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GLOBAL EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR COUPLED NONLINEAR WAVE EQUATIONS WITH DAMPING AND SOURCE TERMS

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Abstract. The initial-boundary value problem for a class of nonlinear higher-order wave equations system with a damping and source terms in bounded domain is studied. We prove the existence of global solutions. Meanwhile, under the condition of the positive initial energy, it is showed that the solutions blow up in the finite time and the lifespan estimate of solutions is also given.

1. Introduction

In this paper, we are concerned with the following initial-boundary value problem for the systems of higher-order nonlinear hyperbolic equations:

$$u_{tt} + A_1 u + a|u_t|^{r_1-2}u_t = g_1(u,v), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$v_{tt} + A_2 v + a|v_t|^{r_2-2}v_t = g_2(u,v), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x) \in H_0^{m_1}(\Omega),\ u_t(x,0) = u_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x) \in H_0^{m_2}(\Omega), \quad v_t(x,0) = v_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.4}$$

$$\frac{\partial^i}{\partial \nu^i} u(x,t) = 0,\ i = 0,1,2,\ldots,m_1-1,\ x \in \partial\Omega,\ t \ge 0, \tag{1.5}$$

$$\frac{\partial^j}{\partial \nu^j} v(x,t) = 0,\ j = 0,1,2,\ldots,m_2-1,\ x \in \partial\Omega,\ t \ge 0, \tag{1.6}$$

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where $A_i = (-\Delta)^{m_i}$, $m_i \geq 1$ $(i = 1, 2)$ are natural numbers, $a > 0$ and $r_i \geq 2$ $(i = 1, 2)$ are real numbers, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied, ν is unit outward normal on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the i-order normal derivation of u. $g_i(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are given functions to be determined later.

For the initial-boundary value problem of a single higher-order nonlinear hyperbolic equation

$$u_{tt} + (-\Delta)^m u + a|u_t|^{r-2}u_t = b|u|^{p-2}u, \; x \in \Omega, \; t > 0, \tag{1.6}$$

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; x \in \Omega, \tag{1.7}$$

$$\frac{\partial^i u}{\partial \nu^i} = 0, \; i = 0, 1, 2, \ldots, m-1, \; x \in \partial\Omega, \; t \geq 0. \tag{1.8}$$

As for $a = 0$, P. Brenner and W. Von Wahl [4] proved the existence and uniqueness of classical solutions to (1.6)-(1.8) in Hilbert space. H. Pecher [9] investigated the existence and uniqueness of Cauchy problem for the equation (1.6) by using the potential well method due to L. Payne and D. H. Sattinger [8] and D. H. Sattinger [11]. Meanwhile, B. X. Wang [13] showed that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s. C. X. Miao [7] obtained the scattering theory at low energy using time-space estimates and nonlinear estimates, and he also gave the global existence and uniqueness of solutions under the condition of low energy.

Quite recently, Y. J. Ye [15] dealt with the existence and asymptotic behavior of global solutions for (1.6)-(1.8). In [2], A. B. Aliev and B. H. Lichaei consider the Cauchy problem of the equation (1.6), and they found the existence and nonexistence criteria of global solutions using the $L^p - L^q$ estimate for the corresponding linear problem and also established the asymptotic behavior of solutions and their derivatives as $t \to +\infty$.

In the case of $m_i = 1$ $(i = 1, 2)$, (1.1)-(1.5) becomes the initial-boundary value problem of the system of wave equations. K. Agre and M. A. Rammaha [3] studied the following system of wave equations:

$$u_{tt} - \Delta u + a|u_t|^{r_1-2}u_t = g_1(u, v), \quad (x, t) \in \Omega \times \mathbb{R}^+, \tag{1.9}$$

$$v_{tt} - \Delta v + a|v_t|^{r_2-2}v_t = g_2(u, v), \quad (x, t) \in \Omega \times \mathbb{R}^+, \tag{1.10}$$

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.11}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \tag{1.12}$$

$$u(x, t) = v(x, t) = 0, \quad x \in \partial\Omega, \; t \geq 0. \tag{1.13}$$

They prove, under some restrictions on the parameters and initial data, several results on local existence and global existence of a weak solution. Meanwhile, they also showed that any weak solution with negative initial energy blows up in finite time. Later, B. Said-Houari [10] investigates the blow-up property of solution for the problem (1.9)-(1.13) provided that the initial data are large enough and the initial energy is positive. This result extends a previous result in [3] to a large class of initial data.

For $m_i > 1$ $(i = 1, 2)$, A. B. Aliev and A. A. Kazimov [1] consider the Cauchy problem of equations (1.1) and (1.2). They obtain the existence and uniqueness of weak global solutions through the use of $L^p - L^q$ type estimate for the corresponding linear parts and also established the uniform decay rates of solutions and their derivatives.

Motivated by the above researches, in this paper, we prove the global existence of the problem (1.1)-(1.5) under the condition $p \le \frac{1}{2}\min\{r_1,\ r_2\}$, where p is refer to (2.1). Meanwhile, for the positive initial energy and $p > \frac{1}{2}\max\{r_1,\ r_2\}$, we give the blow-up result and obtain the lifespan estimates of solutions.

