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THE NUMBER OF PANCYCLIC ARCS CONTAINED IN A HAMILTONIAN CYCLE OF A TOURNAMENT

Michel Surmacs

Abstract. A tournament T is an orientation of a complete graph and an arc in T is called pancyclic if it is contained in a cycle of length l for all $3 \leq l \leq n$, where n is the cardinality of the vertex set of T. In 1994, Moon [5] introduced the graph parameter $h(T)$ as the maximum number of pancyclic arcs contained in the same Hamiltonian cycle of T and showed that $h(T) \geq 3$ for all strong tournaments with $n \geq 3$. Havet [4] later conjectured that $h(T) \geq 2k+1$ for all k-strong tournaments and proved the case $k = 2$. In 2005, Yeo [7] gave the lower bound $h(T) \geq \frac{k+5}{2}$ for all k-strong tournaments T. In this note, we will improve his bound to $h(T) \geq \frac{2k+7}{3}$.

1. Introduction and terminology

We use Bang-Jensen and Gutin [1] for terminology and notation not defined here. A *tournament* T is an orientation of a complete graph. We denote by $V(T)$ and $A(T)$ the *vertex set* and *arc set* of T, respectively. For convenience, let n be the cardinality of $V(T)$. If $xy \in A(T)$, we mostly use the notation $x \to y$ to denote this arc.

Let X be a subset of $V(T)$. The subdigraph of T induced by X is denoted by $T[X]$. Instead of $T[V(T) \setminus X]$, we write $T - X$ (or $T - x$ if X contains only a single vertex x). For a vertex $x \in X$, the *out-neighborhood* (*in-neighborhood*, respectively) in $T[X]$ is the set $N^+_{T[X]}(x) = \{y \mid xy \in A(T[X])\}$ ($N^-_{T[X]}(x) = \{y \mid yx \in A(T[X])\}$, respectively). Instead of N^+_T and N^-_T we use N^+ and N^-, respectively. For a subset Y of X, we define $N^+_{T[X]}(Y) = \bigcup_{x\in Y} N^+_{T[X]}(x) \setminus Y$ and $N^-_{T[X]}(Y) = \bigcup_{x\in Y} N^-_{T[X]}(x) \setminus Y$. We call $d^+_{T[X]}(x) = |N^+_{T[X]}(x)|$ ($d^+(x)$, respectively) the *out-degree* and $d^-_{T[X]}(x)$ ($d^-(x)$, respectively) the *in-degree* of a vertex $x \in X$. Also, $\delta^+ = \delta^+(T) = \min\{d^+(x) \mid x \in V(T)\}$ is the *minimum out-degree* in T.

By a *path* or a *cycle*, we mean a directed path or directed cycle. An l-*cycle* in T is a cycle of length l. An n-cycle is also referred to as a *Hamiltonian cycle*.

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An arc $e = xy \in A(T)$ is called an *out-arc* of the vertex x. Furthermore, it is called *pancyclic* in T, if it is contained in an l-cycle for all $3 \le l \le n$.

A *strong component* H of T is a maximal subdigraph such that for any $x, y \in V(H)$, there is a path from x to y in H and vice versa. If a tournament T has only one strong component, we call it *strongly connected* or *strong*. T is called *k-strong*, if $T - X$ is strong for any $X \subseteq V(T)$ with at most $k-1$ vertices.

In 1994, Moon [5] introduced the graph parameter $h(T)$ as the maximum number of pancyclic arcs contained in the same Hamiltonian cycle of a tournament T and proved the next theorem.

Theorem 1.1 ([5]). *Let T be a strong tournament on $n \ge 3$ vertices. Then*

$$h(T) \ge 3.$$

The subject was studied for k-strong tournaments with $k \ge 2$ by Havet [4], in 2004. With $h_k(n)$ defined as

$$h_k(n) := \min\{h(T) \mid T \text{ is a } k\text{-strong tournament of order } n\}$$

(or $h_k(n) := \infty$, respectively, if there is no k-strong tournament of order n), he gave the following conjecture.

Conjecture 1.2 ([4]). *$h_k(n) \ge 2k + 1$. Given a sufficiently large integer n, $h_k(n) = 3k$ holds.*

Furthermore, he proved his lower bound conjecture for $k = 2$.

Theorem 1.3 ([4]). *Let T be a 2-strong tournament. Then*

$$h(T) \ge 5.$$

The best known lower bound for $k \ge 3$, prior to this note, is due to Yeo [7], who showed the following, in 2005.

Theorem 1.4 ([7]). *$h_k(n) \ge \frac{k+5}{2}$ for $k \ge 1$.*

In this note, we will improve his bound to $h_k(n) \ge \frac{2k+7}{3}$ for $k \ge 1$.
But first we will give some results on tournaments we will use to prove our proposition.

