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ON A NEUMANN PROBLEM AT RESONANCE FOR NONUNIFORMLY SEMILINEAR ELLIPTIC SYSTEMS IN AN UNBOUNDED DOMAIN WITH NONLINEAR BOUNDARY CONDITION

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Abstract. We consider a nonuniformly nonlinear elliptic systems with resonance part and nonlinear Neumann boundary condition on an unbounded domain. Our arguments are based on the minimum principle and rely on a generalization of the Landesman-Lazer type condition.

1. Introduction and preliminaries

Let Ω be an unbounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth and bounded boundary $\partial\Omega$, $\bar{\Omega} = \Omega \cup \partial\Omega$. We consider the existence of weak solutions of Neumann problem for a system of nonuniformly semilinear elliptic equations:

$$\begin{cases} -\mathrm{div}(h_1(x)\nabla u) + a_1(x)u = \lambda_{11}\theta_1(x)u + f(x,u,v) - k_1(x) \\ -\mathrm{div}(h_2(x)\nabla v) + a_2(x)v = \lambda_{21}\theta_2(x)v + g(x,u,v) - k_2(x) \qquad \text{in } \Omega, \end{cases} \tag{1.1}$$

with nonlinear boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial n} = h_2(x)p(x,u,v) \\ \dfrac{\partial v}{\partial n} = h_1(x)q(x,u,v) \end{cases} \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes the derivative with respect to the outward unit normal to $\partial\Omega$ and $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $p, q : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions which will be specified later.

$$h_i(x) \in L^1_{loc}(\bar{\Omega}),\ h_i(x) \geq 1 \ \text{ for a.e } x \in \bar{\Omega},\ i = 1,2, \tag{1.3}$$

$$a_i(x) \in C(\bar{\Omega}),\ a_i(x) \geq a_0 > 0,\ \forall x \in \bar{\Omega},\ a_i(x) \to +\infty \text{ as } |x| \to +\infty,\ i = 1,2, \tag{1.4}$$

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$$\theta_i(x) \in L^\infty(\Omega),\ \theta_i(x) > 0 \text{ for a.e } x \in \bar{\Omega},\ i = 1, 2, \tag{1.5}$$

$$k_i(x) \in L^2(\Omega),\ k_i(x) > 0 \text{ for a.e } x \in \bar{\Omega},\ i = 1, 2. \tag{1.6}$$

λ_{i1} denotes the first eigenvalue of the problem:

$$\begin{cases} -\mathrm{div}(h_i(x)\nabla z) + a_i(x)z = \lambda_{i1}\theta_i(x)z & \text{in} \quad \bar{\Omega} \\ \dfrac{\partial z}{\partial n} = 0 & \text{on} \quad \partial\Omega,\ i = 1, 2 \end{cases} \tag{1.7}$$

in suitable spaces E_i which will be defined below.

We firstly make some comments on the problem (1.1). In the case that Ω is a bounded domain in $\mathbb{R}^N$ and $h(x) = 1$ there were extensive studies dealing with the Neumann problem for nonlinear elliptic equation involving the p-Laplacian, where different techniques of finding weak solutions are illustrated. When Ω is unbounded and $h(x) \in L^1_{loc}(\Omega)$, we refer the reader to [6], where the authors have considered Neumann problem for nonuniformly nonlinear equations involving p-Laplacian type in an unbounded domain $\Omega \subset \mathbb{R}^N$ with smooth and bounded boundary $\partial\Omega$ by using variational techniques via the Mountain Pass Theorem.

On the Landesman-Lazer condition, we refer the reader to [1, 2, 3, 9, 11, 15, 16]. In [1, 2, 3] the authors have considered a resonant problem involving p-Laplacian in a bounded domain $\Omega \subset \mathbb{R}^N$.

$$\Delta_p u = \lambda_1 |u|^{p-2} u + f(x, u) - h(x),$$

and the existence of weak solutions $u \in W_0^{1,p}(\Omega)$ is shown by taking the well-known Landesman-Lazer type condition. In [9, 11] one has extended some results in [1, 2, 3] to resonance problems with Dirichlet condition for nonuniformly nonlinear general elliptic equations in divergence form in bounded domain.

The extension to the case of p-Laplacian systems on resonance, again with Ω bounded and Dirichlet boundary condition, was first considered by N. B. Zographopoulos in [16]. Later in [7] D. A. Kandilakis and M. Magiropoulos have studied a quasilinear elliptic system with resonance part and nonlinear boundary condition in an unbounded domain by assuming the nonlinearities f and g depending only one variable u or v. In [15] Z.-Q. Ou and C.-L. Tang have considered the same system as in [7] with Dirichlet condition in a bounded domain. In these the existence of weak solutions is obtained by critical point theory under a Landesman-Lazer type condition.

In this paper, by introducing a generalization of Landesman-Lazer type condition, we will prove the existence of weak solutions of Neumann problem for a system on resonance of nonuniformly semilinear elliptic equations in an unbounded domain with general nonlinearities.

Recall that due to $h_i(x) \in L^1_{loc}(\Omega)$ $(i = 1, 2)$ the problem (1.1), (1.2) now is nonuniformly in sense that the Euler-Lagrange functional associated to the problem may be infinity at some $w_0 = (u_0, v_0) \in H^1(\Omega) \times H^1(\Omega)$. Hence, we must consider problem (1.1), (1.2) in some suitable subspace of $H^1(\Omega) \times H^1(\Omega)$.

Denotes by

$$C_0^\infty(\bar{\Omega}) = \{u \in C^\infty(\bar{\Omega}) : \mathrm{Supp} u \text{ compact } \subset \bar{\Omega}\},$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$.

Then $H^1(\Omega)$ is a usual Sobolev space which can be defined as the completion of $C_0^\infty(\bar{\Omega})$ under the norm:

$$||u|| = (\int_\Omega |\nabla u|^2 + |u|^2)dx)^{\frac{1}{2}}.$$

We now define following subspaces E_i $(i = 1, 2)$ of $H^1(\Omega)$:

$$E_i = \{u \in H^1(\Omega) : \int_\Omega [h_i(x)|\nabla u|^2 + a_i(x)|u|^2]dx < +\infty\},$$

where $h_i(x)$ and $a_i(x)$ satisfy conditions (1.3), (1.4).

By similar arguments as those used in the proof of Proposition 1.2 in [14], we deduce that E_i $(i = 1, 2)$ are Hilbert spaces with the norms:

$$||u||_{E_i} = (\int_\Omega [h_i(x)|\nabla u|^2 + a_i(x)|u|^2]dx)^{\frac{1}{2}}, \ u \in E_i$$

and the continuous embeddings $E_i \hookrightarrow H^1(\Omega) \hookrightarrow L^q(\Omega)$, $2 \le q \le 2^*$ $(i = 1, 2)$ hold true.

Moreover the embeddings E_i into $L^2(\Omega)$ are compact.

Besides since $\partial\Omega$ is bounded and smooth boundary, hence with $R > 0$ large enough $\partial\Omega \subset B_R(0)$, where $B_R(0)$ is ball of radius R.

Denote $\Omega_R = \bar{\Omega} \cap B_R(0)$, the maps $E_i \hookrightarrow H^1(\Omega_R)$ by $u \to u|_{\Omega_R}$ are continuous.

Therefore, from Theorem A_8 in [12] we deduce that $E_i \hookrightarrow L^2(\partial\Omega)$ compactly, $i = 1, 2$.