We adopt the usual notations and convention. Let $H^m(\Omega)$ denote the Sobolev space with the usual scalar products and norm. Moreover, $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_s$ the Lebesgue space $L^s(\Omega)$ norm and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, we write equivalent norm $\|D^m\cdot\|$ instead of $H_0^m(\Omega)$ norm $\|\cdot\|_{H_0^m(\Omega)}$, where D denotes the gradient operator, that is $Du = \nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})$, and $D^m u = \Delta^j u$ if $m = 2j$ and $D^m u = D\Delta^j u$ if $m = 2j + 1$. In addition, C_i $(i = 0, 1, 2, 3, \dots)$ denote various positive constants which depend on the known constants and may be different at each appearance.

This paper is organized as follows: In the next section, we give some preliminaries. In Section 3, we study the existence of global solutions for problem (1.1)-(1.5). The Section 4 is devote to the study of the blow-up result.

2. Preliminaries

Concerning the functions $g_1(u, v)$ and $g_2(u, v)$, we assume that

$$\begin{aligned} g_1(u,v) &= b_1|u+v|^{2(p-1)}(u+v) + b_2|u|^{p-2}u|v|^p, \\ g_2(u,v) &= b_1|u+v|^{2(p-1)}(u+v) + b_2|v|^{p-2}v|u|^p, \end{aligned} \tag{2.1}$$

where $b_1, b_2 > 0$ and $p > 1$ are constants.

It easy to see that

$$ug_1(u,v) + vg_2(u,v) = 2pG(u,v),\ \forall (u,v) \in \mathbb{R}^2, \tag{2.2}$$

where

$$G(u,v) = \frac{b_1}{2p}|u+v|^{2p} + \frac{b_2}{p}|uv|^p. \tag{2.3}$$

Moreover, a quick computation will show that there exist two positive constants C_0 and C_1 such that the following inequality holds (see [10])

$$\frac{C_0}{2p}(|u|^{2p} + |v|^{2p}) \le G(u,v) \le \frac{C_1}{2p}(|u|^{2p} + |v|^{2p}). \tag{2.4}$$

Now, we define the following energy function associated with a solution $[u, v]$ of the problem (1.1)-(1.5):

$$E(t) = \frac{1}{2}(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2) - \int_\Omega G(u, v)dx \tag{2.5}$$

for $[u, v] \in H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$, and

$$E(0) = \frac{1}{2}(\|u_1\|^2 + \|v_1\|^2 + \|D^{m_1}u_0\|^2 + \|D^{m_2}v_0\|^2) - \int_\Omega G(u_0, v_0)dx \tag{2.6}$$

is the initial total energy.

In order to prove our main result, we need the following lemmas.

Lemma 2.1. *Let s be a number with $2 \le s < +\infty$ if $n \le 2m$ and $2 \le s \le \frac{2n}{n-2m}$ if $n > 2m$. Then there is a constant C depending on Ω and s such that*

$$\|u\|_s \le C\|(-\Delta)^{\frac{m}{2}}u\|, \ \forall u \in H_0^m(\Omega).$$

Lemma 2.2 (Young inequality)**.** *Let X, Y and ε be positive constants and s, $\tau \ge 1$, $\frac{1}{s} + \frac{1}{\tau} = 1$. Then one has the inequality*

$$XY \le \frac{\varepsilon^s X^s}{s} + \frac{Y^\tau}{\tau\varepsilon^\tau}.$$

Lemma 2.3. *Let $[u, v]$ be a solution to the problem* (1.1)-(1.5)*. Then $E(t)$ is a non-increasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -a(\|u_t\|_{r_1}^{r_1} + \|v_t\|_{r_2}^{r_2}) \le 0. \tag{2.7}$$

Multiplying equation (1.1) by u_t and (1.2) by v_t, and integrating over $\Omega \times [0, t]$. Then, adding them together, and integrating by parts, we get

$$E(t) - E(0) = -a\int_0^t (\|u_t(s)\|_{r_1}^{r_1} + \|v_t(s)\|_{r_2}^{r_2})ds \tag{2.8}$$

for $t \ge 0$.

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (2.7) is satisfied.

The local existence and uniqueness of solutions for the problem (1.1)-(1.5) can be obtained by a similar way as done in [3, 5, 14]. The result reads as follows.

Theorem 2.1 (Local existence)**.** *Supposed that*

$$\begin{aligned} &1 < p < +\infty, \ n \le 2\min(m_1, m_2), \\ &1 < p \le \min\left(\frac{n}{n-2m_1}, \ \frac{n}{n-2m_2}\right), \ n > 2\max(m_1, m_2), \end{aligned} \tag{2.9}$$

and $[u_0, v_0] \in H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$. Then there exists $T > 0$ such that the problem (1.1)-(1.5) *has a unique local solution $[u(t), v(t)]$ which satisfies*

$$[u, v] \in C([0, T); \ H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)),$$

$$u_t \in C([0,T);\ L^2(\Omega)) \cap L^{r_1}(\Omega \times [0,T)),$$
$$v_t \in C([0,T);\ L^2(\Omega)) \cap L^{r_2}(\Omega \times [0,T)).$$

Moreover, at least one of the following statements holds true:
(1) $\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2 \to \infty$ *as* $t \to T^-$;
(2) $T = +\infty$.

