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COMMUTATIVE p-SCHUR RINGS OVER NON-ABELIAN GROUPS OF ORDER p^3

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Abstract. Recently, it was proved that every p-Schur ring over an abelian group of order p^3 is Schurian. In this paper, we prove that every commutative p-Schur ring over a non-abelian group of order p^3 is Schurian.

1. Introduction

Let H be a finite group. We denote by $\mathbb{C}H$ the group algebra of H over the complex number field $\mathbb{C}$. For a nonempty subset $T \subseteq H$, we set $\underline{T} := \sum_{t \in T} t$ which is treated as an element of $\mathbb{C}H$.

A subalgebra $\mathcal{A}$ of the group algebra $\mathbb{C}H$ is called a *Schur ring* over H if there exists a partition $\mathrm{Bsets}(\mathcal{A}) := \{T_0, T_1, \ldots, T_r\}$ of H satisfying the following conditions:

(i) $\{\underline{T_i} \mid T_i \in \mathrm{Bsets}(\mathcal{A})\}$ is a linear basis of $\mathcal{A}$;
(ii) $T_0 = \{1_H\}$;
(iii) $T_i^{-1} := \{t^{-1} \mid t \in T_i\} \in \mathrm{Bsets}(\mathcal{A})$ for all $T_i \in \mathrm{Bsets}(\mathcal{A})$.

A Schur ring $\mathcal{A}$ over a p-group H is called a *p-Schur ring* if the size of every element in $\mathrm{Bsets}(\mathcal{A})$ is a power of p, where p is a prime.

Let G be a subgroup of $\mathrm{Sym}(H)$ containing the left regular representation of H. We denote by $T_0 = \{1_H\}$, $T_1, \ldots, T_r$ the orbits of the stabilizer G_{1_H}. The *transitivity module* $V(H, G_{1_H})$ of G is the vector space spanned by $\{\underline{T_i} \mid 0 \le i \le r\}$. It was proved in [16] that $V(H, G_{1_H})$ is a Schur ring over H. Customarily, a Schur ring $\mathcal{A}$ over H is called *Schurian* if $\mathcal{A}$ is the transitivity module $V(H, G_{1_H})$ of some group G containing the left regular representation of H.

A family of Schur rings which are not Schurian was given in [16, Theorem 26.4]. It is known that every Schur ring over a cyclic p-group is Schurian (see [12]). In 1979, M. Klin conjectured that every Schur ring over a cyclic group is

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Schurian. But, it was proved in [3] that there exist non-Schurian Schur rings over cyclic groups.

In [15], Spiga and Wang proved that every p-Schur ring over an elementary abelian p-group of rank 3 is Schurian. Recently, Kim showed that every p-Schur ring over an abelian group of order p^3 is Schurian (see [10]). In this paper, we focus on p-Schur rings over non-abelian groups of order p^3. The following example is a non-Schurian 7-Schur ring over a non-abelian group of order 7^3. We conjecture that such examples can be constructed for each prime $p \geq 7$.

Example 1.1. Let $H = \langle a, b \mid a^{7^2} = b^7 = 1, ab = ba^8\rangle$ be a non-abelian group of order 7^3. Then a partition $\mathrm{Bsets}(\mathcal{A})$ of H determines a non-commutative 7-Schur ring which is not Schurian, where

$$\begin{aligned}\mathrm{Bsets}(\mathcal{A}) = \{\{l\} \mid l \in \langle a^7, b\rangle\} &\bigcup \{a\langle b\rangle (a^7)^i \mid 0 \leq i \leq 6\} \\ &\bigcup \{a^2\langle b(a^7)^2\rangle (a^7)^i \mid 0 \leq i \leq 6\} \bigcup \{a^3\langle b(a^7)^3\rangle (a^7)^i \mid 0 \leq i \leq 6\} \\ &\bigcup \{a^4\langle b(a^7)^6\rangle (a^7)^i \mid 0 \leq i \leq 6\} \bigcup \{a^5\langle b(a^7)^4\rangle (a^7)^i \mid 0 \leq i \leq 6\} \\ &\bigcup \{a^6\langle ba^7\rangle (a^7)^i \mid 0 \leq i \leq 6\}.\end{aligned}$$

So we restrict our attention on commutative p-Schur rings. The following is our main theorem.

