Bull. Korean Math. Soc. **51** (2014), No. 6, pp. 1625–1647
http://dx.doi.org/10.4134/BKMS.2014.51.6.1625

# INCLUSION AND NEIGHBORHOOD PROPERTIES OF CERTAIN SUBCLASSES OF $p$-VALENT ANALYTIC FUNCTIONS OF COMPLEX ORDER INVOLVING A LINEAR OPERATOR

Ashok Kumar Sahoo and Jagannath Patel

Abstract. By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relationships associated with the $(n, \delta)$-neighborhoods of certain subclasses of $p$-valent analytic functions of complex order with missing coefficients, which are introduced here by means of the Saitoh operator. Special cases of some of the results obtained here are shown to yield known results.

## 1. Introduction

Let $\mathcal{A}_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (p, n \in \mathbb{N} = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and $p$-valent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $\mathcal{A}_p(1) = \mathcal{A}_p$ and $\mathcal{A}_1(1) = \mathcal{A}$.

For functions $f$ and $g$, analytic in $\mathbb{U}$, we say that $f$ is subordinate to $g$, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $\omega$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [14]; see also [15]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f_j(z) = \sum_{k=0}^{\infty} a_{k,j} z^k$ $(j = 1, 2)$ analytic in $\mathbb{U}$, we define the Hadamard product (or convolution) of $f_1$ and $f_2$ by

$$(f_1 \star f_2)(z) = \sum_{k=0}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 \star f_1)(z) \quad (z \in \mathbb{U}).$$

Received August 29, 2013; Revised April 6, 2014.
2010 *Mathematics Subject Classification.* 30C45.
*Key words and phrases.* $p$-valent analytic functions, complex order, inclusion relationships, Hadamard product, quasi-Hadamard product, subordination, neighborhood.

©2014 Korean Mathematical Society

A function $f \in \mathcal{A}_p(n)$ is said to be $p$-valently starlike of complex order $b$ and type $\rho$, that is, $f \in \mathcal{S}^*_{p,n}(b,\rho)$, if it satisfies the inequality:

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \rho \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, 0 \leq \rho < p; z \in \mathbb{U}). \tag{1.2}$$

Analogously, a function $f \in \mathcal{A}_p(n)$ is said to be $p$-valently convex of complex order $b$ and type $\rho$, that is, $f \in \mathcal{C}_{p,n}(b,\rho)$, if it satisfies the inequality:

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > \rho \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, 0 \leq \rho < p; z \in \mathbb{U}). \tag{1.3}$$

From (1.2) and (1.3), it follows that

$$f \in \mathcal{C}_{p,n}(b,\rho) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*_{p,n}(b,\rho).$$

In particular, for $p = n = 1$, the classes $\mathcal{S}^*_{p,n}(b,\rho)$ and $\mathcal{C}_{p,n}(b,\rho)$ reduce to the classes $\mathcal{S}^*(b,\rho)$ and $\mathcal{C}(b,\rho)$ of starlike functions of complex order $b$ and type $\rho$, and convex function of complex order $b$ and type $\rho\,(b \in \mathbb{C}^*; 0 \leq \rho < p)$, respectively, which were introduced by Frasin [9].

Setting $\rho = 0$ in $\mathcal{S}^*(b,\rho)$ and $\mathcal{C}(b,\rho)$, we get the classes $\mathcal{S}^*(b)$ and $\mathcal{C}(b)$. These classes of starlike and convex functions of order $b$ were considered earlier by Nasr and Aouf [17] and Wiatrowski [27], respectively (see also [8] and [26]). We further observe that $\mathcal{S}^*_{p,1}(1,\rho) = \mathcal{S}^*_p(\rho)$ and $\mathcal{C}_{p,1}(1,\rho) = \mathcal{C}_p(\rho)$ are, respectively, the classes of $p$-valently starlike and $p$-valently convex functions of order $\rho\,(0 \leq \rho < p)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}^*_1(\rho) = \mathcal{S}^*(\rho)$ and $\mathcal{C}_1(\rho) = \mathcal{C}(\rho)$ are the usual classes of starlike and convex functions of order $\rho\,(0 \leq \rho < 1)$ in $\mathbb{U}$. In the special cases, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$.

Furthermore, let $\mathcal{R}_{p,n}(b,\rho)$ denote the class of functions in $\mathcal{A}_p(n)$ satisfying the condition:

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(\frac{f'(z)}{z^{p-1}} - p\right)\right\} > \rho \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, 0 \leq \rho < p; z \in \mathbb{U}).$$

We note that $\mathcal{R}_{p,n}(1,\rho)$ is a subclass of $p$-valently close-to-convex functions of order $\rho\,(0 \leq \rho < p)$ in the unit disc $\mathbb{U}$.

Let $\varphi_p$ be the incomplete beta function defined by

$$\varphi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (z \in \mathbb{U}), \tag{1.4}$$

where $a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$ and the symbol $(x)_k$ denotes the Pochhammer symbol (or shifted factorial) given by

$$(x)_k = \begin{cases} 1, & (k = 0, x \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}) \\ x(x+1)\cdots(x+k-1), & (k \in \mathbb{N}, x \in \mathbb{C}). \end{cases}$$

With the aid of the function $\varphi_p$, given by (1.4) and the Hadamard product, we consider the linear operator $\mathcal{L}_p(a,c) : \mathcal{A}_p(n) \longrightarrow \mathcal{A}_p(n)$ defined by

$$\mathcal{L}_p(a,c)f(z) = \varphi_p(a,c;z) \star f(z) \qquad (z \in \mathbb{U}). \tag{1.5}$$

If $f$ is given by (1.1), then from (1.5), it readily follows that

$$\mathcal{L}_p(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \qquad (z \in \mathbb{U}). \tag{1.6}$$

The linear operator $\mathcal{L}_p(a,c)$ on the class $\mathcal{A}_p$ was studied by Saitoh [23], which generalizes the linear operator $\mathcal{L}_1(a,c) = \mathcal{L}(a,c)$ introduced by Carlson and Shaffer [7] in their systematic investigation of certain interesting classes of starlike, convex and prestarlike hypergeometric functions.

It follows from (1.6) that

$$z\left(\mathcal{L}_p(a,c)f\right)'(z) = a\mathcal{L}_p(a+1,c)f(z) - (a-p)\mathcal{L}_p(a,c)f(z) \quad (f \in \mathcal{A}_p(n); z \in \mathbb{U}).$$

We also note that for $f \in \mathcal{A}_p$,

(i) $\mathcal{L}_p(a,a)f(z) = f(z)$;

(ii) $\mathcal{L}_p(p+1,p)f(z) = \frac{zf'(z)}{p}$;

(iii) $\mathcal{L}_p(p+2,p)f(z) = \frac{z^2f''(z)+2zf'(z)}{p(p+1)}$;

(iv) $\mathcal{L}_p(m+p,1)f(z) = D^{m+p-1}f(z)$ $(m \in \mathbb{Z},\ m > -p)$, the operator studied by Goel and Sohi [10]. In the case $p = 1$, $D^m f$ is the familiar Ruscheweyh derivative [21] of $f \in \mathcal{A}$.

(v) $\mathcal{L}_p(\nu+p,1)f(z) = D^{\nu,p}f(z)$ $(\nu > -p)$, an extended linear derivative operator of Ruscheweyh type introduced by Raina and Srivastava [20]. In particular, when $\nu = m$, we get the operator $D^{m+p-1}f(z)$ $(m \in \mathbb{Z},\ m > -p)$.

(vi) $\mathcal{L}_p(p+1,m+p)f(z) = \mathcal{I}_{m,p}f(z)$ $(m \in \mathbb{Z},\ m > -p)$, the extended Noor integral operator considered by Liu and Noor [13].