Remark 1.1. With similar arguments as those used in the proof of Lemma 2.3 in [4], we infer that the functional $J_{i0} : E_i \to R$ $(i = 1, 2)$ given by

$$J_{i0}(u) = ||u||^2_{E_i} = \int_\Omega (h_i(x)|\nabla u|^2 + a_i(x)|u|^2)dx, \ u \in E_i$$

is weakly lower semicontinuous on E_i.

Next, we have following proposition which concerns the existence of the first eigenvalue and eigenfunction of the problem (1.7).

Proposition 1.1. *Assume that functions $h_i(x)$, $a_i(x)$, $\theta_i(x)$ $(i = 1, 2)$ satisfy the conditions* (1.3), (1.4), (1.5).

Denotes by

(1.8)

$$\lambda_{i1} = \inf\{\int_\Omega (h_i(x)|\nabla u|^2 + a_i(x)|u|^2)dx : u \in E_i, \int_\Omega \theta_i(x)|u|^2 dx = 1\}, \ i = 1, 2.$$

Then:

(i) $M_i = \{u \in E_i : \int_\Omega \theta_i(x)|u|^2(x)dx = 1\} \neq \phi$.

(ii) *There exists* $\varphi_{i1} \in M_i$, $\varphi_{i1} > 0$ *in* $\bar{\Omega}$ *such that*:

$$\int_\Omega (h_i(x)|\nabla\varphi_{i1}|^2 + a_i(x)|\varphi_{i1}|^2)dx = \lambda_{i1}.$$

Thus $(\lambda_{i1}, \varphi_{i1})$ $(i = 1, 2)$ *are eigenvalues and eigenfunctions associated with* λ_{i1} *of the problem* (1.7) *in* E_i.

Proof. (i) Let $u(x) \in C_0^\infty(\bar{\Omega})$, $u \neq 0$. Then $u \in E_i$ and $\int_\Omega \theta_i(x)|u^2(x)|dx > 0$.

Choose $\bar{u} \in E_i$ as:

$$\bar{u}(x) = \frac{u(x)}{(\int_\Omega \theta_i(x)|u^2(x)|dx)^{\frac{1}{2}}} \quad \text{for } x \in \bar{\Omega}.$$

Then $\int_\Omega \theta_i(x)|\bar{u}(x)|^2 dx = 1$. So $\bar{u} \in M_i$ and $M_i \neq \phi$.

(ii) Let $u_m \subset E_i$ be a minimizing sequence, i.e.,

$$\int_\Omega \theta_i(x)|u_m(x)|^2 dx = 1,\ m = 1, 2, \dots$$

and $\lim_{m\to+\infty} \int_\Omega (h_i(x)|\nabla u_m|^2 + a_i(x)|u_m|^2)dx = \lambda_{i1}$. So $\{u_m\}$ is bounded in E_i.

Then, there exists a subsequence $\{u_{m_k}\}_k$ such that $\{u_{m_k}\}_k$ converges weakly to $\widehat{u}$ in E_i. Since the embedding E_i into $L^2(\Omega)$ is compact, the subsequence $\{u_{m_k}\}$ converges strongly to $\widehat{u}$ in $L^2(\Omega)$.

Moreover since $\theta_i(x) \in L^\infty(\Omega)$, we infer that:

$$1 = \lim_{k\to+\infty} \int_\Omega \theta_i(x)|u_{m_k}|^2 dx = \int_\Omega \theta_i(x)|\widehat{u}|^2 dx.$$

So $\widehat{u} \in M_i$.

By the minimizing properties and the weakly lower semicontinuity of the functional $J_{i0}(u) = \int_\Omega (h_i(x)|\nabla u|^2 + a_i(x)|u|^2)dx$ on E_i (see Remark 1.1), we have:

$$\begin{aligned}\lambda_{i1} &= \liminf_{k\to+\infty} \int_\Omega (h_i(x)|\nabla u_{m_k}|^2 + a_i(x)|u_{m_k}|^2)dx \\ &\geq \int_\Omega (h_i(x)|\nabla\widehat{u}|^2 + a_i(x)|\widehat{u}|^2)dx \geq \lambda_{i1}.\end{aligned}$$

So we obtain

$$\lambda_{i1} = \int_\Omega (h_i(x)|\nabla\widehat{u}|^2 + a_i(x)|\widehat{u}|^2)dx.$$

Thus $\widehat{u}$ is a minimizer of (1.8).

Observe further that since $\widehat{u} \in E_i \subset H^1(\Omega)$ then $|\widehat{u}| \in H^1(\Omega)$ (see [5, p. 152, Lemma 7.6]). Moreover,

$$\int_\Omega (h_i(x)|\nabla\widehat{|u|}|^2 + a_i(x)||\widehat{u}||^2)dx = \int_\Omega (h_i(x)|\nabla\widehat{u}|^2 + a_i(x)|\widehat{u}|^2)dx < +\infty$$

and

$$\int_\Omega (\theta_i(x)||\widehat{u}||^2)dx = \int_\Omega (\theta_i(x)\widehat{|u|}^2)dx = 1.$$

So $|\widehat{u}| \in M_i$ is a minimizer too and

$$\int_\Omega (h_i(x)|\nabla\widehat{|u|}|^2 + a_i(x)|\widehat{|u|}|^2)dx = \lambda_{i1}.$$

Applying the Lagrange multiplier rule, we deduce that

$$\int_\Omega (h_i(x)\nabla|\widehat{u}| \cdot \nabla v + a_i(x)|\widehat{u}| \cdot v)dx - \lambda_{i1}\int_\Omega \theta_i(x)|\widehat{u}| \cdot vdx = 0, \ \forall v \in E_i.$$

This implies that

$$\begin{cases} -\mathrm{div}(h_i(x)\nabla|\widehat{u}|) + a_i(x)|\widehat{u}| = \lambda_{i1}\theta_i(x)|\widehat{u}| & \text{in } \Omega, \\ \dfrac{\partial|\widehat{u}|}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, for any Ω' compact $\subset \Omega$, $h_i(x) \in L^1(\Omega')$, $a_i(x) \in L^\infty(\Omega')$, $|\widehat{u}| \geq 0$ in Ω' and

$$-\mathrm{div}(h_i(x)\nabla|\widehat{u}|) + a_i(x)|\widehat{u}| = \lambda_1\theta_i(x)|\widehat{u}|, \text{ in } \Omega'.$$

So by the Harnack inequality (see [5, Theorem 8.19 or Theorem 8.20 and Corollary 8.21]), it follows that $|\widehat{u}| > 0$ in Ω'. This implies that $|\widehat{u}| > 0$ in $\bar{\Omega}$.

Denotes $\varphi_{i1}(x) = |\widehat{u}|$, then $\varphi_{i1}(x) > 0$ in Ω and φ_{i1} is an eigenfunction of the problem (1.7). The proof of Proposition 1.1 is complete. □

On the other hand by similar argument, we also show that the eigenfunctions of λ_{i1} are either positive or negative in $\bar{\Omega}$. Hence, by the compact embedding E_i into $L^2(\Omega)$ and the standard spectral theory for compact, self-adjoint operators we can infer that for any $i = 1, 2$ the λ_{i1}-eigenfunction φ_{i1} is unique (up to a multiplicitive constant) and

$$\lambda_{i1} = \inf_{0\neq u \in E_i} \frac{\int_\Omega (h_i(x)|\nabla u|^2 + a_i(x)|u|^2)dx}{\int_\Omega (\theta_i(x)|u|^2)dx}, \ i = 1, 2.$$

In order to state our main results, let us introduce following some hypotheses on nonlinearities:

(H_1)

(i) $f, g : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}, \ p, q : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions:

$$f(x, 0, 0) = 0, \ g(x, 0, 0) = 0.$$

(ii) There exist positive functions $\tau_1(x) \in L^2(\Omega), \ \tau_2(x) \in L^2(\partial\Omega)$ such that: for all $(s, t) \in \mathbb{R}^2$, we have

$|f(x, s, t)| \leq \tau_1(x), \ |g(x, s, t)| \leq \tau_1(x)$ for a.e $x \in \Omega$,

$|p(x, s, t)| \leq \tau_2(x), \ |q(x, s, t)| \leq \tau_2(x)$ for a.e $x \in \partial\Omega$.