Theorem 1.2. *Every commutative p-Schur ring over a non-abelian group of order p^3 is Schurian.*

Note that every 2-Schur ring over a group of order 8 is commutative and Schurian (see [6]).

This paper is organized as follows. In Section 2, we review notations and known facts about Schur rings. In Section 3, we give a proof of the main theorem.

2. Preliminaries

Throughout this paper, we use the notations given in [12].

Let $\mathcal{A}$ be a Schur ring over H. We say that a subgroup K of H is an $\mathcal{A}$-*subgroup* if $\underline{K} \in \mathcal{A}$. For each $\mathcal{A}$-subgroup E of H, one can define a subring $\mathcal{A}_E$ by setting $\mathcal{A}_E = \mathcal{A} \cap \mathbb{C}E$. It is easy to see that $\mathcal{A}_E$ is a Schur ring over E and $\mathrm{Bsets}(\mathcal{A}_E) = \{T \mid T \in \mathrm{Bsets}(\mathcal{A}), T \subseteq E\}$.

For a group H, we denote by R_H the set of all binary relations on H that invariant with respect to the left regular representation of H. Then the mapping

$$2^H \to R_H \ (T \mapsto R_H(T)),$$

where $R_H(T) = \{(h, ht) \mid h \in H, t \in T\}$, is a bijection. If $\mathcal{A}$ is a Schur ring over H, then the pair

$$\mathcal{C}(\mathcal{A}) = (H, R_H(\mathrm{Bsets}(\mathcal{A}))),$$

where $R_H(\mathrm{Bsets}(\mathcal{A}))=\{R_H(T) \mid T\in \mathrm{Bsets}(\mathcal{A})\}$, is called a *Cayley* (*association*) *scheme* over H. (See [18] for association schemes.)

Let $\mathcal{C} = (H, R)$ be a Cayley scheme. For each $r \in R$, we set $r(1_H) = \{h \in H \mid (1_H, h) \in r\}$. Then the vector space spanned by $\{\underline{r(1_H)} \mid r \in R\}$ is a Schur ring over H.

Theorem 2.1 ([11]). *The correspondence $\mathcal{A} \mapsto \mathcal{C}(\mathcal{A})$, $\mathcal{C}(\mathcal{A}) \mapsto \mathcal{A}$ induces a bijection between the Schur rings and Cayley schemes over the group H that preserves the natural partial orders on these sets.*

The following propositions are results in [16, 18].

Proposition 2.2. *Let $\mathcal{A}$ be a Schur ring over H. If $T \in \mathrm{Bsets}(\mathcal{A})$, then the stabilizer $\mathrm{St}(T) := \{h \in H \mid Th = T = hT\}$ is an $\mathcal{A}$-subgroup of H.*

Proposition 2.3. *Let $\mathcal{A}$ be a Schur ring over H and m an element of H. If $T, \{m\} \in \mathrm{Bsets}(\mathcal{A})$, then $Tm = \{tm \mid t \in T\}$ lies in $\mathrm{Bsets}(\mathcal{A})$.*

Proposition 2.4. *Let $\mathcal{A}$ be a p-Schur ring over a group H of order p^m. Then*

(i) *the group $\mathbf{O}_\theta(\mathcal{A}) := \{h \in H \mid \{h\} \in \mathrm{Bsets}(\mathcal{A})\}$ is a non-trivial $\mathcal{A}$-subgroup*;

(ii) *the group $\mathbf{O}^\theta(\mathcal{A}) := \langle\{T^{-1}T \mid T \in \mathrm{Bsets}(\mathcal{A})\}\rangle$ is a proper $\mathcal{A}$-subgroup*;

(iii) *there exists a series $H_0 = \{1_H\} < H_1 < \cdots < H_m = H$ of $\mathcal{A}$-subgroups such that $[H_{i+1} : H_i] = p$ for $i = 0, 1, \ldots, m-1$.*

Proposition 2.5 ([8]). *Let $\mathcal{A}$ be a Schur ring over an abelian group H of order p^m. If there exists $T \in \mathrm{Bsets}(\mathcal{A})$ with size p^{m-1}, then $\mathrm{Bsets}(\mathcal{A}) = \mathrm{Bsets}(\mathcal{A}_{\mathbf{O}^\theta(\mathcal{A})}) \cup \{T^{(i)} \mid 1 \le i \le p-1\}$, where $T^{(i)} = \{t^i \mid t \in T\}$.*

The following lemma follows straightforwardly from Propositions 2.4 and 2.5.