(vii) $\mathcal{L}_p(p+1,p+1-\lambda)f(z) = \Omega_z^{(\lambda,p)}f(z)$ $(-\infty < \lambda < p+1)$, the extended fractional differintegral operator considered by Patel and Mishra [19]. Note that

$$\Omega_z^{(0,p)}f(z) = f(z), \Omega_z^{(1,p)}f(z) = \frac{zf'(z)}{p} \quad \text{and} \quad \Omega_z^{(2,p)}f(z) = \frac{z^2f''(z)}{p(p-1)}\ (p \geq 2).$$

Now, by using the operator $\mathcal{L}_p(a,c)$, we introduce the following new subclasses of $p$-valent analytic functions in the unit disk $\mathbb{U}$.

**Definition 1.** A function $f \in \mathcal{A}_p(n)$ is said to be in the class $\mathcal{S}^b_{p,n}(a,c,\rho)$, if it satisfies the following inequality:

$$\left|\frac{1}{b}\left\{\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{\mathcal{L}_p(a,c)f(z)} - p\right\}\right| < p - \rho \quad (b \in \mathbb{C}^*, 0 \leq \rho < p; z \in \mathbb{U}). \tag{1.7}$$

Using the definition of subordination, it is easily seen that (1.7) is equivalent to the following subordination condition:

$$\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{\mathcal{L}_p(a,c)f(z)} \prec p+b(p-\rho)z \quad (b\in\mathbb{C}^*, 0\le\rho<p; z\in\mathbb{U}).$$

**Definition 2.** A function $f\in\mathcal{A}_p(n)$ is said to be in the class $\mathcal{R}^b_{p,n}(a,c,\mu,\rho)$, if it satisfies the following inequality:

$$\left|\frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\mu\frac{\left(L_p(a,c)f\right)'(z)}{z^{p-1}}-p\right\}\right|<p-\rho \tag{1.8}$$
$$(b\in\mathbb{C}^*, 0\le\mu\le 1, 0\le\rho<p; z\in\mathbb{U}).$$

Analogously, (1.8) is equivalent to the following subordination condition:

$$\left\{p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\mu\frac{\left(L_p(a,c)f\right)'(z)}{z^{p-1}}\right\}\prec p+b(p-\rho)z$$
$$(b\in\mathbb{C}^*, 0\le\mu\le 1, 0\le\rho<p; z\in\mathbb{U}).$$

It may be noted that for suitable choices of the parameters involved in Definition 1 and Definition 2, the classes $\mathcal{S}^b_{p,n}(a,c,\rho)$ and $\mathcal{R}^b_{p,n}(a,c,\lambda,\rho)$ extend several subclasses of $p$-valent analytic functions in the unit disc $\mathbb{U}$. For instance,

**Example 1.**

$$\begin{aligned}
&\mathcal{S}^b_{p,n}(p+1,p+1-\lambda,\rho)\\
&=\mathcal{S}^b_{p,n}(\lambda,\rho)\ (b\in\mathbb{C}^*, -\infty<\lambda<p+1,\ 0\le\rho<p)\\
&=\left\{f\in\mathcal{A}_p(n):\left|\frac{1}{b}\left(\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}-p\right)\right|<p-\rho,\ z\in\mathbb{U}\right\},
\end{aligned}$$

which reduces to the class $\mathcal{K}_n(p,\lambda,b,\beta)$ $(b\in\mathbb{C}^*,\ 0\le\lambda\le1,\ 0<\beta\le1)$ studied by Aouf [3] for $\rho=p-\beta$, the class

$$\mathcal{S}^b_{p,n}(\rho)=\left\{f\in\mathcal{A}_p(n):\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right|<p-\rho,\ b\in\mathbb{C}^*,\ 0\le\rho<p; z\in\mathbb{U}\right\}$$

for $\lambda=0$ and the class

$$\mathcal{C}^b_{p,n}(\rho)=\left\{f\in\mathcal{A}_p:\left|\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right|<p-\rho,\ b\in\mathbb{C}^*,\ 0\le\rho<p; z\in\mathbb{U}\right\}$$

for $\lambda=1$. The classes $\mathcal{S}^b_{p,n}(\rho)$ and $\mathcal{C}^b_{p,n}(\rho)$ are the subclasses of $p$-valently starlike and $p$-valently convex functions of complex order $b$ and type $\rho\,(b\in\mathbb{C}^*,\ 0\le\rho<p)$ in $\mathbb{U}$.

**Example 2.**

$$\begin{aligned}
&\mathcal{R}^b_{p,n}(p+1,p+1-\lambda,\mu,\rho)\\
&=\mathcal{R}^b_{p,n}(\lambda,\mu,\rho)\ (b\in\mathbb{C}^*,\ -\infty<\lambda<p,\ 0\le\mu,\ 0\le\rho<p)
\end{aligned}$$

$$= \left\{ f \in \mathcal{A}_p(n) : \left| \left( (p(1-\mu)+\mu\lambda) \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} + \mu(p-\lambda) \frac{\Omega_z^{(1+\lambda,p)} f(z)}{z^p} - p \right) \right| < p-\rho; z \in \mathbb{U} \right\}$$

$$= \left\{ f \in \mathcal{A}_p(n) : \left| \frac{1}{b} \left( p(1-\mu) \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} + \mu \frac{(\Omega_z^{(\lambda,p)} f)'(z)}{z^p} - p \right) \right| < p-\rho; z \in \mathbb{U} \right\},$$

which yields the class considered by Aouf [3] for $\rho = p - \beta\,(0 < \beta \leq 1)$.

Special cases of the parameters $p, \lambda$ and $\rho$ in the class $\mathcal{R}^b_{p,n}(\lambda, \mu, \rho)$ yields

(i)

$$\mathcal{R}^b_{p,n}(0, \mu, \rho) = \mathcal{R}^b_{p,n}(\mu, \rho)$$
$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left( p(1-\mu) \frac{f(z)}{z^p} + \mu \frac{f'(z)}{z^{p-1}} - p \right) \right| < p-\rho,\ \mu \geq 0,\ 0 \leq \rho < p; z \in \mathbb{U} \right\}.$$

(ii)

$$\mathcal{R}^b_{p,n}(1, \mu, \rho) = \mathcal{P}^b_{p,n}(\mu, \rho)$$
$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left( (\mu + \mu(1-p)) \frac{f'(z)}{pz^{p-1}} + \mu \frac{f''(z)}{pz^{p-2}} - p \right) \right| < p-\rho,\ \mu \geq 0,\ 0 \leq \rho < p; z \in \mathbb{U} \right\}.$$

(iii)

$$\mathcal{R}^b_{1,n}(1, \mu, 1-\beta) = \mathcal{R}^b_n(\mu, \beta)$$
$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left( f'(z) + \mu z f''(z) - 1 \right) \right| < \beta, \mu \geq 0, 0 < \beta \leq 1; z \in \mathbb{U} \right\}.$$

The class $\mathcal{R}^b_n(\mu, \beta)$ was studied by Altintas et al. [6].

Let $\mathcal{T}_p(n)$ be the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0;\ p, n \in \mathbb{N}). \tag{1.9}$$

We write $\mathcal{T}_1(1) = \mathcal{T}$.

We denote by $\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$, $\widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho)$, $\widetilde{\mathcal{S}}^b_{p,n}(\rho)$ and $\widetilde{\mathcal{C}}^b_{p,n}(\rho)$, respectively, the classes obtained by taking the intersections of $\mathcal{S}^b_{p,n}(a, c, \rho)$, $\mathcal{R}^b_{p,n}(a, c, \mu, \rho)$, $\mathcal{S}^b_{p,n}(\rho)$ and $\mathcal{C}^b_{p,n}(\rho)$ with $\mathcal{T}_p(n)$. We also observe that

$$\widetilde{\mathcal{S}}^1_{1,1}(\rho) = \widetilde{\mathcal{S}}(\rho) \quad \text{and} \quad \widetilde{\mathcal{C}}^1_{1,1}(\rho) = \widetilde{\mathcal{C}}(\rho)\ (0 \leq \rho < 1)$$

are the subclasses of $\mathcal{T}$ studied by Silverman [24].

Various further subclasses of $\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ and $\widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho)$ were studied in many earlier works (cf., e.g., [5], [6], [16], [20] and [25]; see also the references cited in these earlier works).