(iii) $f(x, \cdot), \ g(x, \cdot), \ p(x, \cdot), \ q(x, \cdot) \in C^1(\mathbb{R}^2)$, and $\forall (s, t) \in \mathbb{R}^2$,

$$\frac{\partial f(x, s, t)}{\partial t} = \frac{\partial g(x, s, t)}{\partial s} \text{ for a.e } x \in \Omega, \tag{1.9}$$

$$\frac{\partial p(x,s,t)}{\partial t} = \frac{\partial q(x,s,t)}{\partial s} \text{ for a.e } x \in \partial\Omega. \tag{1.10}$$

Denotes $\forall (u,v) \in \mathbb{R}^2$,

(1.11)

$$H(x,u,v) = \frac{1}{2}\int_0^u [f(x,s,v) + f(x,s,0)]ds + \frac{1}{2}\int_0^v [g(x,u,t) + g(x,0,t)]dt$$
$$\text{for a.e } x \in \Omega,$$
$$R(x,u,v) = \frac{1}{2}\int_0^u [p(x,s,v) + p(x,s,0)]ds + \frac{1}{2}\int_0^v [q(x,u,t) + q(x,0,t)]dt$$
$$\text{for a.e } x \in \partial\Omega.$$

Remark 1.2. By hypotheses (1.9), (1.10) and (1.11) with some standard computations we deduce that, for all $(u,v) \in \mathbb{R}^2$

$$\begin{aligned} &\frac{\partial H(x,u,v)}{\partial u} = f(x,u,v),\ \frac{\partial H(x,u,v)}{\partial v} = g(x,u,v) \text{ a.e } x \in \Omega, \\ &\frac{\partial R(x,u,v)}{\partial u} = p(x,u,v),\ \frac{\partial R(x,u,v)}{\partial v} = q(x,u,v) \text{ a.e } x \in \partial\Omega. \end{aligned} \tag{1.12}$$

Now we define, for $i,j = 1,2$:

$$\begin{aligned} F_{ij}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [f(x,(-1)^{1+i}y\varphi_{11},(-1)^{1+j}\tau\varphi_{21}) \\ &\qquad + f(x,(-1)^{1+i}y\varphi_{11},0)]dy,\ x \in \Omega, \\ G_{ij}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [g(x,(-1)^{1+i}\tau\varphi_{11},(-1)^{1+j}y\varphi_{21}) \\ &\qquad + g(x,0,(-1)^{1+i}y\varphi_{21})]dy,\ x \in \Omega, \\ P_{ij}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [p(x,(-1)^{1+i}y\varphi_{11},(-1)^{1+j}\tau\varphi_{21}) \\ &\qquad + p(x,(-1)^{1+i}y\varphi_{11},0)]dy,\ x \in \partial\Omega, \\ Q_{ij}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [q(x,(-1)^{1+i}\tau\varphi_{11},(-1)^{1+j}y\varphi_{21}) \\ &\qquad + q(x,0,(-1)^{1+i}y\varphi_{21})]dy,\ x \in \partial\Omega, \end{aligned} \tag{1.13a}$$

$$\begin{aligned} F_{i0}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [f(x,(-1)^{1+i}y\varphi_{11}(x),0)]dy,\ x \in \Omega, \\ G_{0j}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [g(x,0,(-1)^{1+j}y\varphi_{21}(x))]dy,\ x \in \Omega, \\ P_{i0}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [p(x,(-1)^{1+i}y\varphi_{11}(x),0)]dy,\ x \in \partial\Omega, \\ Q_{0j}(x) &= \limsup_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [q(x,0,(-1)^{1+j}y\varphi_{21}(x))]dy,\ x \in \partial\Omega. \end{aligned} \tag{1.13b}$$

Set $h(x) = h_1(x)h_2(x)$ and

$$
\begin{aligned}
L_{11} = & \int_\Omega (F_{11}(x)\varphi_{11}(x) + G_{11}(x)\varphi_{21}(x))dx \\
& + \int_{\partial\Omega} h(x)[P_{11}(x)\varphi_{11}(x) + Q_{11}(x)\varphi_{21}(x)]ds, \\
L_{22} = & \int_\Omega (F_{22}(x)\varphi_{11}(x) + G_{22}(x)\varphi_{21}(x))dx \\
& + \int_{\partial\Omega} h(x)[P_{22}(x)\varphi_{11}(x) + Q_{22}(x)\varphi_{21}(x)]ds, \\
L_{12} = & \int_\Omega (F_{12}(x)\varphi_{11}(x) - G_{12}(x)\varphi_{21}(x))dx \\
& + \int_{\partial\Omega} h(x)[P_{12}(x)\varphi_{11}(x) - Q_{12}(x)\varphi_{21}(x)]ds, \\
L_{21} = & \int_\Omega (F_{21}(x)\varphi_{11}(x) - G_{21}(x)\varphi_{21}(x))dx \\
& + \int_{\partial\Omega} h(x)[P_{21}(x)\varphi_{11}(x) - Q_{21}(x)\varphi_{21}(x)]ds,
\end{aligned}
\tag{1.14a}
$$

$$
\begin{aligned}
L_{i0} &= \int_\Omega F_{i0}(x)\varphi_{11}(x)dx + \int_{\partial\Omega} h(x)P_{i0}(x)\varphi_{11}(x)ds,\ i = 1, 2, \\
L_{0j} &= \int_\Omega G_{0j}(x)\varphi_{21}(x)dx + \int_{\partial\Omega} h(x)Q_{0j}(x)\varphi_{21}(x)ds,\ i = 1, 2.
\end{aligned}
\tag{1.14b}
$$

(H_2) Assume that the following potential Landesman-Lazer type conditions hold:

(i)

$$L_{11} < 2\int_\Omega [k_1(x)\varphi_{11}(x) + k_2(x)\varphi_{21}(x)]dx < L_{22}, \tag{1.15a}$$

$$L_{12} < 2\int_\Omega [k_1(x)\varphi_{11}(x) - k_2(x)\varphi_{21}(x)]dx < L_{21}. \tag{1.16a}$$

(ii)

$$L_{10} < \int_\Omega k_1(x)\varphi_{11}(x)dx < L_{20}, \tag{1.15b}$$

$$L_{01} < \int_\Omega k_2(x)\varphi_{21}(x)dx < L_{02}, \tag{1.16b}$$

where φ_{i1} are eigenfunctions associated with λ_{i1} of the problem (1.7) in E_i, $i = 1, 2$.