3. Global solutions

The following theorem shows that the solution obtained in Theorem 2.1 is a global solution if $p \le \frac{1}{2}\min\{r_1,\ r_2\}$.

Theorem 3.1. *Assume that* (2.1)-(2.4) *and* (2.9) *hold and* $p \le \frac{1}{2}\min\{r_1,\ r_2\}$. *Then the local solutions* $[u(t), v(t)]$ *furnished in Theorem 2.1 are global solutions and* T *may be taken arbitrarily large.*

Proof. Let $[u, v]$ be a weak solution to the initial-boundary value problem (1.1)-(1.5) defined on $[0, T]$ as furnished by Theorem 2.1. We define

$$E_1(t) = \frac{1}{2}(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2), \tag{3.1}$$

and

$$E_2(t) = E(t) + 2\int_\Omega G(u(t), v(t))dx, \tag{3.2}$$

where $E(t)$ is defined by (2.5). Then we easily see from (3.1) and (3.2) that

$$E_1(t) \le E_2(t). \tag{3.3}$$

Our aim is to prove the following inequality holds for all $t \in [0, T]$.

$$\begin{aligned}&\frac{1}{2}(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}u\|^2)\\ &+ \int_\Omega G(u(t), v(t))dx + a\int_0^t (\|u_t(s)\|_{r_1}^{r_1} + \|v_t(s)\|_{r_2}^{r_2}) \le C_T,\end{aligned} \tag{3.4}$$

where C_T depends on $\|D^{m_1}u_0\|,\ \|D^{m_2}v_0\|,\ \|u_1\|,\ \|v_1\|$ and $T > 0$ is arbitrary. We have from (2.4), Lemma 2.1, (3.1) and (3.3) that

$$\begin{aligned}&\frac{C_0}{2p}(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) \le \int_\Omega G(u, v) \le \frac{C_1}{2p}(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p})\\ &\le \frac{B^{2p}C_1}{2p}(\|D^{m_1}u\|^{2p} + \|D^{m_2}v\|^{2p}) \le C_2E_1(t)^p \le C_2E_2(t)^p,\end{aligned} \tag{3.5}$$

where $C_2 = \frac{2^{p-1}B^{2p}C_1}{p}$ and $B = \max(B_1, B_2)$, and B_i $(i = 1, 2)$ is the optimal Sobolev's constant from $H_0^{m_i}(\Omega)$ $(i = 1, 2)$ to $L^{2p}(\Omega)$.

By (3.2), we see that

$$\frac{C_0}{2p}(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) \le \int_\Omega G(u, v) \le E_2(t). \tag{3.6}$$

Put $Q_t = \Omega \times [0,t]$, then it yields from the energy identity (2.8) that

$$E_2(t) + a\int_0^t (\|u_t(s)\|_{r_1}^{r_1} + \|v_t(s)\|_{r_2}^{r_2})ds = E_2(0) + 2\int_{Q_t} \frac{\partial}{\partial s} G(u(s), v(s))dxds. \tag{3.7}$$

In order to estimate the last term in (3.7), we set

$$\begin{aligned} Q_{11} &:= \{(x,s) \in Q_t :\ |u(x,s)| \ge 1,\ |v(x,s)| \ge 1\},\\ Q_{12} &:= \{(x,s) \in Q_t :\ |u(x,s)| \ge 1,\ |v(x,s)| \le 1\},\\ Q_{21} &:= \{(x,s) \in Q_t :\ |u(x,s)| \le 1,\ |v(x,s)| \ge 1\},\\ Q_{22} &:= \{(x,s) \in Q_t :\ |u(x,s)| \le 1,\ |v(x,s)| \le 1\}. \end{aligned}$$

From (2.1) and (2.3) we conclude that

$$\begin{aligned} &\left|2\int_{Q_t} \frac{\partial}{\partial s} G(u(s), v(s))dxds\right|\\ &= 2\left|\int_{Q_t} \frac{\partial G}{\partial u} u_t(s) + \frac{\partial G}{\partial v} v_t(s)dxds\right|\\ &\le 2\int_{Q_t} (\left|\frac{\partial G}{\partial u}\right||u_t(s)| + \left|\frac{\partial G}{\partial v}\right||v_t(s)|)dxds \le C_3(I(t)+J(t)), \end{aligned} \tag{3.8}$$

where $C_3 = 2\max\{2^{2(p-1)}b_1,\ b_2\}$ and

$$I(t) = \int_{Q_t} (|u|^{2p-1} + |v|^{2p-1} + |u|^{p-1}|v|^p)|u_t|dxds, \tag{3.9}$$

$$J(t) = \int_{Q_t} (|u|^{2p-1} + |v|^{2p-1} + |v|^{p-1}|u|^p)|v_t|dxds. \tag{3.10}$$