2. Preliminaries

We begin with a well-known theorem by Camion [2].

Theorem 2.1 ([2]). *A tournament is strong if and only if it has a Hamiltonian cycle.*

Theorem 2.2 ([6]). *If e is an arc of a 3-strong tournament, then e is contained in a Hamiltonian cycle of T.*

Lemma 2.3 ([7]). *Let T be a 2-strong tournament, containing an arc $e = xy$, such that $d^+(x) \le d^+(y)$. Then e is pancyclic in T.*

Theorem 2.4 ([7])**.** *Let T be a 3-strong tournament, containing an arc $e = xy$, such that $d^+(x) = \delta^+(T)$ and $d^+(y) = \min\{d^+(w) \mid w \in N^+(x)\}$. Then all out-arcs of x and all out-arcs of y are pancyclic.*

Lemma 2.5 ([3])**.** *Let T be a 3-strong tournament, containing an arc $e = xy$, such that $d^+(x) = \delta^+(T)$, $d^+(y) = \min\{d^+(w) \mid w \in N^+(x)\}$ and $|\{v \in V(T) \mid d^+(v) = \delta^+(T)\}| \leq 2$. If*

A. $N^+(x) \cap N^+(y) \neq \emptyset$ *or*

B. $N^+(x) \cap N^+(N^+(y)) \neq \emptyset$

then there exists a vertex $z \in N^+(x) \setminus \{y\}$, such that all out-arcs of z are pancyclic.

Lemma 2.6. *Let T be a k-strong tournament and let S be a subset of $V(T)$ with $s = |S| \leq k - 2$. If e is a pancyclic arc in $T - S$, then e is pancyclic in T.*

Proof. Since e is pancyclic in $T - S$, e is contained in cycles of length l for all $3 \leq l \leq n - s$. For $n - s + 1 \leq l \leq n$, let S' be a subset of S with $|S'| = n - l \leq n - (n - s + 1) = s - 1 \leq k - 3$. Then $T - S'$ is 3-strong and therefore, by Theorem 2.2, e is contained in a Hamiltonian cycle of $T - S'$, which is a cycle of length l in T. □

Lemma 2.7. *Let T be a k-strong ($k \geq 5$) tournament, containing three vertices x_0, x_1, x_2, such that all out-arcs of x_0, x_1 and x_2 are pancyclic. If $T[\{x_0, x_1, x_2\}]$ is a 3-cycle and $h(T - \{x_0, x_1, x_2\}) \geq \frac{2(k-3)+7}{3}$, then $h(T) \geq \frac{2k+7}{3}$.*

Proof. Without loss of generality, we may assume that $C_3 = x_0x_1x_2x_0$ is a 3-cycle in T. As a direct consequence of $h(T - \{x_0, x_1, x_2\}) \geq \frac{2(k-3)+7}{3}$, there is a Hamiltonian cycle $C_H = v_0v_1 \cdots v_{n-4}v_0$ of $T - \{x_0, x_1, x_2\}$ that contains at least $\frac{2(k-3)+7}{3}$ arcs which are pancyclic in $T - \{x_0, x_1, x_2\}$. By Lemma 2.6, these arcs are also pancyclic in T.

Suppose that there are indices $i \in \{0, \ldots, n-4\}$ and $j \in \{0, 1, 2\}$, such that $v_i \to x_j$ and $x_{j+2 \mod 3} \to v_{i+1 \mod n-3}$. Without loss of generality, we may assume that $i = j = 0$ (up to rotation and relabeling of C_H and C_3). Then $v_0x_0x_1x_2v_1 \cdots v_{n-4}v_0$ is a Hamiltonian cycle of T that contains at least $\frac{2(k-3)+7}{3} - 1 + 3 = \frac{2k+7}{3}$ pancyclic arcs, since we lose only the arc v_0v_1 and gain 3 pancyclic out-arcs of x_0, x_1 and x_2 in comparison to C_H.

Suppose now that there are no such indices and let us denote this property by ($\star$). If $|N^+(v_{i_0}) \cap \{x_0, x_1, x_2\}| = 3$ for some index $i_0 \in \{0, \ldots, n-4\}$, we have $v_i \to \{x_0, x_1, x_2\}$ for all $i \in \{0, \ldots, n-4\}$, by ($\star$). Hence, T is not strong, a contradiction. By symmetry, the same contradiction is reached if $|N^+(v_{i_0}) \cap \{x_0, x_1, x_2\}| = 0$ (i.e., $|N^-(v_{i_0}) \cap \{x_0, x_1, x_2\}| = 3$) for some index $i_0 \in \{0, \ldots, n-4\}$. It follows that $|N^+(v_i) \cap \{x_0, x_1, x_2\}| \in \{1, 2\}$ for all $i \in \{0, \ldots, n-4\}$. Without loss of generality, we may assume that $N^+(v_0) \cap \{x_0, x_1, x_2\} = \{x_0, x_1\}$. Otherwise, we consider the converse tournament (where the direction of all arcs is reversed) and/or relabel the vertices x_0,

x_1 and x_2, respectively. Because of $(\star)$ and $|N^+(v_i) \cap \{x_0, x_1, x_2\}| \in \{1, 2\}$, we then have $N^+(v_i) = \{x_{-i \mod 3}, x_{1-i \mod 3}\}$ for all $i \in \{0, \ldots, n-4\}$.