Lemma 2.6. *Let $\mathcal{A}$ be a p-Schur ring over a group H of order p^2. Then $\mathrm{Bsets}(\mathcal{A})$ is either $\{\{h\} \mid h \in H\}$ or $\{\{e\}, T \mid e \in E, T \in (H/E) \setminus \{E\}\}$ for some subgroup E of H.*

Lemma 2.7 ([9]). *Let $\mathcal{A}$ be a commutative p-Schur ring over a group H of order p^3 and L an $\mathcal{A}$-subgroup of order p^2. Then $\{|T| \mid T \in \mathrm{Bsets}(\mathcal{A})\backslash\mathrm{Bsets}(\mathcal{A}_L)\}$ is either $\{p\}$ or $\{p^2\}$.*

Let H be a group and L a subgroup of H. We denote by H/L the set of left cosets. For $h \in H$ we define a permutation h_R as follows:

$$h_R(x) = hx \text{ for each } x \in H.$$

For $h \in H$ and $e \in H/L$ we define a permutation h_e as follows:

$$h_e(x) = \begin{cases} h_R(x) & \text{if } x \in e, \\ x & \text{otherwise.} \end{cases}$$

A relative (m, n, k, λ)-difference set (RDS) in a finite group G of order mn relative to a subgroup N of order n is a k-subset R of G such that every element

$g \in G \setminus N$ has exactly λ representations $g = r_1 r_2^{-1}$ with $r_1, r_2 \in R$, and no non-identity element of N has such a representation.

Proposition 2.8 ([4, 7, 14])**.** *Let R be a $(p,p,p,1)$-RDS in G, where p is an odd prime. Then G is elementary abelian.*

A function $f : \mathbb{F}_p \to \mathbb{F}_p$ is called *planar* if $f(x+a) - f(x)$ is a permutation function of $\mathbb{F}_p$ for each $a \neq 0$. It is known that a planar function over $\mathbb{F}_p$ with odd prime p can be written as the form of a quadratic polynomial (see [4, 14]).

Proposition 2.9 ([13])**.** *A function f is planar if and only if the set $R = \{(x, f(x)) \in \mathbb{F}_p \times \mathbb{F}_p \mid x \in \mathbb{F}_p\}$ is a $(p,p,p,1)$-RDS in $\mathbb{F}_p \times \mathbb{F}_p$ relative to $\{0\} \times \mathbb{F}_p$.*

3. p-Schur rings over non-abelian groups of odd prime-cube order

Let p be an odd prime. It is well known that there are exactly two non-abelian groups of order p^3 up to isomorphism, namely

$$H_1 = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba^{p+1} \rangle \text{ and}$$
$$H_2 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1 \rangle.$$

Remark 3.1. (i) Every $\varphi \in \mathrm{Aut}(H_1)$ is a mapping defined by $a \mapsto a^i b^j$ and $b \mapsto a^{pm} b$, where $i \in \mathbb{Z}_{p^2}$, $i \not\equiv 0 \pmod{p}$ and $j, m \in \mathbb{Z}_p$ (see [5, Section 1.5.1]).

(ii) Every $\varphi \in \mathrm{Aut}(H_2)$ is a mapping defined by $a \mapsto a^i b^j c^k$, $b \mapsto a^l b^m c^n$ and $c \mapsto c^s$, where $i, j, k, l, m, n, s \in \mathbb{Z}_p$ and $s = im - jl \neq 0$ (see [5, Section 1.5.3]).

For the remainder of this section, we assume that $\mathcal{A}$ is a commutative p-Schur ring over H_i $(i = 1, 2)$. For convenience, we often omit the subindex i of H_i.