In order to estimate $I(t)$ and $J(t)$, we write

$$I(t) = I_{11} + I_{12} + I_{21} + I_{22},\ J(t) = J_{11} + J_{12} + J_{21} + J_{22}, \tag{3.11}$$

where

$$I_{ij} = \int_{Q_{ij}} (|u|^{2p-1} + |v|^{2p-1} + |u|^{p-1}|v|^p)|u_t|dxds,\ i,j = 1,2, \tag{3.12}$$

$$J_{ij} = \int_{Q_{ij}} (|u|^{2p-1} + |v|^{2p-1} + |v|^{p-1}|u|^p)|v_t|dxds,\ i,j = 1,2. \tag{3.13}$$

We estimate $I_{11}(t)$ as follows: By noting $|u|, |v| \ge 1$ on Q_{11}, the first two terms in $I_{11}(t)$ are estimated in the same way. From $p \le \frac{r_1}{2}$, we see that $\eta = \frac{r_1 - 2p}{r_1} \ge 0$

and $(\eta+2p-1)\frac{r_1}{r_1-1}=2p$. We have from Lemma 2.2 and (3.6) that

(3.14)
$$\begin{aligned}\int_{Q_{11}}|u_t(s)||v(s)|^{2p-1}dxds&=\int_{Q_{11}}|u_t(s)||v(s)|^{\eta+2p-1}|v(s)|^{-\eta}dxds\\&\le\int_{Q_{11}}|u_t(s)||v(s)|^{\eta+2p-1}dxds\\&\le\varepsilon\int_{Q_{11}}|u_t(s)|^{r_1}dxds+C_\varepsilon\int_{Q_{11}}|v(s)|^{2p}dxds\\&\le\varepsilon\int_{Q_t}|u_t(s)|^{r_1}dxds+C_4\int_0^t E_2(s)ds,\end{aligned}$$

where $\varepsilon>0$ will be determined later and $C_4=\frac{2pC_\varepsilon}{C_0}$. Similarly, we have

$$\int_{Q_{11}}|u_t(s)||u(s)|^{2p-1}dxds\le\varepsilon\int_{Q_t}|u_t(s)|^{r_1}dxds+C_4\int_0^t E_2(s)ds. \tag{3.15}$$

For the last term in I_{11}, we get from Lemma 2.2 and (3.6) that

$$\begin{aligned}\int_{Q_{11}}|u|^{p-1}|v|^p|u_t|dxds&\le\frac{1}{2}\int_{Q_{11}}|v|^{2p}dxds+\frac{1}{2}\int_{Q_{11}}|u|^{2(p-1)}|u_t|^2dxds\\&\le\frac{p}{C_0}\int_0^t E_2(s)ds+\frac{1}{2}\int_{Q_t}|u|^{2(p-1)}|u_t|^2dxds.\end{aligned}\tag{3.16}$$

Let $\theta=\frac{2(r_1-2p)}{r_1}$, then we have from $p\le\frac{r_1}{2}$ that $\theta\ge0$ and $(\theta+2p-2)\frac{r_1}{r_1-2}=2p$. Therefore, we obtain from Lemma 2.2 and (3.6) that

$$\begin{aligned}&\frac{1}{2}\int_{Q_{11}}|u_t(s)|^2|u(s)|^{2p-2}dxds\\&=\frac{1}{2}\int_{Q_{11}}|u_t(s)|^2|u(s)|^{\theta+2p-2}|u(s)|^{-\theta}dxds\\&\le\frac{1}{2}\int_{Q_{11}}|u_t(s)|^2|u(s)|^{\theta+2p-2}dxds\\&\le\varepsilon\int_{Q_{11}}|u_t(s)|^{r_1}dxds+C_\varepsilon\int_{Q_{11}}|u(s)|^{2p}dxds\\&\le\varepsilon\int_{Q_t}|u_t(s)|^{r_1}dxds+C_5\int_0^t E_2(s)ds.\end{aligned}\tag{3.17}$$

We have from (3.14)-(3.17) that

$$I_{11}(t)\le3\varepsilon\int_{Q_t}|u_t(s)|^{r_1}dxds+C_6\int_0^t E_2(s)ds, \tag{3.18}$$

where $C_6=2C_4+C_5+\frac{p}{C_0}$. Similarly, we obtain

$$J_{11}(t)\le3\varepsilon\int_{Q_t}|v_t(s)|^{r_2}dxds+C_6\int_0^t E_2(s)ds, \tag{3.19}$$

For $I_{12}(t)$, by noting that $|u| \geq 1$ and $|v| \leq 1$ on Q_{12}, we have from Lemma 2.2 and (3.2) that

$$\begin{aligned} I_{12}(t) &\leq \int_{Q_{12}} (1+|u|^{2p-1}+|u|^{p-1})|u_t|dxds \\ &\leq \delta|Q_t| + C_\delta \int_{Q_{12}} |u_t(s)|^2 dxds + 2\int_{Q_{12}} |u(s)|^{2p-1}|u_t(s)|dxds \\ &\leq \delta|Q_t| + 2C_\delta \int_0^t E_2(s)ds + 2\int_{Q_{12}} |u(s)|^{2p-1}|u_t(s)|dxds, \end{aligned} \tag{3.20}$$

where $|Q_t|$ denotes the Lebesgue measure of Q_t.