Suppose that there is an index $i \in \{0, \ldots, n-4\}$, such that $v_i v_{i+1 \mod n-3}$ is not pancyclic in T. Without loss of generality, we may assume that $i = 0$. Then $C = v_0 x_0 x_1 v_1 \cdots v_{n-4} v_0$ is a Hamiltonian cycle of $T - x_2$ that contains at least $\frac{2(k-3)+7}{3} + 2 = \frac{2k+7}{3}$ pancyclic arcs. We relabel $C = w_0 \ldots w_{n-2} w_0$. Then, since T is strong, there is an index $j \in \{0, \ldots, n-2\}$, such that $w_j \to x_2 \to w_{j+1 \mod n-1}$. Therefore, we can insert x_2 into C. Thereby, we lose at most one pancyclic arc (specifically $w_j w_{j+1 \mod n-1}$) of C and gain another one (the out-arc of x_2). Thus, we obtain a Hamiltonian cycle of T that still contains at least $\frac{2k+7}{3}$ pancyclic arcs.

If all arcs of C_H are pancyclic, then C_H contains $n-3 \geq 2k+1-3 \geq k+3 \geq \frac{2k+7}{3}$ pancyclic arcs, since T is k-strong ($k \geq 5$). As seen above, we can insert x_0, x_1 and x_2 individually into C_H without reducing the number of pancyclic arcs contained in the resulting cycle. □

3. Main result

Theorem 3.1.

$$h_k(n) \geq \frac{2k+7}{3} \textit{ for } k \geq 1.$$

Proof. We will prove the theorem by induction on k. We already know that it is true for $k \in \{1, 2, 3, 4\}$ from Theorem 1.1 and Theorem 1.3. So let T be a k-strong tournament with $k \geq 5$ and let $M = \{x \in V(T) \mid d^+(x) = \delta^+(T)\}$.

Case 1. $|M| \geq 3$. Let $x_0, x_1, x_2 \in M$.

By Lemma 2.3, we have that all out-arcs of x_1, x_2 and x_3 are pancyclic. Since T is k-strong ($k \geq 5$), $T - \{x_0, x_1, x_2\}$ is a $(k-3)$-strong tournament. Thus, by the induction hypothesis, there is a Hamiltonian cycle $C_H = v_0 v_1 \cdots v_{n-4} v_0$ in $T - \{x_0, x_1, x_2\}$ that contains at least $\frac{2(k-3)+7}{3}$ arcs which are pancyclic in $T - \{x_0, x_1, x_2\}$. By Lemma 2.6, these arcs are also pancyclic in T. If $T[\{x_0, x_1, x_2\}]$ is a 3-cycle, then Lemma 2.7 gives us the result.

Thus, without loss of generality, we may assume that $T[\{x_0, x_1, x_2\}]$ is not a 3-cycle and $x_i \to x_j$ for all $0 \leq i < j \leq 2$. If there is no index $i \in \{0, \ldots, n-4\}$, such that $x_2 \to v_{i+1 \mod n-3}$ and $v_i \to x_0$, then x_0 has at least as many out-neighbors in $V(T) \setminus \{x_0, x_1, x_2\}$ as x_2. Therefore, we have

$$\begin{aligned} d^+(x_0) &= d^+_{T-\{x_1,x_2\}}(x_0) + d^+_{T[\{x_0,x_1,x_2\}]}(x_0) \\ &= d^+_{T-\{x_1,x_2\}}(x_0) + 2 > d^+_{T-\{x_0,x_1\}}(x_2) + 0 \\ &= d^+_{T-\{x_0,x_1\}}(x_2) + d^+_{T-\{x_0,x_1,x_2\}}(x_2) = d^+(x_2), \end{aligned}$$

a contradiction to $d^+(x_0) = d^+(x_2) = \delta^+(T)$. Thus, there is such an index i and we obtain a Hamiltonian cycle of T that contains at least $\frac{2k+7}{3}$ pancyclic arcs as in Lemma 2.7.

Case 2. $|M| \leq 2$.

If $|M| = 2$, let $x_0, x_1 \in M$, such that $x_0 \to x_1$. If $|M| = 1$, let $x_0 \in M$ and $x_1 \in N^+(x_0)$, such that $d^+(x_1) = \min\{d^+(w) \mid w \in N^+(x_0)\}$. From Theorem 2.4 we know that all out-arcs of x_0 and x_1 are pancyclic.

