Now we say that a trans-Sasakian manifold $\bar{M}$ of type (α, β) is an *indefinite trans-Sasakian manifold* if $\bar{M}$ is a semi-Riemannian manifold. Indefinite Sasakian manifold is an important kind of indefinite trans-Sasakian manifold with $\alpha = 1$ and $\beta = 0$. Indefinite cosymplectic manifold is another kind of indefinite trans-Sasakian manifold such that $\alpha = \beta = 0$. Indefinite Kenmotsu manifold is also an example with $\alpha = 0$ and $\beta = 1$.

Alegre, Blair and Carriago [1] introduced generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Indefinite Sasakian space form, indefinite Kenmotsu space form and indefinite cosymplectic space form etc are important kinds of indefinite generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4},\ f_2 = f_3 = \tfrac{c-1}{4};\quad f_1 = \tfrac{c-3}{4},\ f_2 = f_3 = \tfrac{c+1}{4};\quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see two books [5, 9]). Recently many authors have studied lightlike submanifolds of indefinite Sasakian, indefinite Kenmotsu and indefinite cosymplectic manifolds.

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In this paper, we study the geometry of indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a generic lightlike submanifold M subject such that the structure vector field of $\bar{M}(f_1, f_2, f_3)$ is tangent to M. The main result is a classification theorem of such an indefinite generalized Sasakian space form.

2. Indefinite generalized Sasakian space form

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite almost contact metric manifold* ([6] ∼ [18]) if there exist a (1, 1)-type tensor field J, a vector field ζ which is called the *structure vector field* of $\bar{M}$ and a 1-form θ such that, for any vector fields X and Y on $\bar{M}$,

$$J^2X = -X + \theta(X)\zeta, \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \epsilon\theta(X)\theta(Y), \quad \theta(\zeta) = 1, \tag{2.1}$$

where ϵ is the causal character of ζ. In this case, the set $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

In an indefinite almost contact metric manifold, we show that $J\zeta = 0$ and $\theta \circ J = 0$. Such a manifold is said to be an *indefinite contact metric manifold* if $d\theta(X, Y) = \bar{g}(X, JY)$. The indefinite almost contact metric structure of $\bar{M}$ is said to be *normal* if $[J, J](X, Y) = -2d\theta(X, Y)\zeta$, where $[J, J]$ denotes the Nijenhuis (or torsion) tensor field of J given by

$$[J, J](X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY].$$

An indefinite normal contact metric manifold is called an *indefinite Sasakian manifold*. It is well known [9] that an indefinite almost contact metric manifold $\bar{M} = (\bar{M}, \bar{g}, J, \zeta, \theta)$ is indefinite Sasakian if and only if

$$(\bar{\nabla}_X J)Y = \bar{g}(X, Y)\zeta - \epsilon\theta(Y)X,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ with respect to $\bar{g}$.

Definition. An indefinite almost contact metric manifold $\bar{M}$ is called *indefinite trans-Sasakian manifold* [1, 22] if there exist two functions α and β such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\}, \tag{2.2}$$

for any vector fields X and Y on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$ with respect to the semi-Riemannian metric $\bar{g}$. We say that $\{J, \zeta, \theta, \bar{g}\}$ is an*indefinite trans-Sasakian structure of type* (α, β).

Replacing Y by ζ in (2.2), we get

$$\bar{\nabla}_X\zeta = -\epsilon\alpha JX + \epsilon\beta(X - \theta(X)\zeta), \quad d\theta(X, Y) = \bar{g}(X, JY). \tag{2.3}$$

Remark 2.1. If $\beta = 0$, then $\bar{M}$ is said to be an *indefinite α-Sasakian manifold*. Indefinite Sasakian manifolds ([6]∼[18]) appear as examples of indefinite α-Sasakian manifolds, with $\alpha = 1$. Another important kind of indefinite trans-Sasakian manifold is that of indefinite cosymplectic manifolds ([15], [17]) obtained for $\alpha = \beta = 0$. If $\alpha = 0$, then $\bar{M}$ is said to be an *indefinite β-Kenmotsu manifold*. Indefinite Kenmotsu manifolds ([13], [14]) are particular examples of indefinite β-Kenmotsu manifold, with $\beta = 1$.

Definition. An indefinite almost contact metric manifold $(\bar{M},\ J,\ \zeta,\ \theta,\ \bar{g})$ is called *indefinite generalized Sasakian space form* [1, 23] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned}\bar{R}(X,Y)Z = {} & f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + f_2\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & \quad + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\},\end{aligned} \tag{2.4}$$

for any vector fields X, Y and Z on $\bar{M}$, where $\bar{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}(f_1, f_2, f_3)$.

Example. Indefinite Sasakian, Kenmotsu and cosymplectic space forms are important kinds of indefinite generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4},\ f_2 = f_3 = \tfrac{c-1}{4};\ \ f_1 = \tfrac{c-3}{4},\ f_2 = f_3 = \tfrac{c+1}{4};\ \ f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms.