The object of the present paper is to investigate various properties and characteristics of functions belonging to the subclasses $\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ and $\widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho)$, which are introduced here by means of the Saitoh operator. Apart from deriving a set of coefficient inequalities, we establish several inclusion relationships associated with the $(n, \delta)$-neighborhoods of functions belonging to these subclasses of $p$-valent analytic functions (with missing coefficients) of complex order. Relevant connections of the results obtained here with some earlier investigations are also pointed out.

## 2. Preliminaries

To prove our main results, we need the following definition and lemmas.

**Definition 3.** A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if for any $g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{C}$

$$\sum_{k=1}^{\infty} c_k d_k z^k \prec g(z) \quad (d_1 = 1; z \in \mathbb{U}).$$

Wilf [28] established the following criterion for a sequence of complex numbers to be a subordinating factor sequence.

**Lemma 1.** *A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \quad (z \in \mathbb{U}). \tag{2.1}$$

We give a necessary and sufficient condition for a function in $\mathcal{T}_p(n)$ to be in the class $\widetilde{\mathcal{S}}_{p,n}^{b}(a, c, \rho)$.

**Lemma 2.** *Let the function $f$ be given by* (1.9). *Then $f \in \widetilde{\mathcal{S}}_{p,n}^{b}(a, c, \rho)$ if and only if*

$$\sum_{k=n}^{\infty} \frac{k + (p-\rho)|b|}{(p-\rho)|b|} \frac{|(a)_k|}{|(c)_k|} a_{p+k} \leq 1. \tag{2.2}$$

*The result in* (2.2) *is sharp.*

*Proof.* Suppose $f$, given by (1.9) belongs to the class $\widetilde{\mathcal{S}}_{p,n}^{b}(a, c, \rho)$. Then from (1.7), it follows that

$$\operatorname{Re}\left\{\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{\mathcal{L}_p(a,c)f(z)} - p\right\} > -(p-\rho)|b| \quad (0 \leq \rho < p; z \in \mathbb{U}),$$

and by using the series expansion of $\mathcal{L}_p(a, c)f(z)$ (c.f., Eqn.(1.6)) in the above expression, we obtain

$$\operatorname{Re}\left\{\frac{-\displaystyle\sum_{k=n}^{\infty} k\frac{(a)_k}{(c)_k} a_{p+k} z^k}{1 - \displaystyle\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^k}\right\} > -(p-\rho)|b| \quad (z \in \mathbb{U}). \tag{2.3}$$

Setting $|z| = r(0 \leq r < 1)$ in (2.3) and noting the fact that for $r = 0$, the resulting expression in the denominator is positive, and remains so for all $r \in$

$(0,1)$, the desired inequality (2.2) follows upon letting $r \to 1^-$ through real values.

To prove the converse, we let $|z| = 1$. Then by using (1.6), we find that

$$\begin{aligned}
\left|\left\{\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{\mathcal{L}_p(a,c)f(z)} - p\right\}\right| &= \left\{\frac{-\sum\limits_{k=n}^{\infty} k\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}{1-\sum\limits_{k=n}^{\infty}\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}\right\} \\
&\leq \left\{\frac{\sum\limits_{k=n}^{\infty} k\dfrac{|(a)_k|}{|(c)_k|}a_{p+k}}{1-\sum\limits_{k=n}^{\infty}\dfrac{|(a)_k|}{|(c)_k|}a_{p+k}}\right\} \quad (z \in \mathbb{U}).
\end{aligned} \tag{2.4}$$

The expression in (2.4) is bounded by $(p-\rho)|b|$, provided

$$\sum_{k=n}^{\infty} \frac{k\,|(a)_k|}{|(c)_k|}a_{p+k} \leq (p-\rho)|b|\left(1-\sum_{k=n}^{\infty}\frac{|(a)_k|}{|(c)_k|}a_{p+k}\right)$$

which is certainly true by using the maximum modulus theorem and the assertion (2.2). Thus, in view of (1.7), we deduce that $f \in \widetilde{\mathcal{S}}^b_{p,n}(a,c,\rho)$, which evidently completes the proof of Lemma 2. $\square$

The result in (2.2) is sharp for the functions

$$f_k(z) = z^p - \frac{(p-\rho)|b||(c)_k|}{k+(p-\rho)|b||(a)_k|}z^{p+k} \quad (k \geq n, a, c \in \mathbb{R}\setminus\mathbb{Z}_0^-; z \in \mathbb{U}).$$

*Remark* 1. (i) A special case of Lemma 2 when $a = \lambda + 1$ $(\lambda > -1)$ and $p = n = b = c = 1$ was given earlier by Ahuja [1]. Further, if in Lemma 2 with $a = 2$, $n = p = b = 1$, we set $c = 2$ and $c = 1$, we shall obtain the familiar results of Silverman [24].

(ii) Putting $a = p+1$, $c = p+1-\lambda$ $(-\infty < \lambda < p+1)$ and $\rho = (p-\beta)$ $(0 < \beta \leq 1)$ in Lemma 2, we get the result obtained by Aouf [3, Lemma 1]. Also, in Lemma 2 with $a = p+1$, $n = b = 1$, if we set $c = p+1$ and $c = p$, we obtain the results of Owa [18].

(iii) Setting $a = \lambda + 1$ $(\lambda > -1)$, $c = p = 1$ and $\rho = 1 - \beta$ $(0 < \beta \leq 1)$ in Lemma 2, we get the result obtained by Murugusundarmoorthy and Srivastava [16].

Our proof of Lemma 3 given below is much akin to that of Lemma 2, so we omit the details.

**Lemma 3.** *Let the function $f$ be defined by* (1.9)*. Then $f \in \widetilde{\mathcal{R}}_{p,n}^{b}(a,c,\mu,\rho)$, if and only if*

$$\sum_{k=n}^{\infty} \frac{(p+\mu k)}{(p-\rho)|b|} \frac{|(a)_k|}{|(c)_k|} a_{p+k} \leq 1. \tag{2.5}$$

*The result in* (2.5) *is sharp for the functions*

$$f_k(z) = z^p - \frac{(p-\rho)|b||(c)_k|}{(p+\mu k)|(a)_k|} z^{p+k} \quad (k \geq n, a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; z \in \mathbb{U}).$$

*Remark* 2. (i) If in Lemma 3 with $a = p+1$, $b = n = 1$, we set $c = p+1$ and $c = p$, our result corresponds to the work of Lee et al. [11, Lemma 2] and Aouf [2, Theorem 1], respectively.

(ii) Putting $a = \lambda + 1$, $c = p = 1$ and $\rho = 1 - \beta$ $(0 < \beta \leq 1)$ in Lemma 3, we get the result due to Murugusundarmoorthy and Srivastava [15, Lemma 2].

(iii) As a special case of Lemma 3 when $a = p+1$, $c = p+1-\lambda$ $(-\infty < \lambda < p)$ and $\rho = p(1-\beta)$ $(0 < \beta \leq 1)$, we get the corresponding result obtained by Aouf [3, Lemma 2].

## 3. Inclusion relationships involving the classes $\widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\rho)$ and $\widetilde{\mathcal{R}}_{p,n}^{b}(a,c,\mu,\rho)$

Unless otherwise mentioned, we assume throughout the sequel that

$$b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},\ a, c > 0,\ \mu \geq 0 \text{ and } 0 \leq \rho < p.$$

We first prove:

**Theorem 1.** *If*

$$\varkappa = p - \frac{a(p-\rho)}{(a+n)+(p-\rho)|b|},$$

*then*

$$\widetilde{\mathcal{S}}_{p,n}^{b}(a+1,c,\rho) \subset \widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\varkappa).$$

*The result is the best possible.*

*Proof.* Let the function $f$, given by (1.9) be in the class $\widetilde{\mathcal{S}}_{p,n}^{b}(a+1,c,\rho)$. Then by using (2.2), we obtain

$$\sum_{k=n}^{\infty} \frac{k+(p-\rho)|b|}{(p-\rho)|b|} \frac{(a+1)_k}{(c)_k} a_{p+k} \leq 1. \tag{3.1}$$

To show that $f \in \widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\varkappa)$, in view of (3.1), we need to find the best possible value of $\varkappa$ such that

$$\frac{k+(p-\varkappa)|b|}{(p-\varkappa)|b|} \frac{(a)_k}{(c)_k} \leq \frac{k+(p-\rho)|b|}{(p-\rho)|b|} \frac{(a+1)_k}{(c)_k} \quad (k \geq n)$$

which is equivalent to

$$\varkappa \leq p - \frac{a(p-\rho)}{a+k+(p-\rho)|b|} \quad (k \geq n). \tag{3.2}$$

Since the right hand side of (3.2) is increases with $k$, letting $k = n$ in (3.2), we get the required result. □

It is easily seen that the result is the best possible for the function

$$f(z) = z^p - \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a+1)_n} z^{p+n} \quad (z \in \mathbb{U}). \tag{3.3}$$

Setting $a = c = p$ in Theorem 1, we get the following result which yields the corresponding work of Silverman [24, Theorem 7] for $p = n = b = 1$.