Lemma 3.2. *Let $\mathcal{A}$ be a commutative p-Schur ring over H. Then there exists a series of $\mathcal{A}$-subgroups of H. Moreover, by replacing the generators if necessary, it is one of the following types*:

(Type(1)) $\{1\} < \langle a^p \rangle < \langle a \rangle < H_1$,

(Type(2)) $\{1\} < \langle a^{ip} b^j \rangle < \langle a^p, b \rangle < H_1$,

(Type(3)) $\{1\} < \langle a^i c^j \rangle < \langle a, c \rangle < H_2$, *where $i, j \in \mathbb{Z}_p$.*

Proof. By Proposition 2.4(iii), there exists a series of $\mathcal{A}$-subgroups of H, i.e., $\{1\} < L < M < H$.

When $H = H_1$, M is either $\langle a^i b^j \rangle$ $(i \neq 0)$ or $\langle a^p, b \rangle$. If $M = \langle a^i b^j \rangle$, then replacing the generator $a^i b^j$ by a, we have Type(1). If $M = \langle a^p, b \rangle$, then we have Type(2).

When $H = H_2$, M is either $\langle a, c \rangle$ or $\langle b, c \rangle$. If $M = \langle b, c \rangle$, then using an automorphism of H $(b \mapsto a, a \mapsto b^{p-1}, c \mapsto c)$, we have Type(3). □

Lemma 3.3. *Let $\mathcal{A}$ be a commutative p-Schur ring over H. Suppose that there exists an element $T \in \mathrm{Bsets}(\mathcal{A})$ with size p^2. Then $\mathcal{A}$ is Schurian.*

Proof. By Proposition 2.4(iii), there exists an $\mathcal{A}$-subgroup L of order p^2. By Lemma 2.7, every element of $\mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_L)$ has size p^2.

First of all, we claim that each element of $\mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_L)$ belongs to $(H/L) \setminus \{L\}$. Suppose $hl_1, h^j l_2 \in T$, where $l_1, l_2 \in L, h \in H \setminus L$ and $j \neq 1$. Then $(hl_1)^{-1}h^j l_2 \in T^{-1}T$. By Proposition 2.4(ii), this is a contradiction.

By Lemma 2.6, we divide our consideration into two cases.

(i) $\mathrm{Bsets}(\mathcal{A}_L) = \{\{l\} \mid l \in L\}$.

Define a subgroup G of $\mathrm{Sym}(H)$ by $\langle l_f, h_R \mid l \in L, h \in H, f \in H/L\rangle$. Clearly, G contains the left regular representation of H. It is easy to see that, for given l_f, h_R, we have $h_R^{-1} l_f h_R = l'_{f'}$ for some $l' \in L, f' \in H/L$. So, we can check $G_1 = \langle l_f \mid l \in L, f \in (H/L) \setminus \{L\}\rangle$. Thus, the set of orbits of G_1 is $\mathrm{Bsets}(\mathcal{A})$.

(ii) $\mathrm{Bsets}(\mathcal{A}_L) = \{\{e\}, T \mid T \in (L/E) \setminus \{E\},\ e \in E\}$, where E is an $\mathcal{A}$-subgroup of order p.

For fixed $e \in E$, $f \in L \setminus E$ and $g \in H$, we set $x := (1\ e \cdots\ e^{p-1})$, $y := (1\ f \cdots\ f^{p-1})(e\ fe\ \cdots\ f^{p-1}e) \cdots (e^{p-1}\ fe^{p-1} \cdots f^{p-1}e^{p-1})$, $z := z_1 z_2 \cdots z_p$, where $z_1 = (1\ g \cdots\ g^{p-1})\ (f\ gf \cdots\ g^{p-1}f)\ (f^2\ gf^2 \cdots\ g^{p-1}f^2) \cdots (f^{p-1}\ gf^{p-1} \cdots\ g^{p-1}f^{p-1})$, $z_2 = (e\ ge\ \cdots\ g^{p-1}e)\ (fe\ gfe\ \cdots\ g^{p-1}fe)\ (f^2e\ gf^2e \cdots g^{p-1}f^2e) \cdots (f^{p-1}e\ gf^{p-1}e \cdots g^{p-1}f^{p-1}e), \ldots, z_p = (e^{p-1}\ ge^{p-1} \cdots\ g^{p-1}e^{p-1})\ (fe^{p-1}\ gfe^{p-1} \cdots\ g^{p-1}fe^{p-1}) \cdots (f^{p-1}e^{p-1}\ gf^{p-1}e^{p-1}\ \cdots\ g^{p-1}f^{p-1}e^{p-1})$.