Suppose that $N^+(x_0) \cap N^+(x_1) \neq \emptyset$ or $N^+(x_0) \cap N^+(N^+(x_1)) \neq \emptyset$. Then Lemma 2.5 gives us the existence of an $x_2 \in N^+(x_0) \setminus \{x_1\}$, such that all out-arcs of x_2 are pancyclic. Without loss of generality, we may assume that $x_1 \to x_2$. Otherwise, we simply swap x_1 and x_2. Thus, we have $x_i \to x_j$ for all $0 \leq i < j \leq 2$, a Hamiltonian cycle $C_H = v_0 v_1 \cdots v_{n-4} v_0$ of $T - \{x_0, x_1, x_2\}$ that contains at least $\frac{2(k-3)+7}{3}$ pancyclic arcs (by the induction hypothesis) and $d^+(x_0) \leq d^+(x_2)$. Our proposition follows as in Case 1.

Suppose now that $N^+(x_0) \cap N^+(x_1) = N^+(x_0) \cap N^+(N^+(x_1)) = \emptyset$ and let us denote this property by $(\star\star)$. Let $x_2 \in N^+(x_1)$, such that $d^+(x_2) = \min\{d^+(w) \mid w \in N^+(x_1)\}$ and let $C_H = v_0 v_1 \cdots v_{n-4} v_0$ be a Hamiltonian cycle of $T - \{x_0, x_1, x_2\}$ that contains at least $\frac{2(k-3)+7}{3}$ pancyclic arcs, whose existence is guaranteed by the induction hypothesis. By $(\star\star)$, we then have $x_2 \to x_0$. From $d^+(x_2) > \delta^+(T) = d^+(x_0)$, it follows that x_2 has more out-neighbors in $V(T) \setminus \{x_0, x_1, x_2\}$ than x_0. Consequently, as seen above, there is an index $i \in \{0, \ldots, n-4\}$, such that $x_2 \to v_{i+1 \mod n-3}$ and $v_i \to x_0$. Without loss of generality, we may assume that $i = 0$ (up to rotation and relabeling of C_H).

Let us now consider the Hamiltonian cycle $C = v_0 x_0 x_1 x_2 v_1 \cdots v_{n-4} v_0$ of T. If $v_0 v_1$ is not pancyclic in $T - \{x_0, x_1, x_2\}$, then C contains at least $\frac{2(k-3)+7}{3} + 2 = \frac{2k+7}{3}$ pancyclic arcs (those of C_H plus the out-arcs of x_0 and x_1). Suppose that $v_0 v_1$ is pancyclic in $T - \{x_0, x_1, x_2\}$ (and therefore in T). We will show that the arc $x_2 v_1$, then, is also pancyclic in T. Consequently, C then contains at least $\frac{2(k-3)+7}{3} - 1 + 3 = \frac{2k+7}{3}$ pancyclic arcs and we have finished.

The pancyclicity of $v_0 v_1$ guarantees the existence of an l-cycle

$$v_0 v_1 w_2 \cdots w_{l-1} v_0$$

in $T - \{x_0, x_1, x_2\}$ for all $3 \leq l \leq n-3$. Then $x_2 v_1 w_2 \cdots w_{l-1} v_0 x_0 x_1 x_2$ is an $(l+3)$-cycle in T that contains $x_2 v_1$ for all $3 \leq l \leq n-3$. Thus, all that remains to be shown is that $x_2 v_1$ is contained in cycles of length 3 to 5 in T.

If $x_1 \to v_1$, then we have $d^+(v_1) \geq d^+(x_2) = \min\{d^+(w) \mid w \in N^+(x_1)\}$ and thus, $x_2 v_1$ is already pancyclic, by Lemma 2.3. Hence, we may assume that $v_1 \to x_1$. Then $x_2 v_1$ is contained in the 3-cycle $x_2 v_1 x_1 x_2$. Furthermore, $(\star\star)$ gives us $v_1 \to x_0$ and thus, $x_2 v_1$ is contained in the 4-cycle $x_2 v_1 x_0 x_1 x_2$. Finally, if $v_2 \to x_0$, then $x_2 v_1$ is contained in the 5-cycle $x_2 v_1 v_2 x_0 x_1 x_2$. Otherwise, i.e., $x_0 \to v_2$, we have $v_2 \to x_1$ by $(\star\star)$, and therefore $x_2 v_1$ is contained in the 5-cycle $x_2 v_1 x_0 v_2 x_1 x_2$. □

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Lehrstuhl C für Mathematik
RWTH Aachen University
52062 Aachen, Germany
E-mail address: michel.surmacs@rwth-aachen.de