We conclude from (3.15) and (3.20) that

$$I_{12}(t) \leq \delta|Q_t| + 2(C_4+C_\delta)\int_0^t E_2(s)ds + 2\varepsilon\int_{Q_t} |u_t(s)|^{r_1}dxds. \tag{3.21}$$

Likewise, we easily get

$$J_{12}(t) \leq \delta|Q_t| + 2(C_4+C_\delta)\int_0^t E_2(s)ds + 2\varepsilon\int_{Q_t} |v_t(s)|^{r_2}dxds. \tag{3.22}$$

Using the same way in (3.21) and (3.22), we have

$$I_{21}(t) \leq \delta|Q_t| + 2(C_4+C_\delta)\int_0^t E_2(s)ds + 2\varepsilon\int_{Q_t} |u_t(s)|^{r_1}dxds, \tag{3.23}$$

$$J_{21}(t) \leq \delta|Q_t| + 2(C_4+C_\delta)\int_0^t E_2(s)ds + 2\varepsilon\int_{Q_t} |v_t(s)|^{r_2}dxds. \tag{3.24}$$

For $I_{22}(t)$, we get from (3.3) and Lemma 2.2 that

$$\begin{aligned} I_{22}(t) &\leq 3\int_{Q_{22}} |u_t(s)|dxds \\ &\leq \delta|Q_t| + C_\delta\int_{Q_{22}} |u_t(s)|^2dxds \leq \delta|Q_t| + C_\delta\int_0^t E_2(s)ds \end{aligned} \tag{3.25}$$

for some $\delta > 0$. Similarly, we have

$$J_{22}(t) \leq \delta|Q_t| + C_\delta\int_{Q_{22}} |v_t(s)|^2dxds \leq \delta|Q_t| + C_\delta\int_0^t E_2(s)ds. \tag{3.26}$$

Combining (3.18)-(3.19), (3.21)-(3.22) and (3.23)-(3.26), we have

$$I(t)+J(t) \leq 6\delta|Q_t| + 7\varepsilon\int_{Q_t} |u_t(s)|^{r_1}dxds + 7\varepsilon\int_{Q_t} |v_t(s)|^{r_2}dxds + C_7\int_0^t E_2(s)ds, \tag{3.27}$$

where $C_7 = 2(C_6+4C_4+5C_\delta)$.

Choosing $\varepsilon > 0$ small enough such that $\varepsilon < \frac{a}{7C_3}$, we have from (3.7), (3.8) and (3.27) that

(3.28)
$$E_2(t)+(a-7C_3\varepsilon)\int_0^t(\|u_t(s)\|_{r_1}^{r_1}+\|v_t(s)\|_{r_2}^{r_2})ds = E_2(0)+C_8|Q_t|+C_9\int_0^t E_2(s)ds,$$

where $C_8 = 6C_3\delta > 0$ and $C_9 = C_3C_7 > 0$.

It follows from Gronwall inequality and (3.28) that

$$E_2(t) \le (E_2(0) + C_8|Q_t|)e^{C_9 t}. \tag{3.29}$$

We have from (3.28) and (3.29) that

$$E_2(t)+(a-7C_3\varepsilon)\int_0^t(\|u_t(s)\|_{r_1}^{r_1}+\|v_t(s)\|_{r_2}^{r_2})ds \le (E_2(0)+C_8|Q_T|)e^{C_9T} \tag{3.30}$$

for all $0 < t \le T$, where T is arbitrary. Thus, (3.4) follows from (3.2) and (3.30). By standard continuation argument, the local solutions $[u(t), v(t)]$ obtained in Theorem 2.1 are global. This finishes the proof of Theorem 3.1. □

4. The result of blow-up

By Minkowski's inequality, Lemma 2.1 and (2.9), we get that

$$\|u+v\|_{2p}^2 \le 2(\|u\|_{2p}^2 + \|v\|_{2p}^2) \le 2B^2(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2). \tag{4.1}$$

Also, we have from Hölder inequality, Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}\|u(t)v(t)\|_p &\le \|u(t)\|_{2p}\cdot\|v(t)\|_{2p} \le \frac{1}{2}(\|u(t)\|_{2p}^2 + \|v(t)\|_{2p}^2)\\ &\le \frac{B^2}{2}(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2).\end{aligned} \tag{4.2}$$

We get from (2.3), (4.1) and (4.2) that

$$\int_\Omega G(u,v)dx \le \frac{C_{10}B^{2p}}{p}(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2)^p, \tag{4.3}$$

where $C_{10} = 2^{p-1}b_1 + \frac{b_2}{2^p}$.