Let (M, g) be an m-dimensional lightlike submanifold immersed in an $(m+n)$-dimensional indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m,\,n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which called the *screen* and *co-screen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM)\,, \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.5}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. We use the following range of indices:

$$i,\, j,\, k,\, \ldots \in \{1,\, \ldots\, ,\, r\}, \qquad a,\, b,\, c,\, \ldots \in \{r\,+\,1,\, \ldots\, ,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively and let $\{N_1,\, \ldots,\, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1,\, \ldots,\, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned}T\bar{M} &= TM \oplus tr(TM) \\ &= \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}).\end{aligned} \tag{2.6}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike* if $1 \leq r < \min\{m,\, n\}$;
(2) *co-isotropic* if $1 \leq r = n < m$;
(3) *isotropic* if $1 \leq r = m < n$;
(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$, respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \ldots, \xi_r\,,\; N_1, \ldots, N_r\,,\; F_{r+1}, \ldots, F_m\,,\; W_{r+1}, \ldots, W_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{W_{r+1}, \ldots, W_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Now we set $\epsilon_a = \bar{g}(W_a, W_a)$ is the sign of W_a.

Let P be the projection morphism of TM on $S(TM)$ with respect to the first decomposition in (2.5). For any r-lightlike submanifold, the local Gauss-Weingarten formulas of M and $S(TM)$ are given, respectively, by

$$\bar{\nabla}_X Y \;=\; \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y) N_i + \sum_{a=r+1}^{n} h_a^{s}(X,Y) W_a, \tag{2.7}$$

$$\bar{\nabla}_X N_i \;=\; -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) W_a, \tag{2.8}$$

$$\bar{\nabla}_X W_a \;=\; -A_{W_a} X + \sum_{i=1}^{r} \phi_{ai}(X) N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b, \tag{2.9}$$

$$\nabla_X PY \;=\; \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.10}$$

$$\nabla_X \xi_i \;=\; -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on TM, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, $A^*_{\xi_i}$ and A_{W_a} are linear operators on TM and τ_{ij}, ρ_{ia}, ϕ_{ai} and $\sigma_{\alpha\beta}$ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both h_i^{ℓ} and h_a^s are symmetric. From the fact $h_i^{\ell}(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$, we know that each h_i^{ℓ} are independent of the choice of $S(TM)$. The above three local second fundamental forms are related to their shape operators by

$$g(A^*_{\xi_i} X, Y) = h_i^{\ell}(X,Y) + \sum_{j=1}^{r} h_j^{\ell}(X, \xi_i)\eta_j(Y), \quad \bar{g}(A^*_{\xi_i} X, N_j) = 0, \tag{2.12}$$

$$(2.13)\qquad g(A_{W_a}X,Y)=\epsilon_a h^s_a(X,Y)+\sum_{i=1}^{r}\phi_{ai}(X)\eta_i(Y),$$

$$\bar{g}(A_{W_a}X,N_i)=\epsilon_a\rho_{ia}(X),\quad \epsilon_b\sigma_{ab}=-\epsilon_a\sigma_{ba},$$

$$(2.14)\qquad g(A_{N_i}X,PY)=h^*_i(X,PY),\quad \eta_j(A_{N_i}X)+\eta_i(A_{N_j}X)=0,$$

where $X,\,Y\in\Gamma(TM)$ and η_is are the 1-forms such that

$$\eta_i(X)=\bar{g}(X,N_i),\quad \forall\, X\in\Gamma(TM).$$

Denote by $(\cdot.\cdot)_i$ the i-th equation of $(\cdot.\cdot)$. We use same notations for any others. Replacing Y by ξ_j to $(2.12)_1$, we have

$$(2.15)\qquad h^\ell_i(X,\xi_j)+h^\ell_j(X,\xi_i)=0,\quad h^\ell_i(X,\xi_i)=0,\quad h^\ell_i(\xi_j,\xi_k)=0.$$

For any r-lightlike submanifold, replacing Y by ξ_i to (2.13), we have

$$(2.16)\qquad h^s_a(X,\xi_i)\,=\,-\epsilon_a\phi_{ai}(X),\ \forall\, X\in\Gamma(TM).$$

We need the following Gauss-Codazzi equations for M and $S(TM)$ (for a full set of these equations see [3, Chapter 5]). Denote by R and R^* the curvature tensors of the induced connection ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$(2.17)\qquad \bar{g}(\bar{R}(X,Y)Z,\,\xi_i)=(\nabla_X h^\ell_i)(Y,\,Z)-(\nabla_Y h^\ell_i)(X,\,Z)$$
$$+\sum_{j=1}^{r}\{h^\ell_j(Y,\,Z)\tau_{ji}(X)-h^\ell_j(X,\,Z)\tau_{ji}(Y)\}$$
$$+\sum_{a=r+1}^{n}\{h^s_a(Y,\,Z)\phi_{ai}(X)-h^s_a(X,\,Z)\phi_{ai}(Y)\},$$