Definition 1.1. Function $w = (u, v) \in E = E_1 \times E_2$ is called a weak solution of the problem (1.1), (1.2) if and only if

(1.17)
$$\begin{aligned}
&\int_\Omega (h_1(x)\nabla u\nabla\varphi(x) + a_1(x)u\varphi(x))dx + \int_\Omega (h_2(x)\nabla v\nabla\psi(x) + a_2(x)v\psi(x))dx \\
&- \lambda_{11}\int_\Omega \theta_1(x)u\varphi(x)dx - \lambda_{21}\int_\Omega \theta_2(x)v\psi(x)dx \\
&- \int_\Omega [f(x,u,v)\varphi + g(x,u,v)\psi(x)]dx + \int_\Omega [k_1(x)\varphi(x) + k_2(x)\psi(x)]dx \\
&- \int_{\partial\Omega} h(x)[p(x,u,v)\varphi(x) + q(x,u,v)\psi(x)]ds = 0
\end{aligned}$$

for all $(\varphi,\psi) \in C_0^\infty(\bar\Omega) \times C_0^\infty(\bar\Omega)$.

Remark 1.3. If $(u_0, v_0) \in C^2(\bar\Omega) \times C^2(\bar\Omega)$ satisfies the condition (1.17), hence (u_0, v_0) is a classical solution of the problem (1.1), (1.2). (See Remark 1.1 in [6].)

Our main result is given by the following theorem.

Theorem 1.1. *Assuming conditions* (H_1)*,* (H_2) *are fulfilled. Then the problem* (1.1)*,* (1.2) *admits at least a nontrivial weak solution in* $E = E_1 \times E_2$*.*

Proof of Theorem 1.1 is based on variational techniques and the Minimum Principle.

2. Proof of the main result

The Euler-Lagrange functional associated to the problem (1.1), (1.2), $I : E \to R$ is given by

(2.1)
$$\begin{aligned}
I(w) =&\frac{1}{2}\int_\Omega (h_1(x)|\nabla u|^2 + a_1(x)|u|^2)dx + \frac{1}{2}\int_\Omega (h_2(x)|\nabla v|^2 + a_2(x)|v|^2)dx \\
&- \frac{\lambda_{11}}{2}\int_\Omega \theta_1(x)|u|^2dx - \frac{\lambda_{21}}{2}\int_\Omega \theta_2(x)|v|^2dx - \int_\Omega H(x,u,v)dx \\
&+ \int_\Omega (k_1(x)u + k_2(x)v)dx - \int_{\partial\Omega} h(x)R(x,u,v)ds,\ \forall w = (u,v) \in E,
\end{aligned}$$

where $h(x) = h_1(x)h_2(x)$, $H(x,u,v)$, $R(x,u,v)$ are given by (1.11).

Denotes

$$\begin{aligned}
J(w) = &\frac{1}{2}\int_\Omega (h_1(x)|\nabla u|^2 + a_1(x)|u|^2)dx \\
&+ \frac{1}{2}\int_\Omega (h_2(x)|\nabla v|^2 + a_2(x)|v|^2)dx,\ w = (u,v) \in E.
\end{aligned} \tag{2.2}$$

(2.3)

$$T(w) = -\frac{\lambda_{11}}{2}\int_\Omega \theta_1(x)|u|^2dx - \frac{\lambda_{21}}{2}\int_\Omega \theta_2(x)|v|^2dx - \int_\Omega H(x,u,v)dx \\ + \int_\Omega (k_1(x)u + k_2(x)v)dx - \int_{\partial\Omega} h(x)R(x,u,v)ds, \ w = (u,v) \in E.$$

By hypotheses (H_1) the functions J, T and then $I = J + T$ are well-defined on H.

Remark 2.1. By Remark 1.1, the functional $J(w), w \in E$ given by (2.1) is weak lower semicontinuous on E. Moreover from hypothesis (H_1) with some standard arguments we deduce that the functional $T(w), w \in E$ given by (2.3) is also weak lower semicontinuous on E. Thus the functional $I = J+T$ is weak lower semicontinuous on E.

The following proposition which concerns the smoothness of the functional I on E.

Proposition 2.1. *The Euler-Lagrange functional I given by* (2.1) *is Fréchet differentiable on E and we have*:

(2.4)

$$(I'(w), \bar{w}) = \int_\Omega (h_1(x)\nabla u\nabla\bar{u} + a_1(x)u\bar{u})dx + \int_\Omega (h_2(x)\nabla v\nabla\bar{v} + a_2(x)v\bar{v})dx \\ - \lambda_{11}\int_\Omega \theta_1(x)u\bar{u}dx - \lambda_{21}\int_\Omega \theta_2(x)v\bar{v}dx \\ - \int_\Omega [f(x,w)\bar{u} + g(x,w)\bar{v}]dx + \int_\Omega (k_1(x)\bar{u} + k_2(x)\bar{v})dx \\ - \int_{\partial\Omega} h(x)[p(x,w)\bar{u} + q(x,w)\bar{v}]ds, \ \forall w = (u,v), \bar{w} = (\bar{u},\bar{v}) \in E.$$

Proof. With similar arguments as those used in the proof of Proposition 2.2(iii) in [14] we deduce that the functional J given by (2.2) is Gateaux differentiable on E and whose the Gateaux derivative is given by:

$$(J'(w), \bar{w}) = \int_\Omega (h_1(x)\nabla u\nabla\bar{u} + a_1(x)u\bar{u})dx \\ + \int_\Omega (h_2(x)\nabla v\nabla\bar{v} + a_2(x)v\bar{v})dx, \ w = (u,v), \ \bar{w} = (\bar{u},\bar{v}) \in E.$$

Now let $\{w_m = (u_m, v_m)\}$ be a sequence converging to $w = (u,v)$ in E, i.e.,

$$\lim_{m\to+\infty} ||w_m - w||_E^2 = \lim_{m\to+\infty}\{\int_\Omega (h_1(x)|\nabla(u_m - u)|^2 + a_1(x)|u_m - u|^2)dx \\ + \int_\Omega (h_2(x)|\nabla(v_m - v)|^2 + a_2(x)|v_m - v|^2)dx\} = 0.$$

Then by some simple computations we have:

$$\begin{aligned}|(J'(w_m) - J'(w)), \bar{w}| = |&\int_\Omega (h_1(x)\nabla(u_m - u)\nabla\bar{u} + a_1(x)(u_m - u)\bar{u})dx \\ &+ \int_\Omega (h_2(x)\nabla(v_m - v)\nabla\bar{v} + a_2(x)(v_m - v)\bar{v})dx| \\ &\leq 4\,||\bar{w}||_E \cdot ||w_m - w||_E \text{ for all } \bar{w} = (\bar{u}, \bar{v}) \in E.\end{aligned}$$

This implies that

$$||J'(w_m) - J'(w)||_{E^*} \leq C||w_m - w||_E.$$

Let $m \to +\infty$ we obtain: $\lim_{m\to+\infty} J'(w_m) = J'(w)$ in E^*.

Hence J' is continuous on E. Thus $J \in C^1(E, R)$.

Besides, from hypotheses (H_1) and (1.5), (1.6) on the functions f, g, p, q, θ_1, θ_2 and $k_1(x)$, $k_2(x)$, for some standard computations we infer that the functional T given by (2.3) is Fréchet differentiable on E and we get:

$$\begin{aligned}(T'(w), \bar{w}) = &-\lambda_{11}\int_\Omega \theta_1(x)u\bar{u}dx - \lambda_{21}\int_\Omega \theta_1(x)v\bar{v}dx \\ &- \int_\Omega [f(x,w)\bar{u} + g(x,w)\bar{v}]dx + \int_\Omega (k_1(x)\bar{u} + k_2(x)\bar{v})dx \\ &- \int_{\partial\Omega} h(x)[p(x,w)\bar{u} + q(x,w)\bar{v}]ds,\ \forall w = (u,v),\ \bar{w} = (\bar{u},\bar{v}) \in E.\end{aligned}$$

Finally, the functional $I = J + T \in C^1(E, R)$ and we have (2.4). Proposition 2.1 is proved. □

Remark 2.2. By Proposition 2.1 the critical points of the Euler-Lagrange functional I are precisely the weak solutions of the problem (1.1), (1.2).