Note that we have from (2.5) that

$$E(t) \ge \frac{1}{2}(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2) - \int_\Omega G(u,v)dx. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned}E(t) &\ge \frac{1}{2}(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2)\\ &\quad - \frac{C_{10}B^{2p}}{p}(\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2)^p\\ &= Q\big(\sqrt{\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2}\big),\end{aligned} \tag{4.5}$$

where

$$Q(\lambda) = \frac{1}{2}\lambda^2 - \frac{C_{10}B^{2p}}{p}\lambda^{2p}.$$

Therefore, we get that

$$Q'(\lambda) = \lambda - 2C_{10}B^{2p}\lambda^{2p-1}, \quad Q''(\lambda) = 1 - 2(2p-1)C_{10}B^{2p}\lambda^{2(p-1)}.$$

Let $Q'(\lambda) = 0$, which implies that $\lambda_1 = (\frac{1}{2C_{10}B^{2p}})^{\frac{1}{2(p-1)}}$. As $\lambda = \lambda_1$, an elementary calculation shows that $Q''(t) = -2p < 0$. Thus, $Q(\lambda)$ has the maximum at λ_1 and the maximum value is

$$d = Q(\lambda_1) = \left(\frac{1}{2} - \frac{1}{2p}\right)\left(\frac{1}{2C_2B^{2p}}\right)^{\frac{1}{p-1}} = \left(\frac{1}{2} - \frac{1}{2p}\right)\lambda_1^2. \tag{4.6}$$

Applying the idea of E. Vitillaro [12], we have the following lemma.

Lemma 4.1. *Let* $[u, v]$ *be a solution of* (1.1)-(1.5). *Assume that* (2.9) *holds. If* $0 < E(0) < d$ *and* $\|D^{m_1}u_0\|^2 + \|D^{m_2}v_0\|^2 > \lambda_1^2$, *then there exists* $\lambda_2 > \lambda_1$ *such that*

$$\|D^{m_1}u(t)\|^2 + \|D^{m_2}v(t)\|^2 \geq \lambda_2^2 \tag{4.7}$$

and

$$\int_\Omega G(u, v)dx \geq \frac{C_{10}B^{2p}}{p}\lambda_2^{2p} \tag{4.8}$$

for $t \geq 0$.

The detail proof of Lemma 4.1 see [10].

Theorem 4.1. *Assume that* (2.9) *holds and that* $p > \frac{1}{2}\max\{r_1, r_2\}$. *If* $[u_0, v_0] \in H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$, *then any solution of* (1.1)-(1.5) *with initial data satisfying* $0 < E(0) < d$ *and* $\|D^{m_1}u_0\|^2 + \|D^{m_2}v_0\|^2 > \lambda_1^2$ *blows up at a finite time.*

Proof. Let

$$H(t) = d - E(t), \ t \geq 0. \tag{4.9}$$

We see from (2.7) in Lemma 2.3 that $H'(t) \geq 0$. Thus we obtain from (2.5) and (4.9) that

$$\begin{aligned} &0 < H(0) \leq H(t) \\ &= d - \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) + \int_\Omega G(u, v)dx. \end{aligned} \tag{4.10}$$

We obtain from (4.6) and (4.7) that

$$d - \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) < d - \frac{1}{2}\lambda_1^2 = -\frac{1}{2p}\lambda_1^2 < 0.$$

Therefore, we have from (4.10) that

$$0 < H(0) \leq H(t) \leq \int_\Omega G(u, v)dx. \tag{4.11}$$

Now, we define $L(t)$ as follows.

$$L(t) = H(t)^{1-\alpha} + \delta \int_\Omega (uu_t + vv_t)dx, \ \forall t \geq 0, \tag{4.12}$$

for δ small to be chosen later and

$$0 < \alpha \leq \left\{ \frac{p-1}{2p}, \ \frac{2p-r_1}{2p(r_1-1)}, \ \frac{2p-r_2}{2p(r_2-1)} \right\}. \tag{4.13}$$

By differentiating both sides of (4.13) on t, we get from (1.1) and (1.2) that

$$\begin{aligned} L'(t) = {} & (1-\alpha)H(t)^{-\alpha}H'(t) + \delta(\|u_t\|^2 + \|v_t\|^2) \\ & - \delta(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) + 2p\delta \int_\Omega G(u,v)dx \\ & - a\delta \int_\Omega (|u_t|^{r_1-2}u_t u + |v_t|^{r_2-2}v_t v)dx. \end{aligned} \tag{4.14}$$

By exploiting (2.5) and (4.9), the equation (4.14) takes the following form

$$\begin{aligned} L'(t) = {} & (1-\alpha)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) \\ & + 2\delta H(t) - 2\delta d + 2\delta(p-1) \int_\Omega G(u,v)dx \\ & - a\delta \int_\Omega (|u_t|^{r_1-2}u_t u + |v_t|^{r_2-2}v_t v)dx. \end{aligned} \tag{4.15}$$