**Corollary 1.** *We have*

$$\widetilde{\mathcal{C}}^b_{p,n}(\rho) \subset \widetilde{\mathcal{S}}^b_{p,n}(\kappa),$$

*where*

$$\kappa = \frac{p(n+\rho+(p-\rho)|b|)}{p+n+(p-\rho)|b|}.$$

*The result is the best possible.*

Similarly, we can prove the following result.

**Theorem 2.** *If*

$$\xi = p - \frac{a(p-\rho)}{a+n},$$

*then*

$$\widetilde{\mathcal{R}}^b_{p,n}(a+1, c, \mu, \rho) \subset \widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \xi).$$

*The result is the best possible for the function*

$$f(z) = z^p - \frac{(p-\rho)|b|(c)_n}{(p+\mu n)(a+1)_n} z^{p+n} \quad (z \in \mathbb{U}).$$

For $\lambda > -p$, we define a linear operator $\mathcal{F}_{\lambda,p} : \mathcal{A}_p(n) \longrightarrow \mathcal{A}_p(n)$ by

$$\mathcal{F}_{\lambda,p}(f)(z) = \frac{p+\lambda}{z^\lambda} \int_0^z t^{\lambda-1} f(t)dt \quad (f \in \mathcal{A}_p(n); z \in \mathbb{U}). \tag{3.4}$$

If $f$ is given by (1.9), then it follows from (3.4) that

$$\mathcal{F}_{\lambda,p}(f)(z) = z^p - \sum_{k=n}^{\infty} \frac{p+\lambda}{p+k+\lambda} a_{p+k} z^{p+k} \quad (\lambda > -p; z \in \mathbb{U}). \tag{3.5}$$

Now, by employing the techniques that proved Theorem 1 and using (3.5), it can be shown that:

**Theorem 3.** *For $f \in \mathcal{A}_p(n)$, if $\mathcal{F}_{\lambda,p}(f)$ is given by* (3.4) *and*

$$\tau = p - \frac{a(p+\lambda)(p-\rho)}{a(p+\lambda)+(a+n+p+\lambda)(n+(p-\rho)|b|)},$$

*then*

$$\mathcal{F}_{\lambda,p}\left(\widetilde{\mathcal{S}}^b_{p,n}(a+1,c,\rho)\right) \in \widetilde{\mathcal{S}}^b_{p,n}(a,c,\tau).$$

*The result is the best possible for the function $f$ given by* (3.3).

## 4. Inclusion relationships involving neighborhoods

Following the earlier investigations by Goodman [11], Ruscheweyh [22] and others including Altintas and Owa [4], Altintas et al. ([5] and [6]), we first define the $(n,\delta)$-neighborhood of a function $f \in \mathcal{A}_p(n)$, given by (1.1) as follows:

$$T_{n,\delta}(f) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \text{ and} \right.$$

$$\left. \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_n} |b_{p+k} - a_{p+k}| \le \delta; \delta > 0 \right\}, \tag{4.1}$$

$$N_{n,\delta}(f) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \text{ and} \right.$$

$$\left. \sum_{k=n}^{\infty} (p+k) |b_{p+k} - a_{p+k}| \le \delta; \delta > 0 \right\}. \tag{4.2}$$

In particular, for the identity function $e(z) = z^p$ $(p \in \mathbb{N}; z \in \mathbb{U})$, we immediately have

$$N_{n,\delta}(e) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \text{ and} \right.$$

$$\left. \sum_{k=n}^{\infty} (p+k)|b_{p+k}| \le \delta; \delta > 0 \right\}. \tag{4.3}$$

Throughout this presentation, we shall make use of the following simplified notations:

$T^+_{n,\delta}(f) = T_{n,\delta}(f) \cap T_p(n)$, $N^+_{n,\delta}(f) = N_{n,\delta}(f) \cap T_p(n)$ and $N^+_{n,\delta}(e) = N_{n,\delta}(e) \cap T_p(n)$.

To establish our results, we need the following lemma.

**Lemma 4.** *Let the function $f \in \mathcal{A}_p(n)$ be given by* (1.1). *Then $f \in \mathcal{S}^b_{p,n}(a,c,\rho)$ if and only if*

(4.4)
$$\frac{\mathcal{L}_p(a,c)f(z)}{z^p} \star \left\{ \frac{1 - \dfrac{(1+(p-\rho)b\,x)z}{(p-\rho)b\,x}}{(1-z)^2} \right\} \neq 0 \quad (|x|=1,\ x\neq 1; 0<|z|<1).$$

*Proof.* From (1.7), it follows that

$$f \in \mathcal{S}^b_{p,n}(a,c,\rho) \Longleftrightarrow \frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{\mathcal{L}_p(a,c)f(z)} \neq p + (p-\rho)b\,x \tag{4.5}$$

for all $z \in \mathbb{U}$ with $|x| = 1$ and $x \neq 1$. Since $z\left(\mathcal{L}_p(a,c)f\right)'(z)/\mathcal{L}_p(a,c)f(z)$ takes the value $p$ at $z = 0$, (4.5) is equivalent to

$$\mathcal{L}_p(a,c)f(z) \star \left\{ \frac{(p-1)z^p}{1-z} + \frac{z^p}{(1-z)^2} \right\} - (p+(p-\rho)b\,x)\mathcal{L}_p(a,c)f(z) \neq 0$$
$$(0<|z|<1)$$

or, equivalently

$$\frac{\mathcal{L}_p(a,c)f(z)}{z^p} \star \left\{ \frac{p-(p-1)z}{(1-z)^2} - \frac{p+(p-\rho)b\,x}{1-z} \right\} \neq 0 \quad (0<|z|<1),$$

which reduces to our assertion (4.4). The converse part follows easily by retracing back the steps that proved (4.4). This completes the proof of Lemma 4. □

**Theorem 4.** *If the function $f \in \mathcal{A}_p(n)$ satisfy*

$$\frac{f(z)+\varepsilon z^p}{1+\varepsilon} \in \mathcal{S}^b_{p,n}(a,c,\rho) \quad (\varepsilon \in \mathbb{C}, |\varepsilon| < \delta; \delta > 0), \tag{4.6}$$

*then*

$$T_{n,\delta}(f) \subset \mathcal{S}^b_{p,n}(a,c,\rho).$$

*Proof.* In view of Lemma 4, we note that a function $g \in \mathcal{S}^b_{p,n}(a,c,\rho)$ if and only if

$$\frac{(g \star h)(z)}{z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{4.7}$$

where for convenience

$$h(z) = z^p + \sum_{k=1}^{\infty} c_{p+k} z^{p+k} \ \text{ with } \ c_{p+k} = -\frac{k-(p-\rho)b\,x(a)_k}{(p-\rho)b\,x(c)_k} \ (|x|=1, x\neq 1, k \geq 1).$$

It is easily seen that

$$|c_{p+k}| \leq \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} \quad (k \geq 1).$$

Using (4.6) and (4.7), we deduce that

$$\frac{\dfrac{f(z)+\varepsilon z^p}{1+\varepsilon} * h(z)}{z^p} \neq 0 \quad (z \in \mathbb{U})$$

or, $(f \star h)(z)/z^p \neq -\varepsilon$, which is equivalent to

$$\left|\frac{(f * h)(z)}{z^p}\right| \geq \delta \quad (\delta > 0; z \in \mathbb{U}). \tag{4.8}$$