Proposition 2.2. *The functional I given by* (2.1) *is coercive on E provided that conditions* (H_1) *and* (H_2) *hold true.*

Proof. By contradiction we assume that the functional I is not coercive on E. Then there exists a sequence $\{w_m = (u_m, v_m)\}_{m=1}^\infty$ in E such that

$$||w_m||_E \to +\infty \quad \text{as } m \to +\infty \quad \text{and} \quad I(w_m) \leq c, \tag{2.5}$$

where c is positive constant.

Let $\widehat{w}_m = \frac{w_m}{||w_m||_E} = (\widehat{u}_m, \widehat{v}_m)$, that is $\widehat{u}_m = \frac{u_m}{||w_m||_E}$, $\widehat{v}_m = \frac{v_m}{||w_m||_E}$, $m = 1, 2, \ldots$. Thus $\{\widehat{w}_m\}_m$ is a bounded sequence in E.

Then there exists a subsequence $\{\widehat{w}_{m_k}\}_k = \{(\widehat{u}_{m_k}, \widehat{v}_{m_k})\}$ which converge weakly to some $\widehat{w} = (\widehat{u}, \widehat{v})$ in $E = E_1 \times E_2$.

Since the embeddings $E_i, (i = 1, 2)$ into $L^2(\Omega)$ and $L^2(\partial\Omega)$ are compact, the subsequences $\{\widehat{u}_{m_k}\}$, $\{\widehat{v}_{m_k}\}$ converge strongly respective to $\widehat{u}$, $\widehat{v}$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$.

From (2.5) dividing by $||w_{m_k}||_E^2$ we deduce that

(2.6)
$$\begin{aligned}&\limsup_{k\to+\infty}\{\frac{1}{2}\int_\Omega (h_1(x)\,|\nabla\widehat{u}_{m_k}|^2+a_1(x)\,|\widehat{u}_{m_k}|^2)dx-\frac{\lambda_{11}}{2}\int_\Omega \theta_1(x)\,|\widehat{u}_{m_k}|^2\,dx\\&+\frac{1}{2}\int_\Omega (h_2(x)\,|\nabla\widehat{v}_{m_k}|^2+a_2(x)\,|\widehat{v}_{m_k}|^2)dx-\frac{\lambda_{21}}{2}\int_\Omega \theta_2(x)\,|\widehat{v}_{m_k}|^2\,dx\\&-\int_\Omega \frac{H(x,w_{m_k})}{||w_{m_k}||_E^2}dx+\int_\Omega \frac{k_1(x)\widehat{u}_{m_k}+k_2(x)\widehat{v}_{m_k}}{||w_{m_k}||_E}dx-\int_{\partial\Omega} h(x)\frac{R(x,w_{m_k})}{||w_{m_k}||_E^2}ds\}\\&\le 0.\end{aligned}$$

By (H_1)-(ii) and (1.11) we have:

$$|H(x,w_{m_k})|\le 2\tau_1(x)(|u_{m_k}|+|v_{m_k}|),\ \tau_1(x)\in L^2(\Omega).$$

Hence

$$\left|\int_\Omega \frac{H(x,w_{m_k})}{||w_{m_k}||_E^2}dx\right|\le \frac{2}{||w_{m_k}||_E}\,||\tau_1||_{L^2(\Omega)}\,(||\widehat{u}_{m_k}||_{L^2(\Omega)}+||\widehat{v}_{m_k}||_{L^2(\Omega)}). \tag{2.7}$$

Remark that $\{(\widehat{u_{mk}}\}, \{\widehat{v_{mk}})\}$ converge strongly in $L^2(\Omega)$, hence they are bounded in $L^2(\Omega)$.

Letting $k\to+\infty$, since $||w_{m_k}||_E\to+\infty$, we obtain:

$$\limsup_{k\to+\infty}\int_\Omega \frac{H(x,w_{m_k})}{||w_{m_k}||_E^2}dx=0. \tag{2.8}$$

Similarly, we also obtain

$$\begin{aligned}&\limsup_{k\to+\infty}\int_\Omega \frac{k_1(x)\widehat{u}_{m_k}+k_2(x)\widehat{v}_{m_k}}{||u_{m_k}||_E}dx=0,\\&\limsup_{k\to+\infty}\int_{\partial\Omega} h(x)\frac{R(x,w_{m_k})}{||w_{m_k}||_E^2}ds=0.\end{aligned} \tag{2.9}$$

Moreover, we have

$$\begin{aligned}&\lim_{k\to+\infty}\int_\Omega \theta_1(x)|\widehat{u}_{m_k}|^2dx=\int_\Omega \theta_1(x)|\widehat{u}|^2dx,\\&\lim_{k\to+\infty}\int_\Omega \theta_2(x)|\widehat{v}_{m_k}|^2dx=\int_\Omega \theta_2(x)|\widehat{v}|^2dx.\end{aligned} \tag{2.10}$$

Then from (2.6) and (2.8), (2.9), (2.10), we deduce that

$$\begin{aligned}&\limsup_{k\to+\infty}\{\int_\Omega (h_1(x)\,|\nabla\widehat{u}_{m_k}|^2+a_1(x)\,|\widehat{u}_{m_k}|^2)dx\\&\quad+\int_\Omega (h_2(x)\,|\nabla\widehat{v}_{m_k}|^2+a_2(x)\,|\widehat{v}_{m_k}|^2)dx\}\\&\le \lambda_{11}\int_\Omega \theta_1(x)\,|\widehat{u}|^2\,dx+\lambda_{21}\int_\Omega \theta_2(x)\,|\widehat{v}|^2\,dx.\end{aligned}$$

By Remark 1.1 and the variational characterization of λ_{11}, λ_{21}, we get:

$$\begin{aligned}
&\lambda_{11}\int_\Omega \theta_1(x)\,|\widehat{u}|^2\,dx+\lambda_{21}\int_\Omega \theta_2(x)\,|\widehat{v}|^2\,dx\\
\leq\ &\int_\Omega (h_1(x)\,|\nabla\widehat{u}|^2+a_1(x)\,|\widehat{u}|^2)dx+\int_\Omega (h_2(x)\,|\nabla\widehat{v}|^2+a_2(x)\,|\widehat{v}|^2)dx\\
\leq\ &\liminf_{k\to+\infty}\{\int_\Omega (h_1(x)\,|\nabla\widehat{u}_{m_k}|^2+a_1(x)\,|\widehat{u}_{m_k}|^2)dx\\
&+\int_\Omega (h_2(x)\,|\nabla\widehat{v}_{m_k}|^2+a_2(x)\,|\widehat{v}_{m_k}|^2)dx\}\\
\leq\ &\limsup_{k\to+\infty}\{\int_\Omega (h_1(x)\,|\nabla\widehat{u}_{m_k}|^2+a_1(x)\,|\widehat{u}_{m_k}|^2)dx\\
&+\int_\Omega (h_2(x)\,|\nabla\widehat{v}_{m_k}|^2+a_2(x)\,|\widehat{v}_{m_k}|^2)dx\}\\
\leq\ &\lambda_{11}\int_\Omega \theta_1(x)\,|\widehat{u}|^2\,dx+\lambda_{21}\int_\Omega \theta_2(x)\,|\widehat{v}|^2\,dx.
\end{aligned}$$