It is known that $\langle x, y, z\rangle$ is a Sylow p-subgroup of $\mathrm{Sym}(H)$ (see Exercise 2.6.10 of [2]). This implies that $\mathrm{Bsets}(\mathcal{A})$ is the set of orbits of a Sylow p-subgroup of $\mathrm{Sym}(H)$. $\square$

By Lemma 3.3, from now on, we assume that every element of $\mathrm{Bsets}(\mathcal{A})$ has at most size p.

Lemma 3.4. *Let $\mathcal{A}$ be a commutative p-Schur ring over H such that $|\mathbf{O}_\theta(\mathcal{A})| = p$ and L an $\mathcal{A}$-subgroup of order p^2. If T is an element of* $\mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_L)$ *such that* $\mathrm{St}(T) = \{1\}$, *then* $\underline{T^{-1}} \cdot \underline{T} = p\underline{1} + \sum_{T' \in I} \underline{T'}$, *where* $I = \mathrm{Bsets}(\mathcal{A}_L) \setminus \{\{h\} \mid h \in \mathbf{O}_\theta(\mathcal{A})\}$.

Proof. Since $p \geq 3$, we have $T^{-1} \cap T = \emptyset$. By Proposition 2.4(ii), we have $\underline{T^{-1}} \cdot \underline{T} = p\underline{1} + \sum_{T' \in \mathrm{Bsets}(\mathcal{A}_L) \setminus \{\{1\}\}} c_{T'} \underline{T'}$. Since $\mathrm{St}(T) = \{1\}$, we have $c_{T'} = 0$ for each $T' \in \mathbf{O}_\theta(\mathcal{A}) \setminus \{1\}$. Thus, we have $\underline{T^{-1}} \cdot \underline{T} = p\underline{1} + \sum_{T' \in I} c_{T'} \underline{T'}$.

We claim that $c_{T'} = 1$ for each $T' \in I$.

First of all, we show that $T = \{bx_0, bx_1c, \ldots, bx_{p-1}c^{p-1}\}$, where $b \in H \setminus L$, $c \in L \setminus \mathbf{O}_\theta(\mathcal{A})$, $x_i \in \mathbf{O}_\theta(\mathcal{A})$. By Proposition 2.4(ii), all elements of T belong to a coset in H/L, i.e., $T = \{ba_0, ba_1, \ldots, ba_{p-1}\}$, where $b \in H \setminus L$, $a_i \in L$. Suppose that, for distinct i, j, a_i and a_j belong to the same coset in $L/\mathbf{O}_\theta(\mathcal{A})$. Then $a_i d = a_j$ for some $d \in \mathbf{O}_\theta(\mathcal{A})$. By Proposition 2.3, we have $Td = T$, a contradiction.

Next, we calculate $\underline{T^{-1}} \cdot \underline{T} = (x_0^{-1}b^{-1} + c^{-1}x_1^{-1}b^{-1} + \cdots + cx_{p-1}^{-1}b^{-1})(bx_0 + bx_1c + \cdots + bx_{p-1}c^{p-1})$. Using the fact that L is abelian, we can check that every element of I should appear in $\underline{T^{-1}} \cdot \underline{T}$.

Thus, the size of T implies that $c_{T'} = 1$ for each $T' \in I$. $\square$

Lemma 3.5. *Let $\mathcal{A}$ be a commutative p-Schur ring over H and L an $\mathcal{A}$-subgroup of order p^2. If there exists $T \in \mathrm{Bsets}(\mathcal{A}) \backslash \mathrm{Bsets}(\mathcal{A}_L)$ such that $\mathrm{St}(T) \neq \{1\}$, then $\mathbf{O}^\theta(\mathcal{A})$ is the center of H.*

Proof. We consider three types of $\mathcal{A}$-subgroup series. Fix an element $T \in \mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_L)$ such that $\mathrm{St}(T) \neq \{1\}$.