$$(2.18)\qquad \epsilon_a\bar{g}(\bar{R}(X,Y)Z,\,W_a)=(\nabla_X h^s_a)(Y,Z)-(\nabla_Y h^s_a)(X,Z)$$
$$+\sum_{i=1}^{r}\{h^\ell_i(Y,\,Z)\rho_{ia}(X)-h^\ell_i(X,\,Z)\rho_{ia}(Y)\}$$
$$+\sum_{\beta=r+1}^{n}\{h^s_b(Y,\,Z)\sigma_{ba}(X)-h^s_b(X,\,Z)\sigma_{ba}(Y)\},$$

$$(2.19)\qquad \bar{g}(\bar{R}(X,Y)Z,\,N_i)=\bar{g}(R(X,Y)Z,\,N_i)$$
$$+\sum_{j=1}^{r}\{h^\ell_j(X,\,Z)\eta_i(A_{N_j}Y)-h^\ell_j(Y,\,Z)\eta_i(A_{N_j}X)\},$$
$$+\sum_{a=r+1}^{n}\epsilon_a\{h^s_a(X,\,Z)\rho_{ia}(Y)-h^s_a(Y,\,Z)\rho_{ia}(X)\},$$

$$(2.20)\qquad g(R(X,Y)PZ,\,N_i)=(\nabla_X h^*_i)(Y,\,PZ)-(\nabla_Y h^*_i)(X,\,PZ)$$

$$+\sum_{j=1}^{r}\{h_j^*(X,\,PZ)\tau_{ij}(Y)-h_j^*(Y,\,PZ)\tau_{ij}(X)\}.$$

3. Classification theorem

In the entire discussion of this paper, we shall assume that ζ is tangent to M, such M is called a *tangential lightlike submanifold* of $\bar{M}$. Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which many authors assumed in their works [7, 8, 9, 12, 16, 18]. We also assume this result. Therefore

$$\theta(\xi_i)=\epsilon g(\zeta,\xi_i)=0,\ \theta(N_i)=\epsilon g(\zeta,N_i)=0,\ \theta(W_a)=\epsilon g(\zeta,W_a)=0. \tag{3.1}$$

In case g is non-degenerate, there exists a class of submanifolds of an almost complex manifold $\bar{M}$. We say that M is a generic (anti-holomorphic) submanifold of $\bar{M}$ if the normal bundle $TM^{\perp}$ of M is mapped into the tangent bundle TM by action of the structure tensor J of $\bar{M}$, i.e., $J(TM^{\perp})\subset TM$ [19, 20].

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^*=TM/Rad(TM)$ considered by Kupeli [21]. Thus all screen distributions $S(TM)$ are mutually isomorphic. Moreover, while TM is lightlike, all $S(TM)$ are non-degenerate. Due to these reasons, we defined generic lightlike submanifolds of an almost complex manifold $\bar{M}$ as follow:

Definition. We say that an r-lightlike submanifold M of an indefinite almost complex manifold $\bar{M}$ is a *generic r-lightlike submanifold* [6, 17] if there exist a screen distribution $S(TM)$ of M such that

$$J(S(TM)^{\perp})\subset S(TM). \tag{3.2}$$

Example. Any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$ is a generic lightlike submanifold of $\bar{M}$ [10, 13, 15, 16]. Also, any 1-lightlike submanifold M of codimension 2 of an indefinite almost contact metric manifold $\bar{M}$ is a generic lightlike submanifold of $\bar{M}$ [11, 12, 14, 18].

For the rest of this section, a generic lightlike submanifold we shall mean a tangential generic r-lightlike submanifold unless otherwise specified.

For any generic lightlike submanifold M, from (3.2) we show that the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$. In this case, there exists a non-degenerate almost complex distribution H_o with respect to the structure tensor field J, i.e., $J(H_o)=H_o$, such that

$$S(TM)=\{J(Rad(TM))\oplus J(ltr(TM))\}\oplus_{orth}J(S(TM^{\perp}))\oplus_{orth}H_o.$$

Denote by H the almost complex distribution with respect to J such that

$$H=Rad(TM)\oplus_{orth}J(Rad(TM))\oplus_{orth}H_o.$$

Therefore the general decomposition $(2.5)_1$ of TM is reduced to

$$TM=H\oplus J(ltr(TM))\oplus_{orth}J(S(TM^{\perp})). \tag{3.3}$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields E_a for each a, and their 1-forms u_i, v_i and e_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad E_a = -JW_a, \tag{3.4}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad e_a(X) = \epsilon_a g(X, E_a). \tag{3.5}$$

These vector fields U_i, V_i and E_a on $S(TM)$ satisfy

$$g(U_i, U_j) = g(V_i, V_j) = g(U_i, E_a) = g(V_i, E_a) = 0,$$
$$g(V_i, U_j) = \delta_{ij}, \quad g(E_a, E_b) = \epsilon_a \delta_{ab}.$$

Denote by S the projection morphism of TM on H with respect to (3.3). Then, for any vector field X on M, JX is expressed as follow:

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} e_a(X)W_a, \tag{3.6}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$.