Thus, these inequalities are indeed equalities and we have

$$\begin{aligned}
&\lim_{k\to+\infty}\{\int_\Omega (h_1(x)\,|\nabla\widehat{u}_{m_k}|^2+a_1(x)\,|\widehat{u}_{m_k}|^2)dx\\
&\quad+\int_\Omega (h_2(x)\,|\nabla\widehat{v}_{m_k}|^2+a_2(x)\,|\widehat{v}_{m_k}|^2)dx\}\\
&=\int_\Omega (h_1(x)\,|\nabla\widehat{u}|^2+a_1(x)\,|\widehat{u}|^2)dx+\int_\Omega (h_2(x)\,|\nabla\widehat{v}|^2+a_2(x)\,|\widehat{v}|^2)dx\\
&=\lambda_{11}\int_\Omega \theta_1(x)\,|\widehat{u}|^2\,dx+\lambda_{21}\int_\Omega \theta_2(x)\,|\widehat{v}|^2\,dx.
\end{aligned}\tag{2.11}$$

On the other hand from (2.11) and remark that $||\widehat{w}_{m_k}||_E=1$ we infer that:

$$\lim_{k\to+\infty}||\widehat{w}_{m_k}||_E=||\widehat{w}||_E=1.$$

Hence $\widehat{w}=(\widehat{u},\widehat{v})\neq 0$.

By again the variational characterization of λ_{11}, λ_{21} we deduce that

$$\int_\Omega (h_1(x)\,|\nabla\widehat{u}|^2+a_1(x)\,|\widehat{u}|^2)dx=\lambda_{11}\int_\Omega \theta_1(x)\,|\widehat{u}|^2\,dx,$$

$$\int_\Omega (h_2(x)\,|\nabla\widehat{v}|^2+a_2(x)\,|\widehat{v}|^2)dx=\lambda_{21}\int_\Omega \theta_2(x)\,|\widehat{v}|^2\,dx.$$

This implies from definition of eigenfunctions $\varphi_{11}(x)$ and $\varphi_{21}(x)$ that:

- If $\widehat{u}\neq 0$, $\widehat{v}\neq 0$, then $\widehat{u}(x)=\pm\varphi_{11}(x)$, $\widehat{v}(x)=\pm\varphi_{21}(x)$.
- If $\widehat{u}\neq 0$, $\widehat{v}=0$, then $\widehat{u}(x)=\pm\varphi_{11}(x)$.
- If $\widehat{u}=0$, $\widehat{v}\neq 0$, then $\widehat{v}(x)=\pm\varphi_{21}(x)$.

Next, we will consider the following cases:

Let $\widehat{u}_{m_k}\to\widehat{u}=\varphi_{11}$, $\widehat{v}_{m_k}\to\widehat{v}=\varphi_{21}$ as $k\to+\infty$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$.

Firstly by the variational characterization of λ_{11} and λ_{21} we have

$$\int_\Omega (h_1(x)\left|\nabla u_{m_k}\right|^2 + a_1(x)\left|u_{m_k}\right|^2)dx + \int_\Omega (h_2(x)\left|\nabla v_{m_k}\right|^2 + a_2(x)\left|v_{m_k}\right|^2)dx$$

$$\geq \lambda_{11}\int_\Omega \theta_1(x)\left|u_{m_k}\right|^2 dx + \lambda_{21}\int_\Omega \theta_2(x)\left|v_{m_k}\right|^2 dx,\ m = 1, 2, \ldots.$$

Hence from (2.5) one get

$$\begin{aligned} &-\int_\Omega H(x, w_{m_k})dx - \int_{\partial\Omega} h(x)R(x, w_{m_k})ds \\ &+ \int_\Omega [k_1(x)u_{m_k}(x) + k_2(x)v_{m_k}(x)]dx \\ &\leq I(w_{m_k}) \leq c,\ k = 1, 2, \ldots. \end{aligned} \tag{2.12}$$

After dividing (2.12) by $||w_{m_k}||_E$, letting $\limsup_{k\to+\infty}$ and remark that

$$\lim_{k\to+\infty}\int_\Omega [k_1(x)\widehat{u}_{m_k}(x) + k_2(x)\widehat{v}_{m_k}(x)]dx = \int_\Omega [k_1(x)\varphi_{11}(x) + k_2(x)\varphi_{21}(x)]dx.$$

We get

$$\begin{aligned} &\limsup_{k\to+\infty}\{\int_\Omega \frac{H(x, w_{m_k})}{||w_{m_k}||_E}dx + \int_{\partial\Omega} h(x)\frac{R(x, w_{m_k})}{||w_{m_k}||_E}ds\} \\ &\geq \int_\Omega [k_1(x)\varphi_{11}(x) + k_2(x)\varphi_{21}(x)]dx. \end{aligned} \tag{2.13}$$

Lemma 2.1. *Assume that* $\widehat{u}_{m_k} \to \varphi_{11}$, $\widehat{v}_{m_k} \to \varphi_{21}$ *in* $L^2(\Omega)$ *and* $L^2(\partial\Omega)$ *as* $k \to +\infty$. *Then*:

$$\text{i)}\ \limsup_{k\to+\infty}\int_\Omega \frac{H(x, w_{m_k})}{||w_{m_k}||_E}dx = \frac{1}{2}\int_\Omega [F_{11}(x)\varphi_{11} + G_{11}\varphi_{21}]dx, \tag{2.14}$$

(2.15)

$$\text{ii)}\ \limsup_{k\to+\infty}\int_{\partial\Omega} h(x)\frac{R(x, w_{m_k})}{||w_{m_k}||_E}ds = \frac{1}{2}\int_{\partial\Omega} h(x)[P_{11}(x)\varphi_{11} + Q_{11}\varphi_{21}]ds.$$

Proof. By (1.11) we have

(2.16)

$$2H(x, w_{m_k}) = \int_0^{u_{m_k}} [f(x, s, v_{m_k}) + f(x, s, 0)]ds + \int_0^{v_{m_k}} [g(x, u_{m_k}, t) + g(x, 0, t)]dt.$$

Set $l_k = ||w_{m_k}||_E \to +\infty$ as $k \to +\infty$.

Observe that by hypotheses (H_1) on $f(x, w)$, $g(x, w)$ we have

$$\begin{aligned} &\left|\int_0^{u_{m_k}} f(x, s, v_{m_k})ds - \int_0^{l_k\varphi_{11}} f(x, s, l_k\varphi_{21})ds\right| \\ &\leq \left|\int_0^{u_{m_k}} [f(x, s, v_{m_k}) - f(x, s, l_k\varphi_{21})]ds\right| + \left|\int_{l_k\varphi_{11}}^{u_{m_k})} f(x, s, l_k\varphi_{21})ds\right| \\ &\leq \left|\int_0^{u_{m_k}} \frac{\partial f}{\partial t}(x, s, l_k\varphi_{21} + \delta(v_{m_k} - l_k\varphi_{21}))\cdot(v_{m_k} - l_k\varphi_{21})ds\right| \end{aligned}$$

$$
\begin{aligned}
&+ \tau_1(x)|u_{m_k} - l_k\varphi_{11}| \\
&\leq |\int_0^{u_{m_k}} \frac{\partial g}{\partial s}(x, s, l_k\varphi_{21} + \delta(v_{m_k} - l_k\varphi_{21}))ds \cdot (v_{m_k} - l_k\varphi_{21})| \\
&+ \tau_1(x)|u_{m_k} - l_k\varphi_{11}| \\
&\leq 2\tau_1(x)|v_{m_k} - l_k\varphi_{21}| + \tau_1(x)|u_{m_k} - l_k\varphi_{11}|, \ \delta \in (0,1).
\end{aligned}
$$