We obtain from (4.8) and (4.15) that

$$\begin{aligned} L'(t) \geq {} & (1-\alpha)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) + 2\delta H(t) \\ & + C_{11}\delta \int_\Omega G(u,v)dx - a\delta \int_\Omega (|u_t|^{r_1-2}u_t u + |v_t|^{r_2-2}v_t v)dx, \end{aligned} \tag{4.16}$$

where $C_{11} = 2(p-1-\frac{dp}{C_{10}(B\lambda_2)^{2p}})$. By (4.6) and $\lambda_2 > \lambda_1$, we see that $C_{11} > 0$. We get from Lemma 2.2 that

$$a\left|\int_\Omega |u_t|^{r_1-2}u_t u dx\right| \leq \frac{\sigma_1^{r_1}}{r_1}\|u\|_{r_1}^{r_1} + \frac{r_1-1}{r_1}\sigma_1^{-r_1/(r_1-1)}\|u_t\|_{r_1}^{r_1}, \ \forall \sigma_1 > 0, \tag{4.17}$$

and

$$a\left|\int_\Omega |v_t|^{r_2-2}v_t v dx\right| \leq \frac{\sigma_2^{r_2}}{r_2}\|v\|_{r_2}^{r_2} + \frac{r_2-1}{r_2}\sigma_2^{-r_2/(r_2-1)}\|v_t\|_{r_2}^{r_2}, \ \forall \sigma_2 > 0. \tag{4.18}$$

It follows from (4.16)-(4.18) that

$$\begin{aligned} L'(t) \geq {} & (1-\alpha)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) + 2\delta H(t) \\ & + C_{11}\delta \int_\Omega G(u,v)dx - \delta\frac{\sigma_1^{r_1}}{r_1}\|u\|_{r_1}^{r_1} - \delta\frac{r_1-1}{r_1}\sigma_1^{-r_1/(r_1-1)}\|u_t\|_{r_1}^{r_1} \\ & - \delta\frac{\sigma_2^{r_2}}{r_2}\|v\|_{r_2}^{r_2} - \delta\frac{r_2-1}{r_2}\sigma_2^{-r_2/(r_2-1)}\|v_t\|_{r_2}^{r_2}. \end{aligned} \tag{4.19}$$

Choosing σ_1 and σ_2 such that

$$\sigma_1^{-r_1/(r_1-1)} = \Pi_1 H(t)^{-\alpha}, \quad \sigma_2^{-r_2/(r_2-1)} = \Pi_2 H(t)^{-\alpha}, \tag{4.20}$$

where Π_1 and Π_2 are large constants to be fixed latter. Hence, we have from (2.4), (4.19) and (4.20) that

$$\begin{aligned} L'(t) \geq\ & (1-\alpha-\Pi\delta)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) + 2\delta H(t) \\ & + C_{12}\delta(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) - \frac{\delta}{r_1}\Pi_1^{-(r_1-1)}H(t)^{\alpha(r_1-1)}\|u\|_{r_1}^{r_1} \\ & - \frac{\delta}{r_2}\Pi_2^{-(r_2-1)}H(t)^{\alpha(r_2-1)}\|v\|_{r_2}^{r_2}, \end{aligned} \tag{4.21}$$

where $\Pi = (r_1-1)/r_1\Pi_1 + (r_2-1)/r_2\Pi_2$ and $C_{12} = \frac{C_{11}C_0}{2p}$ is a positive constant. By $p > \frac{1}{2}\max\{r_1,\ r_2\}$, we obtain from (2.4) and (4.11) that

$$H(t)^{\alpha(r_1-1)}\|u\|_{r_1}^{r_1} \leq C_{13}(\|u\|_{2p}^{2\alpha p(r_1-1)+r_1} + \|v\|_{2p}^{2\alpha p(r_1-1)}\|u\|_{r_1}^{r_1}), \tag{4.22}$$

and

$$H(t)^{\alpha(r_2-1)}\|v\|_{r_2}^{r_2} \leq C_{13}(\|v\|_{2p}^{2\alpha p(r_2-1)+r_2} + \|u\|_{2p}^{2\alpha p(r_2-1)}\|v\|_{r_2}^{r_2}). \tag{4.23}$$

From (4.13) and the following algebraic inequality

$$z^\mu \leq z+1 \leq (1+\frac{1}{k})(z+k),\ \forall z\geq 0,\ 0<\mu\leq 1,\ k\geq 0, \tag{4.24}$$

we have

$$\|u\|_{2p}^{2\alpha p(r_1-1)+r_1} \leq \beta(\|u\|_{2p}^{2p} + H(0)) \leq \beta(\|u\|_{2p}^{2p} + H(t)), \tag{4.25}$$

where $\beta = 1 + 1/H(0)$. Similarly,

$$\|v\|_{2p}^{2\alpha p(r_2-1)+r_2} \leq \beta(\|v\|_{2p}^{2p} + H(t)). \tag{4.26}$$

Also,using the inequality $(X+Y)^s \leq C(X^s+Y^s)$, $X, Y \geq 0$, $s > 0$, we conclude from (4.13) and (4.24) that

$$\|v\|_{2p}^{2\alpha p(r_1-1)}\|u\|_{r_1}^{r_1} \leq C_{14}(\|v\|_{2p}^{2p} + \|u\|_{r_1}^{2p}) \leq C_{15}(\|v\|_{2p}^{2p} + \|u\|_{2p}^{2p}), \tag{4.27}$$

$$\|u\|_{2p}^{2\alpha p(r_2-1)}\|v\|_{r_2}^{r_2} \leq C_{14}(\|u\|_{2p}^{2p} + \|v\|_{r_2}^{2p}) \leq C_{15}(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}). \tag{4.28}$$