Let $g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \in \mathcal{T}_{n,\delta}(f)$. Then

$$\left|\frac{((g-f) \star h)(z)}{z^p}\right| = \left|\sum_{k=n}^{\infty} (b_{p+k} - a_{p+k}) c_{p+k} z^k\right|$$

$$\leq |z|^k \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} |b_{p+k} - a_{p+k}| \leq \delta \quad (z \in \mathbb{U}). \tag{4.9}$$

Now, with the aid of (4.8) and (4.9), we obtain

$$\left|\frac{(g * h)(z)}{z^p}\right| \geq \left|\frac{(f \star h)(z)}{z^p}\right| - \left|\frac{((g-f) \star h)(z)}{z^p}\right| > 0 \quad (z \in \mathbb{U}),$$

which in view of (4.7) implies that $g \in \mathcal{S}_{p,n}^b(a,c,\rho)$ and the proof of Theorem 4 is thus completed. □

**Theorem 5.** *If $f \in \widetilde{\mathcal{S}}_{p,n}^b(a+1,c,\rho)$ and $\delta_1 = n/(a+n)$, then*

$$T_{n,\delta_1}^{+}(f) \subset \widetilde{\mathcal{S}}_{p,n}^b(a,c,\rho).$$

*The result is the best possible in the sense that $\delta_1$ cannot be increased.*

*Proof.* Let $f$, given by (1.9) be in the class $\widetilde{\mathcal{S}}_{p,n}^b(a+1,c,\rho)$. Then by (2.2), we have

$$\sum_{k=n}^{\infty} \frac{k+(p-\rho)|b|}{(p-\rho)|b|} \frac{(a+1)_k}{(c)_k} a_{p+k} \leq 1$$

so that

$$\sum_{k=n}^{\infty} \frac{k+(p-\rho)|b|}{(p-\rho)|b|} \frac{(a)_k}{(c)_k} a_{p+k} \leq \frac{a}{a+n}. \tag{4.10}$$

Assuming that the function $g \in \mathcal{T}_p(n)$ defined in $\mathbb{U}$ by

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \quad (z \in \mathbb{U}) \tag{4.11}$$

is in the set $T_{n,\delta_1}^{+}(f)$, we deduce from (4.1) that

$$\sum_{k=n}^{\infty} \frac{(k+(p-\alpha)|b|)(a)_k}{(p-\rho)|b|(c)_k} |b_{p+k} - a_{p+k}| \leq \delta_1. \tag{4.12}$$

Now, with the aid of (4.10) and (4.12), we obtain

$$\begin{aligned}
&\sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} b_{p+k} \\
&\le \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} |b_{p+k}-a_{p+k}| + \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} a_{p+k} \\
&\le \frac{a}{a+n} + \delta_1 = 1,
\end{aligned}$$

which shows that $g \in \widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\rho)$.

To see that the result is the best possible, we consider the function $f$, given by (3.3) and the function $g$ defined by

$$g(z) = z^p - \left[\frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a+1)_n} + \frac{(p-\rho)|b|(c)_n \delta'}{(n+(p-\rho)|b|)(a)_n}\right] z^{p+n}$$

$$(\delta' > \delta_1; z \in \mathbb{U}).$$

It is easily seen that $f \in \widetilde{\mathcal{S}}_{p,n}^{b}(a+1,c,\rho), g \in T_{n,\delta'}^{+}(f)$, but $g \notin \widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\rho)$. This completes the proof of Theorem 5. □

**Theorem 6.** *Let $f \in \widetilde{\mathcal{S}}_{p,n}^{b}(a+1,c,\rho)$ and $\mathcal{F}_{\lambda,p}(f)$ be given by* (3.4). *If*

$$\delta_2 = \frac{n(n+a+p+\lambda)}{a(p+\lambda)+n(n+a+p+\lambda)},$$

*then*

$$T_{n,\delta_2}^{+}(\mathcal{F}_{\lambda,p}(f)) \subset \widetilde{\mathcal{S}}_{p,n}^{b}(a,c,\rho).$$

*The result is the best possible in the sense that $\delta_2$ cannot be increased.*

*Proof.* Let the function $f$ be given by (1.9). Then by using Lemma 2 and Theorem 3, we get

$$\sum_{k=n}^{\infty} \frac{(k+(p-\tau)|b|)(a)_k}{(p-\tau)|b|(c)_k} \frac{p+\lambda}{p+k+\lambda} a_{p+k} \le 1$$

from which it follows that

$$\sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} \frac{p+\lambda}{p+k+\lambda} a_{p+k} \le \frac{(p-\tau)(n+(p-\rho)|b|)}{(p-\rho)(n+(p-\rho)|b|)}. \tag{4.13}$$

Suppose the function $g$, given by (4.11) belongs to the set $T_{n,\delta_2}^{+}(\mathcal{F}_{\lambda,p}(f))$. Then by using (3.5) and (4.1), we deduce that

$$\sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} \left| b_{p+k} - \frac{p+\lambda}{p+\lambda+k} a_{p+k} \right| \le \delta_2. \tag{4.14}$$

Thus, in view of (4.13) and (4.14), we obtain

$$\begin{aligned}
&\sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} b_{p+k} \\
\leq &\sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} \frac{p+\lambda}{p+k+\lambda} a_{p+k} \\
&+ \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} \left| b_{p+k} - \frac{p+\lambda}{p+\lambda+k} a_{p+k} \right| \\
\leq &\frac{(p-\tau)(n+(p-\rho)|b|)}{(p-\rho)(n+(p-\tau)|b|)} + \delta_2 = 1.
\end{aligned}$$

To show that the result is the best possible, we consider the function $f$, given by (3.3) and the function $g$ defined by

$$g(z) = z^p - \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n} \left( \frac{a(p+\lambda)}{(a+n)(n+p+\lambda)} + \delta' \right) z^{p+n}$$
$$(\delta' > \delta_2; z \in \mathbb{U}).$$

It is easily seen that $f \in \widetilde{\mathcal{S}}^b_{p,n}(a+1, c, \rho)$ and $g \in T^+_{n,\delta'}(\mathcal{F}_{p,\lambda}(f))$, but $g \notin \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$. This proves the assertion of Theorem 6. □

**Theorem 7.** *If* $a \geq c > 0, |b| < p/(p-\rho)$ *and*

$$\delta_3 = \frac{(p+n)(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n},$$

*then*

$$\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho) \subset N^+_{n,\delta_3}(e).$$

*Proof.* For a function $f \in \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ of the form (1.9), the assertion (2.2) immediately yields

$$\frac{(n+(p-\rho)|b|)(a)_n}{(c)_n} \sum_{k=n}^{\infty} a_{p+k} \leq (p-\rho)|b|,$$

so that

$$\sum_{k=n}^{\infty} a_{p+k} \leq \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n}. \tag{4.15}$$

Making use of (2.2) again, in conjunction with (4.15) and the fact that $|b| < p/(p-\rho)$, we get

$$\begin{aligned}
\frac{(a)_n}{(c)_n} \sum_{k=n}^{\infty} (p+k) a_{p+k} &\leq (p-\rho)|b| + (p-(p-\rho)|b|) \frac{(a)_n}{(c)_n} \sum_{k=n}^{\infty} a_{p+k} \\
&\leq (p-\rho)|b| + (p-(p-\rho)|b|) \frac{(a)_n}{(c)_n} \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n}
\end{aligned}$$

$$= \frac{(n+p)|b|}{n+(p-\rho)|b|}$$

that is,

$$\sum_{k=n}^{\infty}(p+k)a_{p+k} \leq \frac{(n+p)(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n} = \delta_3,$$

which, in view of (4.3) establishes the inclusion relation asserted by Theorem 7. □

*Remark* 3. (i) As a special case of Theorem 7 when $p = b = 1$, $a = c$ and $p = b = c = 1$, $a = 2$, respectively, we get the results due to Altintaş and Owa [4, Theorem 2.1 and Theorem 2.2].