In Type(1), we have $\mathrm{St}(T) = \langle a^p \rangle$. We claim that, for each element of $\mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_L)$, its stabilizer is $\langle a^p \rangle$. Without loss of generality, we can assume $T = \langle a^p \rangle b$ by replacing the generators if necessary. Then we have $\underline{T} \cdot \underline{T} = p\underline{\langle a^p \rangle b^2}$. This implies $\mathbf{O}^\theta(\mathcal{A}) = \langle a^p \rangle$.

In Type(2), We claim $\mathrm{St}(T) = \langle a^p \rangle$. Suppose $\mathrm{St}(T) = \langle a^p b^j \rangle$ for some $j \neq 0$. Then we can put $T = \langle a^p b^j \rangle a$. Since $\mathcal{A}$ is commutative, we have $\underline{\langle a^p b^j \rangle a} \cdot a^p b^j \neq a^p b^j \cdot \underline{\langle a^p b^j \rangle a}$ by the direct computation. This is a contradiction. Thus, we have $\mathrm{St}(T) = \langle a^p \rangle$. This implies $\mathbf{O}^\theta(\mathcal{A}) = \langle a^p \rangle$.

Type(3) is similar to the second one. $\square$

Now we divide our consideration into cases depending on $|\mathbf{O}^\theta(\mathcal{A})|$. By Proposition 2.4(ii), we have $|\mathbf{O}^\theta(\mathcal{A})| = p$ or p^2.

Proposition 3.6. *If $\mathcal{A}$ is a commutative p-Schur ring over H satisfying one of the following conditions*:

(1) $|\mathbf{O}^\theta(\mathcal{A})| = p$,
(2) $|\mathbf{O}_\theta(\mathcal{A})| = |\mathbf{O}^\theta(\mathcal{A})| = p^2$,

then $\mathcal{A}$ is Schurian.

Proof. If $\mathcal{A}$ satisfies condition(1), then $\mathcal{A}$ is Schurian by the main theorem of [17].

If $\mathcal{A}$ satisfies condition(2), then $\mathbf{O}^\theta(\mathcal{A})$ is either cyclic or elementary abelian Suppose $\mathbf{O}^\theta(\mathcal{A})$ is elementary abelian. By [1, Lemma 3.3], $\mathrm{Bsets}(\mathcal{A})$ has elements with size p^2, a contradiction. Thus, $\mathbf{O}^\theta(\mathcal{A})$ is cyclic. By the main theorem of [17], $\mathcal{A}$ is Schurian. $\square$

Lemma 3.7. *Let $\mathcal{A}$ be a commutative p-Schur ring over H such that $|\mathbf{O}_\theta(\mathcal{A})| = p$ and $|\mathbf{O}^\theta(\mathcal{A})| = p^2$. Then $\mathbf{O}^\theta(\mathcal{A})$ is elementary abelian.*

Proof. Fix an element $T \in \mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_{\mathbf{O}^\theta(\mathcal{A})})$ with size p. By Lemma 3.5, we have $\mathrm{St}(T) = \{1\}$. By Lemma 3.4, we have $\underline{T^{-1}} \cdot \underline{T} = p\underline{1} + \sum_{T' \in I} \underline{T'}$, where $I = \mathrm{Bsets}(\mathcal{A}_{\mathbf{O}^\theta(\mathcal{A})}) \setminus \mathrm{Bsets}(\mathcal{A}_{\mathbf{O}_\theta(\mathcal{A})})$. This implies that there exists a $(p, p, p, 1)$-RDS in $\mathbf{O}^\theta(\mathcal{A})$. By Proposition 2.8, $\mathbf{O}^\theta(\mathcal{A})$ is elementary abelian. $\square$

Proposition 3.8. *Let $\mathcal{A}$ be a commutative p-Schur ring over H such that $|\mathbf{O}_\theta(\mathcal{A})| = p$ and $|\mathbf{O}^\theta(\mathcal{A})| = p^2$. Then $\mathcal{A}$ is Schurian.*

Proof. By Lemma 3.3, we assume that every element of $\mathrm{Bsets}(\mathcal{A})$ has at most size p. By Lemma 3.7, $\mathbf{O}^\theta(\mathcal{A})$ is elementary abelian. Then $\mathcal{A}$-subgroup series is either Type(2) or Type(3). We fix an element $T \in \mathrm{Bsets}(\mathcal{A}) \setminus \mathrm{Bsets}(\mathcal{A}_{\mathbf{O}^\theta(\mathcal{A})})$.