Applying $\bar{\nabla}_X$ to $(3.4)_{1,2,3}$ by turns and using (2.2), (2.7) ~ (2.9), (2.11), (2.12) ~ (2.14), (2.16) and (3.4) ~ (3.6), for all X, $Y \in \Gamma(TM)$, we have

$$\begin{aligned} & h_j^{\ell}(X, U_i) = h_i^{*}(X, V_j), \qquad \epsilon_a h_i^{*}(X, E_a) = h_a^{s}(X, U_i), \\ & h_j^{\ell}(X, V_i) = h_i^{\ell}(X, V_j), \qquad \epsilon_a h_i^{\ell}(X, E_a) = h_a^{s}(X, V_i), \\ & \epsilon_b h_b^{s}(X, E_a) = \epsilon_a h_a^{s}(X, E_b); \end{aligned} \tag{3.7}$$

$$\begin{aligned} \nabla_X U_i = {} & F(A_{N_i}X) + \sum_{j=1}^{r} \tau_{ij}(X)U_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a \\ & - \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \nabla_X V_i = {} & F(A^{*}_{\xi_i}X) - \sum_{j=1}^{r} \tau_{ji}(X)V_j + \sum_{j=1}^{r} h_j^{\ell}(X, \xi_i)U_j \\ & - \sum_{a=r+1}^{n} \epsilon_a \phi_{ai}(X)E_a - \beta u_i(X)\zeta, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \nabla_X E_a = {} & F(A_{W_a}X) + \sum_{i=1}^{r} \phi_{ai}(X)U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b \\ & - \epsilon_a \beta e_a(X)\zeta. \end{aligned} \tag{3.10}$$

Theorem 3.1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian structure of type (α, β), admitting a generic lightlike submanifold satisfies* (1) *α is a constant and* (2) *$\alpha\beta = 0$.*

(i) *In case $\alpha = 0 : \epsilon\zeta[\beta] + \beta^2 = f_3 - \epsilon f_1$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu space form.*

(ii) *In case $\alpha \neq 0 : \alpha^2 = \epsilon f_1 - f_3$, $\beta = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form.*

Proof. Applying $\bar{\nabla}_X$ to the three equations of (3.1) by turns and using (2.1), (2.3), (2.7) ∼ (2.9), (2.11) ∼ (2.14), (3.1), (3.4) and (3.5), we have

$$\begin{aligned} &h_i^\ell(X,\zeta) = -\epsilon\alpha u_i(X), \qquad h_a^s(X,\zeta) = -\epsilon\alpha e_a(X), \\ &h_i^*(X,\zeta) = \epsilon\beta\eta_i(X) - \epsilon\alpha v_i(X), \qquad \forall\, X \in \Gamma(TM). \end{aligned} \tag{3.11}$$

Substituting (3.6) into (2.3) and using (2.7), we have

$$\nabla_X\zeta = -\epsilon\alpha FX + \epsilon\beta(X - \theta(X)\zeta), \quad \forall\, X \in \Gamma(TM). \tag{3.12}$$

Applying $\bar{\nabla}_X$ to $u_i(Y) = g(Y, V_i)$ and using (3.6) and (3.9), we get

$$\begin{aligned} (\nabla_X u_i)(Y) = &-\sum_{j=1}^r u_j(Y)\tau_{ji}(X) + \sum_{j=1}^r v_j(Y)h_j^\ell(X,\xi_i) \\ &- \sum_{a=r+1}^n e_a(Y)\phi_{ai}(X) - \epsilon\beta\theta(Y)u_i(X) - h_i^\ell(X,FY) \end{aligned} \tag{3.13}$$

for all $X, Y \in \Gamma(TM)$. Substituting (2.4) into (2.17), we have

$$\begin{aligned} &f_2\{u_i(Y)\bar{g}(X,JZ) - u_i(X)\bar{g}(Y,JZ) + 2u_i(Z)\bar{g}(X,JY)\} \\ &= (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ &\quad + \sum_{j=1}^r \{h_j^\ell(Y,Z)\tau_{ji}(X) - h_j^\ell(X,Z)\tau_{ji}(Y)\} \\ &\quad + \sum_{a=r+1}^n \{h_a^s(Y,Z)\phi_{ai}(X) - h_a^s(X,Z)\phi_{ai}(Y)\} \end{aligned} \tag{3.14}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Z by ζ to this and using (3.11), we have

$$\begin{aligned} &(\nabla_X h_i^\ell)(Y,\zeta) - (\nabla_Y h_i^\ell)(X,\zeta) + \epsilon\alpha\sum_{j=1}^r\{u_j(X)\tau_{ji}(Y) - u_j(Y)\tau_{ji}(X)\} \\ &+ \epsilon\alpha\sum_{a=r+1}^n\{e_a(X)\phi_{ai}(Y) - e_a(Y)\phi_{ai}(X)\} = 0. \end{aligned}$$