(ii) For $a = \lambda+1(\lambda > -1)$, $c = p = 1$ and $\rho = 1-\beta$ $(0 < \beta \leq 1)$, Theorem 7 corresponds to a result of Murugusundaramoorthy and Srivastava [16, Theorem 1].

(iii) A result due to Aouf [3, Theorem 1] can deduced from Theorem 7 by taking $a = p+1$, $c = p+1-\lambda$ $(0 \leq \lambda \leq 1)$ and $\rho = p-\beta$ $(0 < \beta \leq 1)$, which in turn yields the corresponding work of Altintaş et al. [5, Theorem 1 and Theorem 2] for $p = 1$, $a = c$, $\rho = \beta$ $(0 < \beta \leq 1)$ and $a = 2$, $p = c = 1$, $a = c$, $\rho = \beta$ $(0 < \beta \leq 1)$, respectively.

In an analogous manner, by applying the assertion (2.5) of Lemma 3 instead of the assertion (2.2) of Lemma 2 to the functions of the class $\widetilde{\mathcal{R}}^b_{p,n}(a,c,\mu,\rho)$, we can prove the following inclusion relationship, which in turn yields the corresponding result obtained by Murugusundaramoorthy and Srivastava [16, Theorem 2] for $a = \lambda+1$ $(\lambda > -1)$, $c = p = 1$ and $\rho = 1-\beta$ $(0 < \beta \leq 1)$. A result due to Aouf [3, Theorem 2] can also be deduced by setting $a = p+1$, $c = p+1-\lambda$ $(0 \leq \lambda \leq 1)$ and $\rho = \beta$ $(0 < \beta \leq 1)$.

**Theorem 8.** *If $a \geq c > 0$, $\mu > 1$, $|b| < p/(p-\rho)$ and*

$$\delta_4 = \frac{(p+n)(p-\rho)|b|(c)_n}{(p+\mu n)(a)_n},$$

*then*

$$\widetilde{\mathcal{R}}^b_{p,n}(a,c,\mu,\rho) \subset N_{n,\delta_4}(e).$$

For function $f$, given by (1.9) and $g$ defined by (4.11), we define the modified Hadamard (or quasi-Hadamard) product of $f$ and $g$ by

$$(f \star_q g)(z) = z^p - \sum_{k=n}^{\infty} a_{p+k}b_{p+k}z^{p+k} = (g \star_q f)(z) \quad (p,n \in \mathbb{N}; z \in \mathbb{U}).$$

Using the notion of modified Hadamard product, for subsets $E_1$ and $E_2$ of $\mathcal{T}_p(n)$, we denote

$$E_1 \otimes E_2 = \{f \star_q g : f \in E_1 \text{ and } g \in E_2\}.$$

We now prove:

**Theorem 9.** *If $a \geq c > 0$ and $e(z) = z^p$, then*
(i) $T^{+}_{n,\delta_5}(e) \otimes T^{+}_{n,\delta_5}(e) \subset \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$, *where*

$$\delta_5 = \sqrt{\{(n + (p-\rho)|b|)(a)_n\}/\{(p-\rho)|b|(c)_n\}}.$$

(ii) $T^{+}_{n,\delta_6}(e) \otimes N^{+}_{n,\delta_6}(e) \subset \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$, *where* $\delta_6 = \sqrt{p+n}$.
*The result, in each case is the best possible.*

*Proof.* Let the function $f$ be given by (1.9) and the function $g$ be defined by (4.11). Suppose that $f, g \in T^{+}_{n,\delta_5}(e)$. Then by (4.1), we have
(4.16)

$$\sum_{k=n}^{\infty} \frac{(k + (p-\alpha)|b|)(a)_k}{(p-\rho)|b|(c)_k} a_{p+k} \leq \delta_5 \quad \text{and} \quad \sum_{k=n}^{\infty} \frac{(k + (p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} b_{p+k} \leq \delta_5.$$

For $a \geq c > 0$, we note that $\{(k + (p-\rho)|b|)(a)_k\}/(c)_k$ is an increasing function of $k$ $(k \geq n)$ so that the first estimate in (4.16) immediately yields

$$\sum_{k=n}^{\infty} a_{p+k} \leq \frac{(p-\rho)|b|(c)_n \delta_5}{(n + (p-\rho)|b|)(a)_n}$$

which implies that

$$a_{p+k} \leq \frac{(p-\rho)|b|(c)_n \delta_5}{(n + (p-\rho)|b|)(a)_n} \quad (k \geq n). \tag{4.17}$$

Using (4.17) and the second estimate in (4.16), we get

$$\sum_{k=n}^{\infty} \frac{(k + (p-\rho)|b|)(a)_k}{(p-\rho)|b|)(c)_k} a_{p+k} b_{p+k} \leq \frac{(p-\rho)|b|(c)_n \delta_5^2}{(n + (p-\rho)|b|)(a)_n} = 1,$$

which again in view of the assertion (2.2) implies that $(f \star_q g) \in \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$.

To see that the result in part (i) is the best possible, we consider the functions $f$ and $g$ defined by

$$f(z) = g(z) = z^p - \sqrt{\frac{(p-\rho)|b|(c)_n}{(n + (p-\rho)|b|)(a)_n}} z^{p+n} \quad (a \geq c > 0; z \in \mathbb{U}).$$

Clearly, $f, g \in T^{+}_{n,\delta_5}(e)$ and $(f \star_q g) \in \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$. This proves part (i) of the theorem.

To prove part (ii), we assume that $f \in T^{+}_{n,\delta_6}(e)$ and $g \in N^{+}_{n,\delta_6}(e)$. Then

$$\sum_{k=n}^{\infty} \frac{(k + (p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} a_{p+k} \leq \delta_6 \quad \text{and} \quad \sum_{k=n}^{\infty} (p+k) b_{p+k} \leq \delta_6.$$

Thus, $b_{p+k} \leq \delta_6/(p+n)$ for $k \geq n$ and

$$\sum_{k=n}^{\infty} \frac{(k + (p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} a_{p+k} b_{p+k} \leq \frac{\delta_6^2}{p+n} = 1,$$

which in view of (2.2) implies that $(f \star_q g) \in \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$.

Considering the functions $f, g$ defined in $\mathbb{U}$ by

$$f(z) = z^p - \frac{(p-\rho)|b|\sqrt{p+n}(c)_n}{(n+(p-\rho)|b|)(a)_n} z^{p+n} \ (a \geq c > 0) \quad \text{and} \quad g(z) = z^p - \frac{z^{p+n}}{\sqrt{p+n}},$$

it is easily seen that $f \in T^{+}_{n,\delta_6}(e), g \in N^{+}_{n,\delta_6}(e)$ and $(f \star_q g) \in \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ and the result in part (ii) is the best possible. This completes the proof of Theorem 9. $\square$

Putting $a = c$ in Theorem 9, we have:

**Corollary 2.** *Let the function $f$ be given by* (1.9) *and the function $g$ be defined by* (4.11)*. If*

(i) $\sum_{k=n}^{\infty}(k + (p-\rho)|b|)a_{p+k} \leq \sqrt{(p-\rho)(n+(p-\rho)|b|)|b|}$ *and* $\sum_{k=n}^{\infty}(k + (p-\rho)|b|)b_{p+k} \leq \sqrt{(p-\rho)(n+(p-\rho)|b|)|b|}$, *then* $(f \star_q g) \in \widetilde{\mathcal{S}}^b_{p,n}(\rho)$.

(ii) $\sum_{k=n}^{\infty}(k + (p-\rho)|b|)a_{p+k} \leq (p-\rho)|b|\sqrt{p+n}$ *and* $\sum_{k=n}^{\infty}(p+k)b_{p+k} \leq \sqrt{p+n}$, *then* $(f \star_q g) \in \widetilde{\mathcal{S}}^b_{p,n}(\rho)$. *The result in* (i) *and* (ii) *are the best possible.*

Next, we determine the neighborhood for each of the classes

$$\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho; \eta), \quad \widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho; \eta) \quad \text{and} \quad \widetilde{\mathcal{K}}^b_{p,n}(a, c, \rho; \kappa)$$

which we define as follows:

A function $f \in \mathcal{T}_p(n)$ is said to be in the class $\widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho; \eta)$, if there exists a function $g \in \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \eta \quad (0 < \eta \leq 1; z \in \mathbb{U}). \tag{4.18}$$

Analogously, a function $f \in \mathcal{T}_p(n)$ is said to be in the class $\widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho; \eta)$, if there exists a function $g \in \widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho)$ such that the inequality (4.18) holds true.