From this and remark that $\widehat{u}_{m_k} = \frac{u_{m_k}}{l_k}$, $\widehat{v}_{m_k} = \frac{v_{m_k}}{l_k}$, we get:

$$
\begin{aligned}
(2.17) \qquad &|\frac{1}{l_k}\int_0^{u_{m_k}} f(x, s, v_{m_k})ds - \frac{1}{l_k}\int_0^{l_k\varphi_{11}} f(x, s, l_k\varphi_{21})ds| \\
&\leq 2\tau_1(x)|\widehat{v}_{m_k} - \varphi_{21}| + \tau_1(x)|\widehat{u}_{m_k} - \varphi_{11}|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(2.18) \qquad &|\frac{1}{l_k}\int_0^{u_{m_k}} f(x, s, 0)ds - \frac{1}{l_k}\int_0^{l_k\varphi_{11}} f(x, s, 0)ds| \\
&\leq \tau_1(x)|\widehat{u}_{m_k} - \varphi_{11}|.
\end{aligned}
$$

Combining (2.17), (2.18) we infer that

$$
\begin{aligned}
&|\int_\Omega \{\frac{1}{l_k}\int_0^{u_{m_k}} [f(x, s, v_{m_k}) + f(x, s, 0)]ds \\
&- \frac{1}{l_k}\int_0^{l_k\varphi_{11}} [f(x, s, l_k\varphi_{21}) + f(x, s, 0)]ds\}dx| \\
\leq &\int_\Omega \{2\tau_1(x)|(\widehat{v}_{m_k} - \varphi_{21})| + 2\tau_1(x)|\widehat{u}_{m_k} - \varphi_{11}|\}dx \\
\leq &2||\tau_1(x)||_{L^2(\Omega)} \cdot ||\widehat{v}_{m_k} - \varphi_{21}||_{L^2(\Omega)} + 2||\tau_1(x)||_{L^2(\Omega)} \cdot ||\widehat{u}_{m_k} - \varphi_{11}||_{L^2(\Omega)}.
\end{aligned}
$$

Letting $k \to +\infty$, since

$$
\lim_{k\to+\infty} ||\widehat{v}_{m_k} - \varphi_{21}||_{L^2(\Omega)} = 0 \ , \ \lim_{k\to+\infty} ||\widehat{u}_{m_k} - \varphi_{11}||_{L^2(\Omega)} = 0
$$

we deduce that

$$
\begin{aligned}
&\limsup_{k\to+\infty} \int_\Omega \{\frac{1}{l_k}\int_0^{u_{m_k}} [f(x, s, v_{m_k}) + f(x, s, 0)]ds\}dx \\
= &\limsup_{k\to+\infty} \int_\Omega \{\frac{1}{l_k}\int_0^{l_k\varphi_{11}} [f(x, s, l_k\varphi_{21}) + f(x, s, 0)]ds\}dx.
\end{aligned}
$$

Set $s = y\varphi_{11}(x)$, $ds = \varphi_{11}(x)dy$, we get

$$
\begin{aligned}
&\int_0^{l_k\varphi_{11}} [f(x, s, l_k\varphi_{21}) + f(x, s, 0)]ds \\
= &\int_0^{l_k} [f(x, y\varphi_{11}, l_k\varphi_{21}) + f(x, y\varphi_{11}, 0)]\varphi_{11}dy.
\end{aligned}
$$

Remark that $l_k = ||w_{m_k}||_E \to +\infty$ as $k \to +\infty$, hence

$$(2.19) \qquad \limsup_{k\to+\infty} \int_\Omega \{\frac{1}{l_k}\int_0^{u_{m_k}} [f(x,s,v_{m_k}) + f(x,s,0)]ds\}dx$$

$$= \limsup_{k\to+\infty} \int_\Omega \{\frac{1}{l_k}\int_0^{l_k} [f(x,y\varphi_{11},l_k\varphi_{21}) + f(x,y\varphi_{11},0)]dy\}\varphi_{11}dx$$

$$= \int_\Omega F_{11}(x)\varphi_{11}(x)dx.$$

Similarly, we also derive that

(2.20)

$$\limsup_{k\to+\infty} \int_\Omega \{\frac{1}{l_k}\int_0^{v_{m_k}} [g(x,u_{m_k},t) + g(x,0,t)]ds\}dx = \int_\Omega G_{11}(x)\varphi_{21}(x)dx,$$

where $F_{11}(x)$, $G_{11}(x)$ are given in (1.13a).

Combining (2.19), (2.20) we obtain:

$$(2.21) \qquad \limsup_{k\to+\infty} \int_\Omega 2\frac{H(x,w_{m_k})}{||w_{m_k}||_E}dx = \int_\Omega [F_{11}(x)\varphi_{11}(x) + G_{11}(x)\varphi_{21}(x)]dx.$$

By computations as those above we also have

(2.22)

$$\lim_{k\to+\infty}\sup \int_{\partial\Omega} 2h(x)\frac{R(x,w_{m_k})}{||w_{m_k}||_E}dx = \int_{\partial\Omega} h(x)[P_{11}(x)\varphi_{11}(x) + Q_{11}(x)\varphi_{21}(x)]ds,$$

where $P_{11}(x)$, $Q_{11}(x)$ are given in (1.13a).

Lemma 2.1 is proved. □

By Lemma 2.1, from (2.13) and using (2.14), (2.15), we deduce that:

$$\int_\Omega [F_{11}(x)\varphi_{11}(x) + G_{11}(x)\varphi_{21}(x)]dx$$
$$+ \int_{\partial\Omega} h(x)[P_{11}(x)\varphi_{11}(x) + Q_{11}(x)\varphi_{21}(x)]ds$$
$$\geq 2\int_\Omega [k_1(x)\varphi_{11}(x) + k_2(x)\varphi_{21}(x)]dx.$$

That is

$$L_{11} \geq 2\int_\Omega [k_1(x)\varphi_{11} + k_2(x)\varphi_{21}]dx$$

which contradicts (1.15a).

Now, assume that $\widehat{u}_{m_k} \to \widehat{u} = \varphi_{11}$, $\widehat{v}_{m_k} \to \widehat{v} = -\varphi_{21}$ as $k \to +\infty$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$.

Remark that with the similar computations as those used in the proof of the Lemma 2.1, we get

$$\limsup_{k\to+\infty} \int_\Omega \frac{H(x,w_{m_k})}{||w_{m_k}||_E}dx = \frac{1}{2}\int_\Omega [F_{12}(x)\varphi_{11}(x) - G_{12}(x)\varphi_{21}(x)]dx,$$

$$\limsup_{k\to+\infty}\int_{\partial\Omega} h(x)\frac{R(x,w_{m_k})}{||w_{m_k}||_E}ds = \frac{1}{2}\int_{\partial\Omega} h(x)[P_{12}(x)\varphi_{11}(x) - Q_{12}(x)\varphi_{21}(x)]ds.$$

After dividing (2.12) by $||w_{m_k}||_E$, letting $\limsup_{k\to+\infty}$ and remark that

$$\lim_{k\to+\infty}\int_\Omega [k_1(x)\widehat{u}_{m_k}(x) + k_2(x)\widehat{v}_{m_k}(x)]dx = \int_\Omega [k_1(x)\varphi_{11}(x) - k_2(x)\varphi_{21}(x)]dx.$$

We obtain:

$$\begin{aligned}&\int_\Omega [F_{12}(x)\varphi_{11}(x) - G_{12}(x)\varphi_{21}(x)]dx\\ &+\int_{\partial\Omega} h(x)[P_{12}(x)\varphi_{11}(x) - Q_{12}(x)\varphi_{21}(x)]ds\\ \geq\ & 2\int_\Omega [k_1(x)\varphi_{11}(x) - k_2(x)\varphi_{21}(x)]dx.\end{aligned}$$

This implies that

$$L_{12} \geq 2\int_\Omega [k_1(x)\varphi_{11} - k_2(x)\varphi_{21}]dx$$

which contradicts (1.16a).