Applying ∇_X to $h_i^\ell(Y,\zeta) = -\epsilon\alpha u_i(Y)$ and using (3.12) and (3.13), we get

$$\begin{aligned} (\nabla_X h_i^\ell)(Y,\zeta) = &-\epsilon X[\alpha]u_i(Y) - \epsilon\beta h_i^\ell(X,Y) \\ &+ \alpha\beta\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\} \\ &+ \epsilon\alpha\{\sum_{j=1}^r u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^n e_a(Y)\phi_{ai}(X) \\ &-\sum_{j=1}^r v_j(Y)h_j^\ell(X,\xi_i) + h_i^\ell(X,FY) + h_i^\ell(Y,FX)\} \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Using the last two equations, we have

$$\begin{aligned}&\{\epsilon X[\alpha]+2\alpha\beta\theta(X)\}u_i(Y)-\{\epsilon Y[\alpha]+2\alpha\beta\theta(Y)\}u_i(X)\\ &=\epsilon\alpha\sum_{j=1}^{r}\{v_j(X)h_j^\ell(Y,\xi_i)-v_j(Y)h_j^\ell(X,\xi_i)\},\quad \forall\, X,\,Y\in\Gamma(TM).\end{aligned}$$

Replacing Y by U_i to this equation, for all $X \in \Gamma(TM)$, we obtain

$$X[\alpha]+2\epsilon\alpha\beta\theta(X)=U_i[\alpha]u_i(X)+\alpha\sum_{j=1}^{r}v_j(X)h_j^\ell(U_i,\xi_i).$$

Replacing X by E_a to this equation and using (3.5), we obtain

$$E_a[\alpha]=0,\quad \forall\, a. \tag{3.15}$$

Applying $\bar\nabla_Y$ to $(3.5)_3$ and using $(2.13)_{1,2,3}$, $(3.7)_4$ and (3.10), we get

$$\begin{aligned}(\nabla_X e_a)(Y)=\; &-\sum_{i=1}^{r}u_i(Y)\rho_{ia}(X)-\sum_{b=r+1}^{n}e_b(Y)\sigma_{ba}(X)\\ &-\epsilon\beta\theta(Y)e_a(X)-h_a^s(X,FY)\end{aligned} \tag{3.16}$$

for all $X, Y \in \Gamma(TM)$. Substituting (2.4) into (2.18), we have

$$\begin{aligned}&f_2\{e_a(Y)\bar g(X,JZ)-e_a(X)\bar g(Y,JZ)+2e_a(Z)\bar g(X,JY)\}\\ &=(\nabla_X h_a^s)(Y,Z)-(\nabla_Y h_a^s)(X,Z)\\ &\quad+\sum_{i=1}^{r}\{h_i^\ell(Y,\,Z)\rho_{ia}(X)-h_i^\ell(X,\,Z)\rho_{ia}(Y)\}\\ &\quad+\sum_{\beta=r+1}^{n}\{h_b^s(Y,\,Z)\sigma_{ba}(X)-h_b^s(X,\,Z)\sigma_{ba}(Y)\}\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Z by ζ to this and using (3.11), we get

$$\begin{aligned}&(\nabla_X h_a^s)(Y,\zeta)-(\nabla_Y h_a^s)(X,\zeta)+\epsilon\alpha\sum_{i=1}^{r}\{u_i(X)\rho_{ia}(Y)-u_i(Y)\rho_{ia}(X)\}\\ &+\epsilon\alpha\sum_{b=r+1}^{n}\{e_b(X)\sigma_{ba}(Y)-e_b(Y)\sigma_{ba}(X)\}=0.\end{aligned}$$

Applying ∇_X to $h_a^s(Y,\zeta)=-\epsilon\alpha e_a(Y)$ and using (3.12) and (3.16), we have

$$\begin{aligned}(\nabla_X h_a^s)(Y,\zeta)=\; &-\epsilon X[\alpha]e_a(Y)-\epsilon\beta h_a^s(X,Y)\\ &+\alpha\beta\{\theta(Y)e_a(X)-\theta(X)e_a(Y)\}\\ &+\epsilon\alpha\{\sum_{i=1}^{r}u_i(Y)\rho_{ia}(X)+\sum_{b=r+1}^{n}e_b(Y)\sigma_{ba}(X)\\ &+h_a^s(X,FY)+h_a^s(Y,FX)\}\end{aligned}$$

for all $X,\, Y \in \Gamma(TM)$. Using the last two equations, we have

$$\{\epsilon X[\alpha] + 2\alpha\beta\theta(X)\}e_a(Y) - \{\epsilon Y[\alpha] + 2\alpha\beta\theta(Y)\}e_a(X) = 0.$$

Replacing Y by E_a to this and using (3.15), we obtain

$$X[\alpha] + 2\epsilon\alpha\beta\theta(X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.17}$$

Applying $\bar{\nabla}_X$ to $v_i(Y) = g(Y, U_i)$ and using (3.5) ∼ (3.8), we get

$$\begin{aligned}(\nabla_X v_i)(Y) = &\sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a e_a(Y)\rho_{ia}(X) \\ &- \sum_{j=r+1}^{r} u_j(Y)\eta_j(A_{N_i}X) - g(A_{N_i}X, FY) \\ &- \epsilon\theta(Y)\{\alpha\eta_i(X) + \beta v_i(X)\}, \quad \forall\, X,\, Y \in \Gamma(TM).\end{aligned} \tag{3.18}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.8), we have

$$(\nabla_X \eta_i)Y = -g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{3.19}$$