Furthermore, a function $f \in \mathcal{T}_p(n)$ is said to be in the class $\widetilde{\mathcal{K}}^b_{p,n}(a, c, \rho; \kappa)$, if there exists a function $g \in \widetilde{\mathcal{S}}^b_{p,n}(a+1, c, \rho)$ such that

$$\operatorname{Re}\left\{ \frac{f'(z)}{g'(z)} \right\} > \varrho \quad (0 \leq \varrho < 1; z \in \mathbb{U}).$$

**Theorem 10.** *If $a \geq c > 0$, $g \in \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho)$ and*

$$\delta_7 = \frac{(p+n)\,\eta\{(n+(p-\rho)|b|)(a)_n - (p-\rho)|b|(c)_n\}}{(n+(p-\rho)|b|)(a)_n},$$

*then*

$$N^{+}_{n,\delta_7}(g) \subset \widetilde{\mathcal{S}}^b_{p,n}(a, c, \rho; \eta).$$

*Proof.* Suppose $f$, given by (1.9) belongs to the set $N^+_{n,\delta_7}(g)$, where $g$, given by (4.11) satisfy the condition (4.18). Then

$$\sum_{k=n}^{\infty}(p+k)|a_{p+k}-b_{p+k}| \leq \delta_7,$$

which readily implies that

$$\sum_{k=n}^{\infty}|a_{p+k}-b_{p+k}| \leq \frac{\delta_7}{p+n}.$$

Since $g \in \widetilde{\mathcal{S}}^b_{p,n}(a,c,\rho)$, we have from Lemma 2

$$\sum_{k=n}^{\infty} b_{p+k} \leq \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{k=n}^{\infty}|a_{p+k}-b_{p+k}|}{1-\sum_{k=n}^{\infty}b_{p+k}} \\ &\leq \frac{(n+(p-\rho)|b|)(a)_n\delta_7}{(p+n)\{(n+(p-\rho)|b|(a)_n-(p-\rho)|b|(c)_n\}} \\ &= \eta \quad (z \in \mathbb{U}).\end{aligned}$$

Thus in view of (4.17), $f \in \widetilde{\mathcal{S}}^b_{p,n}(a,c,\rho;\eta)$ and the proof of Theorem 10 is completed. $\square$

Our proof of Theorem 11 given below is much akin to that of Theorem 10, and we omit the details.

**Theorem 11.** *If $a \geq c > 0$, $g \in \widetilde{\mathcal{R}}^b_{p,n}(a,c,\mu,\rho)$ and*

$$\delta_8 = \frac{(p+n)\eta\{(p+\mu n)(a)_n-(p-\rho)|b|(c)_n\}}{(p+\mu n)(a)_n},$$

*then*

$$N^+_{n,\delta_8}(g) \subset \widetilde{\mathcal{R}}^b_{p,n}(a,c,\mu,\rho;\eta).$$

**Theorem 12.** *If $a \geq c > 0$, $|b| < p/(p-\rho)$ and*

$$\delta_9 = p(1-\varrho) + \frac{\varrho(p+n)(p-\rho)|b|(c)_n}{\{n+(p-\rho)|b|\}(a+1)_n},$$

*then*

$$\widetilde{\mathcal{K}}^b_{p,n}(a,c,\rho;\varrho) \subset N_{n,\delta_9}(e).$$

*Proof.* Suppose that the function $f$, given by (1.9) belongs to the class $\widetilde{\mathcal{K}}^b_{p,n}(a,c,\rho;\kappa)$ and the function $g \in \widetilde{\mathcal{S}}^b_{p,n}(a+1,c,\rho)$ is given by (4.11). Then, we have

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} = \operatorname{Re}\left\{\frac{p-\sum_{k=n}^{\infty}(p+k)a_{p+k}z^{k+1}}{p-\sum_{k=n}^{\infty}(p+k)b_{p+k}z^{k+1}}\right\}$$

$$\geq \frac{p-\sum_{k=n}^{\infty}(p+k)a_{p+k}}{p-\sum_{k=n}^{\infty}(p+k)b_{p+k}} > \varrho,$$

which implies that

$$\sum_{k=n}^{\infty}(p+k)a_{p+k} \leq p(1-\varrho)+\kappa\sum_{k=n}^{\infty}(p+k)b_{p+k}. \tag{4.19}$$

Since $a \geq c > 0$ and the function $g \in \widetilde{\mathcal{S}}^{b}_{p,n}(a+1,c,\rho)$, Lemma 2 implies that

$$\sum_{k=n}^{\infty}(p+k)b_{p+k} \leq \frac{(p+n)(p-\rho)|b|(c)_n}{\{n+(p-\rho)|b|\}(a+1)_n}.$$

Thus, by using the above inequality in (4.19), we get the required result. $\square$

*Remark* 4. (i) Letting $a=c=p=b=1$ in Theorem 10 and Theorem 12, we get the corresponding results obtained by Altintas and Owa [4].

(i) Setting $a=\lambda+1$ $(\lambda > -1)$, $c=p=1$, $b=\gamma$, $\rho=1-\beta$ $(0<\beta\leq 1)$ and $\eta = 1-\alpha(0\leq\alpha<1)$ in Theorem 10 and Theorem 11, respectively, we get the results obtained by Murugusundaramoorthy and Srivastava [16, Theorem 3 and Theorem 4].

(ii) Taking $a=p+1$, $c=p+1-\lambda$ $(0\leq\lambda\leq 1)$, $\rho=1-\beta$ $(0<\beta\leq 1)$ and $\eta=p-\alpha$ $(0\leq\alpha<p)$ in Theorem 10 and Theorem 11, respectively, we get the results due to Aouf [3, Theorem 3 and Theorem 4].

## 5. Subordination results

We now prove:

**Theorem 13.** *If* $a \geq c > 0$, $f \in \widetilde{\mathcal{S}}^{b}_{p,n}(a,c,\rho)$, $g \in \mathcal{C}$ *and*

$$\varepsilon = \frac{(n+(p-\rho)|b|)(a)_n}{2\{(n+(p-\rho)|b|)(a)_n+(p-\rho)|b|(c)_n\}},$$

*then*

$$[\varepsilon\, z^{1-p}f(z)]\star g(z) \prec g(z) \quad (z\in\mathbb{U}). \tag{5.1}$$

*Moreover,*

$$\operatorname{Re}\left(\frac{f(z)}{z^{p-1}}\right) > -\frac{1}{2\,\varepsilon} \quad (z\in\mathbb{U}). \tag{5.2}$$

*If $p$ and $n$ are odd, then the constant factor $\varepsilon$ in* (5.1) *and* (5.2) *cannot be replaced by a larger number.*

*Proof.* Let the function $f$ be given by (1.9) and $g(z)=z+\sum_{k=2}^{\infty}c_k z^k \in \mathcal{C}$. Then

$$\{\varepsilon\, z^{1-p}f(z)\}\star g(z) = \sum_{k=1}^{\infty} b_k c_k\, z^k \quad (z\in\mathbb{U}),$$

where

$$b_k = \begin{cases} \varepsilon, & \text{if } k = 1, \\ 0, & \text{if } 2 \le k \le n, \\ -\varepsilon\, a_{p+k-1}, & \text{if } k \ge n+1. \end{cases}$$