Combining (4.21)-(4.28), we get

$$\begin{aligned} L'(t) \geq\ & (1-\alpha-\Pi\delta)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) + 2\delta H(t) \\ & + \delta(C_{12} - C_{16}\Pi_1^{-(r_1-1)} - C_{17}\Pi_2^{-(r_2-1)})(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) \\ & + \delta(2 - C_{18}\Pi_1^{-(r_1-1)} - C_{19}\Pi_2^{-(r_2-1)})H(t). \end{aligned} \tag{4.29}$$

For large values of Π_1 and Π_2, there exist positive constants Θ_1 and Θ_2 such that (4.29) becomes

$$\begin{aligned} L'(t) \geq\ & (1-\alpha-\Pi\delta)H(t)^{-\alpha}H'(t) + 2\delta(\|u_t\|^2 + \|v_t\|^2) \\ & + \delta\Theta_1(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}) + \delta\Theta_2 H(t). \end{aligned} \tag{4.30}$$

Once Π_1 and Π_2 are fixed, we pick δ small enough such that $1-\alpha-\Pi\delta \geq 0$ and

$$L(0) = H(0)^{1-\alpha} + \delta \int_\Omega (u_0u_1 + v_0v_1)dx > 0.$$

Then, we have from (4.30) that

$$L'(t) \geq C_{20}\delta[\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2p}^{2p} + \|v\|_{2p}^{2p} + H(t)]. \tag{4.31}$$

Therefore, $L(t)$ is a nondecreasing function for $t \geq 0$, then we obtain that $L(t) \geq L(0) > 0$ for $t \geq 0$.

Since $0 < \alpha < 1$, it is evident that $\frac{1}{1-\alpha} > 1$. We deduce from (4.12) that

$$L(t)^{\frac{1}{1-\alpha}} \leq C_{21}\left[H(t) + \left(\int_\Omega (uu_t + vv_t)dx\right)^{\frac{1}{1-\alpha}}\right]. \tag{4.32}$$

On the other hand, for $p > 1$, we have from Hölder inequality and Lemma 2.2 that

(4.33)

$$\begin{aligned}\left(\int_\Omega (uu_t + vv_t)dx\right)^{\frac{1}{1-\alpha}} &\leq C_{22}\left(\|u_t\|^{\frac{1}{1-\alpha}}\|u\|_{2p}^{\frac{1}{1-\alpha}} + \|v_t\|^{\frac{1}{1-\alpha}}\|v\|_{2p}^{\frac{1}{1-\alpha}}\right)\\ &\leq C_{23}\left(\|u\|_{2p}^{\frac{\rho}{1-\alpha}} + \|v\|_{2p}^{\frac{\rho}{1-\alpha}} + \|u_t\|^{\frac{\nu}{1-\alpha}} + \|v_t\|^{\frac{\nu}{1-\alpha}}\right),\end{aligned}$$

where $\frac{1}{\rho} + \frac{1}{\nu} = 1$. We take $\nu = 2(1-\alpha)$, then $\frac{\rho}{1-\alpha} = \frac{2}{1-2\alpha}$. It follows from (4.13) and (4.24) that

$$\|u\|_{2p}^{\frac{\rho}{1-\alpha}} = \|u\|_{2p}^{\frac{2}{1-2\alpha}} \leq \beta(\|u\|_{2p}^{2p} + H(t)), \tag{4.34}$$

and

$$\|v\|_{2p}^{\frac{\rho}{1-\alpha}} = \|v\|_{2p}^{\frac{2}{1-2\alpha}} \leq \beta(\|v\|_{2p}^{2p} + H(t)). \tag{4.35}$$

We obtain from (4.33)-(4.35) that

$$\left(\int_\Omega (uu_t + vv_t)dx\right)^{\frac{1}{1-\alpha}} \leq C_{24}[\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2p}^{2p} + \|v\|_{2p}^{2p} + H(t)]. \tag{4.36}$$

Combining (4.32) and (4.36), we find that

$$L(t)^{\frac{1}{1-\alpha}} \leq C_{25}\left[\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2p}^{2p} + \|v\|_{2p}^{2p} + H(t)\right]. \tag{4.37}$$

We have from (4.31) and (4.37) that

$$L'(t) \geq C_{26}L(t)^{\frac{1}{1-\alpha}},\ t \geq 0, \tag{4.38}$$

where $C_{26} = \frac{C_{20}\delta}{C_{25}}$. Integrating both sides of (4.38) over $[0,t]$ yields that

$$L(t) \geq \left(L(0)^{\frac{\alpha}{\alpha-1}} - \frac{C_{26}\alpha}{1-\alpha}t\right)^{-\frac{1-\alpha}{\alpha}}. \tag{4.39}$$

Noting that $L(0) > 0$, then there exists $T* = T_{\max} = \frac{(1-\alpha)L(0)^{\frac{\alpha}{\alpha-1}}}{C_{26}\alpha}$ such that $L(t) \to +\infty$ as $t \to T*$. Namely, the solutions of the problem (1.1)-(1.5) blow up in finite time. □

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