Substituting (2.4) and (2.20) into (2.19) with $Z = PZ$, we have

$$\begin{aligned}&f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\ &+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\} \\ &+ f_3\{\theta(X)\theta(PZ)\eta_i(Y) - \theta(Y)\theta(PZ)\eta_i(X)\} \\ = &(\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ) \\ &+ \sum_{j=1}^{r}\{h_j^*(X, PZ)\tau_{ij}(Y) - h_j^*(Y, PZ)\tau_{ij}(X)\} \\ &+ \sum_{a=r+1}^{n} \epsilon_a\{h_a^s(X, PZ)\rho_{ia}(Y) - h_a^s(Y, PZ)\rho_{ia}(X)\} \\ &+ \sum_{j=1}^{r}\{h_j^\ell(X, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y, PZ)\eta_i(A_{N_j}X)\}.\end{aligned} \tag{3.20}$$

Replacing Z by ζ to the last equation and using (3.11), we have

$$\begin{aligned}&(\epsilon f_1 - f_3)\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\} \\ = &(\nabla_X h_i^*)(Y, \zeta) - (\nabla_Y h_i^*)(X, \zeta) \\ &+ \epsilon\beta\sum_{j=1}^{r}\{\eta_j(X)\tau_{ij}(Y) - \eta_j(Y)\tau_{ij}(X)\} \\ &+ \epsilon\alpha\sum_{j=1}^{r}\{v_j(Y)\tau_{ij}(X) - v_j(X)\tau_{ij}(Y)\}\end{aligned}$$

$$+ \epsilon\alpha \sum_{a=r+1}^{n} \epsilon_a \{e_a(Y)\rho_{ia}(X) - e_a(X)\rho_{ia}(Y)\}$$
$$+ \epsilon\alpha \sum_{j=1}^{r} \{u_j(Y)\eta_i(A_{N_j}X) - u_j(X)\eta_i(A_{N_j}Y)\}.$$

Applying ∇_Y to $(3.11)_3$ and using $(2.14)_2$, (3.12), (3.18) and (3.19), we have

$$\begin{aligned}(\nabla_X h_i^*)(Y,\zeta) = {} & \epsilon X[\beta]\eta_i(Y) - \epsilon X[\alpha]v_i(Y) \\ & + \epsilon\alpha\{g(A_{N_i}X, FY) + g(A_{N_i}Y, FX) - \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) \\ & - \sum_{a=r+1}^{n} \epsilon_a e_a(Y)\rho_{ia}(X) - \sum_{j=1}^{r} u_j(Y)\eta_i(A_{N_j}X)\} \\ & + \epsilon\beta\{-g(A_{N_i}X, Y) - g(A_{N_i}Y, X) + \sum_{j=1}^{r} \eta_j(Y)\tau_{ij}(X)\} \\ & + \alpha\beta\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\} \\ & + \alpha^2\theta(Y)\eta_i(X) + \beta^2\theta(X)\eta_i(Y).\end{aligned}$$

Using the last two equations and (3.17), for any X, $Y \in \Gamma(TM)$, we have

$$\{\epsilon X[\beta] + A\theta(X)\}\eta_i(Y) = \{\epsilon Y[\beta] + A\theta(Y)\}\eta_i(X), \tag{3.21}$$

where $A = \epsilon f_1 - f_3 - \alpha^2 + \beta^2$. Taking $X = \zeta$ and $Y = \xi_i$ to (3.21), we have

$$\epsilon\zeta[\beta] + \{\epsilon f_1 - f_3 - \alpha^2 + \beta^2\} = 0. \tag{3.22}$$

On the other hand, replacing Y by ξ_i to (3.21), we have

$$X[\beta] = \xi_i[\beta]\eta_i(X) - \epsilon A\theta(X).$$

Applying ∇_Y to (3.17) and using (3.17) and the last equation, we have

$$\epsilon XY[\alpha] + 2\alpha\beta X(\theta(Y)) = 2\alpha\epsilon(2\beta^2 + A)\theta(X)\theta(Y) - 2\alpha\xi_i[\beta]\theta(Y)\eta_i(X)$$

for all X, $Y \in \Gamma(TM)$. Using this and the fact $[X, Y] = XY - YX$, we obtain

$$2\alpha\beta\bar{g}(X, JY) = \alpha\xi_i[\beta]\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}$$

for all X, $Y \in \Gamma(TM)$. Taking $X = U_i$ and $Y = \xi_i$ to this, we have $\alpha\beta = 0$. As $\alpha\beta = 0$, we see that α is a constant by (3.17). Therefore $\alpha = 0$ or $\beta = 0$.

(i) In case $\alpha = 0$: From (3.22) we have $\epsilon\zeta[\beta] + \beta^2 = f_3 - \epsilon f_1$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu space form.