In the case of Type(2), we have $\mathrm{Bsets}(\mathcal{A}_{\langle a^p,b\rangle}) = \{\{h\} \mid h \in \langle a^{ip}b^j\rangle\} \cup \{L \mid L \in (\langle a^p, b\rangle/\langle a^{ip}b^j\rangle) \setminus \{\langle a^{ip}b^j\rangle\}\}$.

First of all, we assume $j \neq 0$. Without loss of generality, we can put $\mathbf{O}_\theta(\mathcal{A}) = \langle a^{ip}b\rangle$. Since $\mathrm{St}(T) = \{1\}$, we can assume $T = \{ax_0, ax_1a^p, ax_2a^{p^2}, \ldots, ax_{p-1}a^{p(p-1)}\}$, where $x_l \in \mathbf{O}_\theta(\mathcal{A})$. Since $\mathcal{A}$ is commutative, it must be satisfied $(a^{ip}b)^m T = T(a^{ip}b)^m$ for each $1 \leq m \leq p-1$. Thus, all x_l are same, i.e., $T = \{ab^j, ab^ja^p, \ldots, ab^ja^{p(p-1)}\}$ for some j. This implies $\mathrm{St}(T) = \langle a^p\rangle$, a contradiction.

Next, we assume $j = 0$. Then we have $\mathbf{O}_\theta(\mathcal{A}) = \langle a^p\rangle$. By Lemma 3.4, we can assume $T = \{ay_0, ay_1b, ay_2b, \ldots, ay_{p-1}b^{p-1}\}$, where $y_l \in \mathbf{O}_\theta(\mathcal{A})$. This implies that there exists a $(p,p,p,1)$-RDS in $\langle a^p, b\rangle$. By Proposition 2.9, we have $T = \{ab^ia^{pf(i)} \mid 0 \leq i \leq p-1\}$, where $f(i)$ is a planar function.

Replacing the generator a by $aa^{pf(0)}$, we can assume $f(i)$ such that $f(0) = 0$. By the same argument, i.e., replacing b by $ba^{pf(1)}$, we also assume $f(1) = 0$.

It is well known that $f(x)$ is a quadratic polynomial. So we assume that $f(x) = dx^2 + ex$. It is easy to see that $f(i+1) - f(i) = 2di$ for each $i \in \mathbb{F}_p$.

Now we define $\gamma \in \mathrm{Aut}(H)$ by $a \mapsto ab$ and $b \mapsto (a^p)^{2d}b$. Then $P := \langle h_R \mid h \in H\rangle \rtimes \langle\gamma\rangle$ is a subgroup of $\mathrm{Sym}(H)$. Using $f(i+1) - f(i) = 2di$, we can check $\gamma(ab^ia^{pf(i)}) = ab^{i+1}a^{pf(i+1)}$. Thus, it follows that the set of orbits of P_1 is $\mathrm{Bsets}(\mathcal{A})$.

In the case of Type(3), we have $\mathrm{Bsets}(\mathcal{A}_{\langle a,c\rangle}) = \{\{h\} \mid h \in \langle a^ic^j\rangle\} \cup \{L \mid L \in (\langle a, c\rangle/\langle a^ic^j\rangle) \setminus \{\langle a^ic^j\rangle\}\}$. Using the fact that c corresponds to a^p in Type(2), we can induce $f(i+1) - f(i) = 2di$ as mentioned in Type(2). Defining $\gamma \in \mathrm{Aut}(H)$ by $a \mapsto ac^{2d}$, $b \mapsto ba$ and $c \mapsto c$, we can check that $\mathcal{A}$ is Schurian. $\square$

In conclusion, it is proved that every commutative p-Schur ring over a non-abelian group of order p^3 is Schurian.

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