Similarly, in the cases when $\widehat{u}_{m_k} \to \widehat{u} = -\varphi_{11}$, $\widehat{v}_{m_k} \to \widehat{v} = -\varphi_{21}$ and when $\widehat{u}_{m_k} \to \widehat{u} = -\varphi_{11}$, $\widehat{v}_{m_k} \to \widehat{v} = \varphi_{21}$ we obtain the following respective inequalities

$$L_{22} \leq 2\int_\Omega [k_1(x)\varphi_{11} + k_2(x)\varphi_{21}]dx$$

and

$$L_{21} \geq 2\int_\Omega [k_1(x)\varphi_{11} - k_2(x)\varphi_{21}]dx$$

which contradict (1.15a) and (1.16a).

Now, we consider the case when $\widehat{u}_{m_k} \to \widehat{u} = \varphi_{11}(x)$ and $\widehat{v}_{m_k} \to \widehat{v} = 0$ as $k \to +\infty$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$.

By similar computations as those used in the proof of Lemma 2.1, we obtain

$$\begin{aligned}&\limsup_{k\to+\infty}\int_\Omega \{\frac{1}{l_k}\int_0^{u_{m_k}} [f(x,s,v_{m_k}) + f(x,s,0)]ds\}dx\\ =\ &\limsup_{k\to+\infty}\int_\Omega \{\frac{1}{l_k}\int_0^{l_k} 2f(x,y\varphi_{11},0)dy\}\varphi_{11}(x)dx\end{aligned}$$

and

$$\limsup_{k\to+\infty}\int_\Omega \{\frac{1}{l_k}\int_0^{v_{m_k}} [g(x,u_{m_k},t) + g(x,0,t)]dt\}dx = 0,$$

where $l_k = ||w_{m_k}||_E \to +\infty$ as $k \to +\infty$.

From this and remark (1.11) we arrive at

$$\limsup_{k\to+\infty}\int_\Omega \frac{H(x,w_{m_k})}{||w_{m_k}||_E}dx = \limsup_{k\to+\infty}\int_\Omega \{\frac{1}{l_k}\int_0^{l_k} f(x,y\varphi_{11},0)dy\}\varphi_{11}(x)dx$$

$$= \int_\Omega F_{10}(x)\varphi_{11}(x)dx,$$

where $F_{10}(x)$ is given by (1.13b).

Similarly

$$\limsup_{k\to+\infty} \int_{\partial\Omega} h(x)\frac{R(x,w_{m_k})}{||w_{m_k}||_E}ds = \int_{\partial\Omega} h(x)P_{10}(x)\varphi_{11}(x)ds,$$

where $P_{10}(x)$ is given by (1.13b).

From (2.13) we get

$$\int_\Omega F_{10}(x)\varphi_{11}(x)dx + \int_{\partial\Omega} h(x)P_{10}(x)\varphi_{11}(x)ds \geq \int_\Omega k_1(x)\varphi_{11}(x)dx,$$

which gives

$$\int_\Omega k_1(x)\varphi_{11}(x)dx \leq L_{10}.$$

We get a contradiction with (1.15b).

By same arguments, if $\widehat{u}_{m_k} \to \widehat{u} = -\varphi_{11}(x)$ and $\widehat{v}_{m_k} \to \widehat{v} = 0$ as $k \to +\infty$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$. From (2.12) we obtain:

$$-\int_\Omega F_{20}(x)\varphi_{11}(x)dx - \int_{\partial\Omega} h(x)P_{20}(x)\varphi_{21}(x)ds \geq -\int_\Omega k_1(x)\varphi_{11}(x)dx.$$

This implies

$$L_{20} \leq \int_\Omega k_1(x)\varphi_{11}(x)dx$$

which contradicts (1.15b).

In the cases when $\widehat{u}_{m_k} \to \widehat{u} = 0$ and $\widehat{v}_{m_k} \to \widehat{v} = \pm\varphi_{21}(x)$ as $k \to +\infty$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$, by similar computations used above we arrive at contradictions with (1.16b) as follows:

$$\int_\Omega k_2(x)\varphi_{21}(x)dx \leq L_{01}$$

and

$$\int_\Omega k_2(x)\varphi_{21}(x)dx \geq L_{02}.$$

Thus the functional I given by (2.1) is coercive on E and Proposition 2.2 is proved. □

Proof of Theorem 1.1. By Proposition 2.2 and the weak lower semicontinuity of the functional I (see Remark 2.1), applying the Minimum Principle (see [12, p. 4, Theorem 1.2]), the functional I has a global minimum and by (H_1) the problem (1.1) admits a nontrivial weak solution in E. □

Remark 2.3. Since $||\widehat{w}||_E^2 = ||\widehat{u}||_{E_1}^2 + ||\widehat{v}||_{E_2}^2$, we would also consider more general cases such that $\widehat{u} = a\varphi_{11}$, $\widehat{v} = b\varphi_{21}$, where $a^2 + b^2 = 1$. So the hypotheses (H2) would be changed by following condition more general

$$\begin{aligned}
&\int_\Omega (aF(x)\varphi_{11}(x) + bG(x)\varphi_{21}(x))dx \\
&+ \int_{\partial\Omega} h(x)[aP(x)\varphi_{11}(x) + bQ(x)\varphi_{21}(x)]dx \\
&< 2\int_\Omega (ak_1(x)\varphi_{11}(x) + bk_2(x)\varphi_{21}(x))dx, \ \forall a, b \in R : a^2 + b^2 = 1,
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
F(x) &= \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [f(x, ya\varphi_{11}, \tau b\varphi_{21}) + f(x, ya\varphi_{11}, 0)]dy, \\
G(x) &= \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [g(x, \tau a\varphi_{11}, yb\varphi_{21}) + g(x, 0, yb\varphi_{21})]dy, \\
P(x) &= \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [p(x, ya\varphi_{11}, \tau b\varphi_{21}) + p(x, ya\varphi_{11}, 0)]dy, \\
Q(x) &= \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau [q(x, \tau a\varphi_{11}, yb\varphi_{21}) + q(x, 0, yb\varphi_{21})]dy,
\end{aligned} \tag{2.24}$$

and the proof of Proposition 2.2 is more simple. However, this condition is more strict and difficult to check.

Remark 2.4. Remark if we replace the inequalities in the hypotheses (H_2)-(i), (ii) by following inverse inequalities:

(i)
$$\begin{aligned}
L_{11} &> 2\int_\Omega [k_1(x)\varphi_{11}(x) + k_2(x)\varphi_{21}(x)]dx > L_{22}, \\
L_{12} &> 2\int_\Omega [k_1(x)\varphi_{11}(x) - k_2(x)\varphi_{21}(x)]dx > L_{21}.
\end{aligned}$$

(ii)
$$\begin{aligned}
L_{10} &> \int_\Omega k_1(x)\varphi_{11}(x)dx > L_{20}, \\
L_{01} &> \int_\Omega k_2(x)\varphi_{21}(x)dx > L_{02},
\end{aligned}$$

then by applying Saddle Point Theorem, we can prove that the problem (1.1), (1.2) also has at least weak solution in E.

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