(ii) In case $\alpha \neq 0$: We get $\beta = 0$. Therefore $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form. From (3.22) we show that $\alpha^2 = \epsilon f_1 - f_3$. □

Corollary 3.2. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian structure of type (α, β), admitting a generic lightlike submanifold is either an indefinite β-Kenmotsu space form or an indefinite α-Sasakian space form such that $\alpha^2 = \epsilon f_1 - f_3$.*

Corollary 3.3. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian structure of type (α, β), admitting either a lightlike hypersurface or a codimension 2 half lightlike submanifold satisfies* (1) *α is a constant and* (2) $\alpha\beta = 0$.

(i) *In case $\alpha = 0 : \epsilon\zeta[\beta] + \beta^2 = f_3 - \epsilon f_1$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu space form.*

(ii) *In case $\alpha \neq 0 : \alpha^2 = \epsilon f_1 - f_3$, $\beta = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form.*

4. Additional results

Definition. We say that M is *screen totally umbilical* [4] if there exist a smooth function γ_i on a coordinate neighborhood $\mathcal{U}$ such that

$$h_i^*(X, PY) = \gamma_i g(X, PY), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.1}$$

In case $\gamma_i = 0$ on $\mathcal{U}$, we say that M is *screen totally geodesic.*

Definition. A lightlike submanifold M is said to be *irrotational* [21] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(Rad(TM))$ for all i.

For any r-lightlike submanifold M, the above definition is equivalent to

$$h_j^\ell(X, \xi_i) = 0, \quad h_\alpha^s(X, \xi_i) = \phi_{\alpha i}(X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.2}$$

Theorem 4.1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting an irrotational screen totally umbilical generic lightlike submanifold is a semi-Euclidean space, i.e., $\bar{M}(f_1, f_2, f_3)$ satisfies $f_1 = f_2 = f_3 = 0$.*

Proof. As M is screen totally umbilical, from $(3.11)_3$ and (4.1) we have

$$\gamma_i \theta(X) = \beta\eta_i(X) - \alpha v_i(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = \zeta$, $X = V_i$ and $X = \xi_i$ by turns, we have $\gamma_i = 0$, $\alpha = 0$ and $\beta = 0$, respectively. Thus M is screen totally geodesic and $\bar{M}(f_1, f_2, f_3)$ is an indefinite cosymplectic manifold. As $h_i^* = 0$, (3.20) is reduce to

$$\begin{aligned}
& f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
& + f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\} \\
& + f_3\{\theta(X)\theta(PZ)\eta_i(Y) - \theta(Y)\theta(PZ)\eta_i(X)\} \\
= & \sum_{a=r+1}^{n} \epsilon_a\{h_a^s(X, PZ)\rho_{ia}(Y) - h_a^s(Y, PZ)\rho_{ia}(X)\} \\
& + \sum_{j=1}^{r}\{h_j^\ell(X, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y, PZ)\eta_i(A_{N_j}X)\}
\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Y by ξ_i to this and using (4.2), we get

$$-f_1 g(X,PZ) - f_2\{v_i(X)u_i(PZ) + 2u_i(X)v_i(PZ)\} + f_3\theta(X)\theta(PZ) = \sum_{a=r+1}^{n} \epsilon_a h_a^s(X,PZ)\rho_{ia}(\xi_i) + \sum_{j=1}^{r} h_j^\ell(X,PZ)\eta_i(A_{N_j}\xi_i) \tag{4.3}$$

for all $X, Z \in \Gamma(TM)$. Taking $X = PZ = \zeta$ to this and using the fact $h_j^\ell(\zeta,\zeta) = h_a^s(\zeta,\zeta) = 0$ due to $(3.11)_{1,2}$, we have $\epsilon f_1 = f_3$.

Taking $X = V_k$, $PZ = U_k$ and $X = U_k$, $PZ = V_k$ to (4.3) by turns, we get

$$\sum_{a=r+1}^{n} \epsilon_a h_a^s(V_k,U_k)\rho_{ia}(\xi_i) + \sum_{j=1}^{r} h_j^\ell(V_k,U_k)\eta_i(A_{N_j}\xi_i) = -f_1 - f_2,$$

$$\sum_{a=r+1}^{n} \epsilon_a h_a^s(U_k,V_k)\rho_{ia}(\xi_i) + \sum_{j=1}^{r} h_j^\ell(U_k,V_k)\eta_i(A_{N_j}\xi_i) = -f_1 - 2f_2.$$

From these two equations we show that $f_2 = 0$. As $\bar{M}$ is an indefinite cosymplectic manifold, we have $f_1 = f_2 = f_3 = \frac{c}{4}$ by Example 2.3. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1,f_2,f_3)$ is a semi-Euclidean space. □

Definition. A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is called *screen conformal* if the second fundamental forms h_i^* of $S(TM)$ are conformally related to the corresponding fundamental forms h_i^ℓ of M by

$$h_i^*(X,PY) = \varphi_i h_i^\ell(X,PY), \quad \forall\, X, Y \in \Gamma(TM),\ i \in \{1, \ldots, r\}, \tag{4.4}$$

where φ_is are non-vanishing smooth functions on $\mathcal{U}$ in M.