Thus, by Definition 3, the subordination result (5.1) holds true, if $\{b_k\}_{k=1}^{\infty}$ is the subordinating factor sequence. Since

$$\frac{(k+(p-\rho)|b|)(a)_k}{(c)_k} \ge \frac{(n+(p-\rho)|b|)(a)_n}{(c)_n} > 0 \quad (k \ge n),$$

we have for $|z| = r < 1$,

$$\begin{aligned} &\operatorname{Re}\left\{1 + 2\varepsilon\, z - 2\varepsilon \sum_{k=n} b_{k+1} z^{k+1}\right\} \\ &= \operatorname{Re}\left\{1 + 2\varepsilon\, z - \frac{1}{[(n+(p-\rho)|b|)(a)_n + (p-\rho)|b|(c)_n]} \sum_{k=n}^{\infty} [(n+(p-\rho)|b|)(c)_n] a_{p+k} z^{k+1}\right\} \\ &\ge 1 - \frac{(n+(p-\rho)|b|)(a)_n\, r}{[(n+(p-\rho)|b|)(a)_n + (p-\rho)|b|(c)_n]} \\ &\quad - \frac{(p-\rho)|b|(c)_n\, r}{[(n+(p-\rho)|b|)(a)_n + (p-\rho)|b|(c)_n]} \sum_{k=n}^{\infty} \frac{(k+(p-\rho)|b|)(a)_k}{(p-\rho)|b|(c)_k} a_{p+k}. \end{aligned}$$

Thus, by using Lemma 2, we deduce that

$$\begin{aligned} &\operatorname{Re}\left\{1 + 2\sum_{k=0}^{\infty} b_k z^k\right\} \\ &= 1 - \frac{(n+(p-\rho)|b|)(a)_n\, r}{[(n+(p-\rho)|b|)(a)_n + (p-\rho)|b|(c)_n]} - \frac{(p-\rho)|b|(c)_n\, r}{[(n+(p-\rho)|b|)(a)_n + (p-\rho)|b|(c)_n]} \\ &= 1 - r > 0. \end{aligned}$$

This proves the subordination result (5.1).

Letting $g(z) = z/(1-z)$ $(z \in \mathbb{U})$ in (5.1), we easily get the result (5.2), and considering the function

$$f(z) = z^p - \frac{(p-\rho)|b|(c)_n}{(n+(p-\rho)|b|)(a)_n} z^{p+n} \; (a \ge c > 0; z \in \mathbb{U}),$$

it is easily seen that $f \in \widetilde{\mathcal{S}}^b_{p,n}(a,c,\rho)$ and if $p$ and $n$ are odd, then

$$[z^{1-p} f(z)]_{z=-1} = -\frac{1}{2\varepsilon}$$

so that the constant factor $\varepsilon$ cannot be replaced by a larger number. □

Setting $a = c$ and $b = 1$ in Theorem 13, we obtain:

**Corollary 3.** *If* $f \in \widetilde{\mathcal{S}}^b_{p,n}(\rho)$, $g \in \mathcal{C}$ *and*

$$\varepsilon = \frac{n + (p-\rho)}{2\{n + 2(p-\rho)\}},$$

*then*

$$[\varepsilon\, z^{1-p} f(z)] \star g(z) \prec g(z) \quad (z \in \mathbb{U}).$$

*Moreover,*

$$Re\left(\frac{f(z)}{z^{p-1}}\right) > -\frac{1}{2\,\varepsilon} \quad (z \in \mathbb{U}).$$

*If* $p$ *and* $n$ *are odd, then the constant factor* $\varepsilon$ *cannot be replaced by a larger number.*

The proof of the following theorem is similar to that of Theorem 13, and we omit the details.

**Theorem 14.** *If* $a \geq c > 0$, $f \in \widetilde{\mathcal{R}}^b_{p,n}(a, c, \mu, \rho)$, $g \in \mathcal{C}$ *and*

$$\sigma = \frac{(p + \mu\, n)(a)_n}{2\{(p + \mu\, n)(a)_n + (p-\rho)|b|(c)_n\}},$$

*then*

$$\{\sigma\, z^{1-p} f(z)\} \star g(z) \prec g(z) \quad (z \in \mathbb{U}). \tag{5.3}$$

*Moreover,*

$$\mathrm{Re}\left(\frac{f(z)}{z^{p-1}}\right) > -\frac{1}{2\,\sigma} \quad (z \in \mathbb{U}). \tag{5.4}$$

*If* $p$ *and* $n$ *are odd, then the constant factor* $\sigma$ *in* (5.3) *or* (5.4) *cannot be replaced by a larger number.*

*Remark* 5. By suitably specializing the various parameters involved, we can derive the results (for example inclusion relations and neighborhood properties) of this paper for many relatively more familiar function classes (see also Example 1, Example 2, Remark 1 and Remark 2).

**Acknowledgement.** The authors would like to thank the referee(s) for their comments and valuable suggestions which improved the presentation of the paper.

## References

[1] O. P. Ahuja, *Hadamard product of analytic functions defined by Ruscheweyh derivatives*, in Current Topics in Analytic Function Theory, H. M. Srivastava and S. Owa (Eds.), pp. 13–28, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[2] M. K. Aouf, *A subclass of p-valent functions with negative coefficients*, Utilitas Math. **46** (1994), 219–231.

[3] ———, *Neighborhoods of a certain family of multivalent functions defined by using a fractional derivative operator*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 1, 31–40.

[4] O. Altintaş and S. Owa, *Neighborhoods of certain analytic functions with negative coefficients*, Internat. J. Math. Math. Sci. **19** (1996), no. 4, 797–800.

[5] O. Altintaş, Ö. Özkan, and H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Lett. **13** (2000), no. 3, 63–67.

[6] ———, *Neighborhoods of a certain family of multivalent functions with negative coefficients*, Comput. Math. Appl. **47** (2004), no. 10-11, 1667–1672.

[7] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **159** (1984), no. 4, 737–745.

[8] P. L. Duren, *Univalent Functions*, A Series of Comprehensive Studies in Mathematics, Vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.

[9] B. A. Frasin, *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi.(N.S.) **22** (2006), no. 2, 179–191.

[10] R. M. Goel and N. S. Sohi, *A new criterion for p-valent functions*, Proc. Amer. Math. Soc. **78** (1980), no. 3, 353–357.

[11] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. **8** (1957), 598–601.

[12] S. K. Lee, S. Owa, and H. M. Srivastava, *Basic properties and characterizations of a certain class of analytic functions with negative coefficients*, Utilitas Math. **36** (1989), 121–128.

[13] J.-L. Liu and K. I. Noor, *Some properties of Noor integral operator*, J. Nat. Geom. **21** (2002), no. 1-2, 81–90.

[14] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), no. 2, 157–172.

[15] ———, *Differential Subordinations*: *Theory and Applications*, in: Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York, 2000.

[16] G. Murugusundaramoorthy and H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure Appl. Math. **5** (2004), no. 2, Article 24, 8 pp.

[17] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, J. Natur. Sci. Math. **25** (1985), no. 1, 1–12.

[18] S. Owa, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin **59** (1985), no. 4, 385–402.

[19] J. Patel and A. K. Mishra, *On certain subclasses of multivalent functions associated with an extended fractional differintegral operator*, J. Math. Anal. Appl. **332** (2007), no. 1, 109–122.

[20] R. K. Raina and H. M. Srivastava, *Inclusion and neighborhood properties of some analytic and multivalent functions*, J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Article 5, 6 pp.

[21] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[22] ———, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **8** (1981), no. 4, 521–527.

[23] H. Saitoh, *A linear operator and its application of first order differential subordinations*, Math. Japon. **44** (1996), no. 1, 31–38.

[24] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[25] ———, *Neighborhoods of classes of analytic functions*, Far East J. Math. Sci. **3** (1995), no. 2, 165–169.

[26] H. M. Srivastava and S. Owa(Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[27] P. Wiatrowski, *The coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Łódz Nauk. Mat.-Przyrod. (2) **39** (1970), 75–85.

[28] H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle*, Proc. Amer. Math. Soc. **12** (1961), 689–693.

Ashok Kumar Sahoo
Department of Mathematics
VSS University of Technology
Burla-768018, Odisha, India
*E-mail address*: ashokuumt@gmail.com

Jagannath Patel
Department of Mathematics
Utkal University
Vani Vihar, Bhubaneswar-751004, India
*E-mail address*: jpatelmath@yahoo.co.in