Theorem 4.2. *Any indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ admitting an irrotational screen conformal generic lightlike submanifold is a semi-Euclidean space, i.e., $\bar{M}(f_1,f_2,f_3)$ satisfies $f_1 = f_2 = f_3 = 0$.*

Proof. As M is screen conformal, from $(3.11)_{1,3}$ and (4.4) we have

$$\alpha v_i(X) - \beta\eta_i(X) = \alpha\varphi_i u_i(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = V_i$ and $X = N_i$ by turns, we have $\alpha = 0$ and $\beta = 0$, respectively. Thus $\bar{M}(f_1,f_2,f_3)$ is an indefinite cosymplectic manifold.

Substituting (4.4) into (3.20) and using (3.14), we have

$$\begin{aligned} &f_1\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\ &+ f_2\{[v_i(Y) - \varphi_i u_i(Y)]\bar{g}(X,JPZ) - [v_i(X) - \varphi_i u_i(X)]\bar{g}(Y,JPZ) \\ &+ 2[v_i(PZ) - \varphi_i u_i(PZ)]\bar{g}(X,JY)\} \\ &+ f_3\{\theta(X)\theta(PZ)\eta_i(Y) - \theta(Y)\theta(PZ)\eta_i(X)\} \\ = &\ X[\varphi_i]h_i^\ell(Y,PZ) - Y[\varphi_i]h_i^\ell(X,PZ) \\ &+ \sum_{j=1}^{r}\{\varphi_i\tau_{ij}(Y) + \varphi_i\tau_{ji}(Y) + \eta_i(A_{N_j}Y)\}h_j^\ell(X,PZ) \end{aligned}$$

$$- \sum_{j=1}^{r} \{\varphi_i \tau_{ij}(X) + \varphi_i \tau_{ji}(X) + \eta_i(A_{N_j} X)\} h_j^\ell(Y, PZ)$$
$$+ \sum_{a=r+1}^{n} \{\epsilon_a \rho_{ia}(Y) + \varphi_i \phi_{ai}(Y)\} h_a^s(X, PZ)$$
$$- \sum_{a=r+1}^{n} \{\epsilon_a \rho_{ia}(X) + \varphi_i \phi_{ai}(X)\} h_a^s(Y, PZ)$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Y by ξ_i to this and using (4.2), we get

$$\begin{aligned} &f_1 g(X, PZ) + f_2\{v_i(X) - \varphi_i u_i(X)\} u_i(PZ) \\ &\quad + 2\{v_i(PZ) - \varphi_i u_i(PZ)\} u_i(X) - f_3 \theta(X)\theta(PZ) \\ &= \xi_i[\varphi_i] h_i^\ell(X, PZ) - \sum_{a=r+1}^{n} \epsilon_a \rho_{ia}(\xi_i) h_a^s(X, PZ) \\ &\quad - \sum_{j=1}^{r} \{\varphi_i \tau_{ij}(\xi_i) + \varphi_i \tau_{ji}(\xi_i) + \eta_i(A_{N_j} \xi_i)\} h_j^\ell(X, PZ) \end{aligned} \tag{4.5}$$

for all $X, Z \in \Gamma(TM)$. Taking $X = PZ = \zeta$ to this equation and using the fact $h_j^\ell(\zeta, \zeta) = h_a^s(\zeta, \zeta) = 0$, we have $\epsilon f_1 = f_3$. Taking $X = V_k$, $PZ = U_k$ and $X = U_k$, $PZ = V_k$ to (4.5) by turns, and then, comparing these resulting two equations, we obtain $f_2 = 0$.

As $\bar{M}$ is an indefinite cosymplectic manifold, we see that $f_1 = f_2 = f_3 = \frac{c}{4}$ by Example 2.3. Thus we have $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is a semi-Euclidean space. □

Definition. An r-lightlike submanifold M of $\bar{M}$ is said to be *totally umbilical* [4] if there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X, Y) = \mathcal{H}\, g(X, Y), \ \forall\, X, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$, we say that M is *totally geodesic*.

It is easy to see [4] that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions A_i and B_α such that

$$h_i^\ell(X, Y) = A_i g(X, Y), \ h_a^s(X, Y) = B_a g(X, Y), \ \forall\, X, Y \in \Gamma(TM). \tag{4.6}$$

Theorem 4.3. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a totally umbilical generic lightlike submanifold is an indefinite β-Kenmotsu space form. In this case M is totally geodesic.*

Proof. From $(3.11)_{1,2}$ and (4.6), we have

$$A_i \theta(X) = -\alpha u_i(X), \quad B_a \theta(X) = -\alpha e_a(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = \zeta$ and $X = U_i$ or E_a by turns, we get $A_i = B_a = 0$ and $\alpha = 0$, respectively. Thus $\bar{M}$ is an indefinite β-Kenmotsu space form and M is totally geodesic. □

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