Bull. Korean Math. Soc. **51** (2014), No. 6, pp. 1615–1623
http://dx.doi.org/10.4134/BKMS.2014.51.6.1615

GENERALIZED MATRIX FUNCTIONS, IRREDUCIBILITY AND EQUALITY

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Abstract. Let $G \leq S_n$ and χ be any nonzero complex valued function on G. We first study the irreducibility of the generalized matrix polynomial $d_\chi^G(X)$, where $X = (x_{ij})$ is an n-by-n matrix whose entries are n^2 commuting independent indeterminates over $\mathbb{C}$. In particular, we show that if χ is an irreducible character of G, then $d_\chi^G(X)$ is an irreducible polynomial, where either $G = S_n$ or $G = A_n$ and $n \neq 2$. We then give a necessary and sufficient condition for the equality of two generalized matrix functions on the set of the so-called χ-singular (χ-nonsingular) matrices.

1. Introduction

Let S_n be the symmetric group of degree n, G an arbitrary subgroup of S_n, and let $\chi : G \to \mathbb{C}$ be a function. Denote by $M_n(\mathbb{C})$ the set of all n-by-n matrices over $\mathbb{C}$ and define the function $d_\chi^G : M_n(\mathbb{C}) \to \mathbb{C}$ as follows:

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$. The function d_χ^G is called the *generalized matrix function* associated with G and χ. Note that if $G = S_n$ and $\chi = 1_G$ is the principal character of G, then $d_\chi^G = \text{per}$ is the permanent, and if $G = S_n$ and $\chi = \varepsilon$ is the alternating character of G, then $d_\chi^G = \det$ is the determinant. We refer the reader to [4] and [5] for more information about generalized matrix functions. Now let $X = (x_{ij})$ be an n-by-n matrix whose entries are n^2 commuting independent indeterminates over $\mathbb{C}$. Therefore,

$$d_\chi^G(X) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}$$

Received August 14, 2013.
2010 *Mathematics Subject Classification.* 15A15, 12E05, 20C15.
Key words and phrases. generalized matrix functions, irreducibility, χ-singular and χ-nonsingular matrices.

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can be viewed as an element of $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$, the polynomial ring in the variables $x_{11}, x_{12}, \ldots, x_{nn}$ with coefficients in $\mathbb{C}$. We call $d_\chi^G(X)$ the *generalized matrix polynomial* associated with G and χ.

Our main aim in this paper is as follows: in Section 2, we first prove some results about the irreducibility of polynomials and then, using them, we obtain the irreducibility of $d_\chi^G(X)$ under some restrictions over G and χ; in Section 3, by using Hilbert's Nullstellensatz and results of Section 2, we give a criterion for the equality of two generalized matrix functions on the set of the so-called χ-singular (χ-nonsingular) matrices.

2. Irreducibility of $d_\chi^G(X)$

First we recall a few standard notation and definitions. Let $\mathbb{C}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in the variables $x_1, x_2, \ldots, x_n$ with coefficients in $\mathbb{C}$. A product of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is called a monomial in $x_1, x_2, \ldots, x_n$, where the α_i are nonnegative integers. The total degree of this monomial is the number $\sum_{i=1}^n \alpha_i$. A monomial is said to be square-free if all the α_i are at most 1. It is obvious that an element f of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ can be uniquely written as a finite linear combination of monomials with coefficients in $\mathbb{C}$. The total degree of f is the maximum total degrees of its monomials. We say that a polynomial f is homogeneous of total degree m if for all $t \in \mathbb{C}$,

$$f(tx_1, tx_2, \ldots, tx_n) = t^m f(x_1, x_2, \ldots, x_n).$$

It can be easily seen that a polynomial is homogeneous if and only if all its monomials have the same total degrees.

The following, perhaps standard, theorem, which we have not found a reference for it, gives some information about the factors of a homogeneous polynomial. We give a proof of it for the convenience of the reader.

Theorem 2.1. *Let $f, g, h \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be nonzero and $f = gh$. If f is homogeneous, then g and h are also homogeneous.*

Proof. Let m, r, and s be the total degrees of f, g, and h, respectively. Then we may write g and h as follows:

$$g = g_r + \cdots + g_0, \quad h = h_s + \cdots + h_0,$$

where g_k and h_k are the homogeneous parts of g and h of total degree k, respectively. Now let i and j be the least integers such that $g_i \neq 0 \neq h_j$. Therefore,

$$f = gh = g_i h_j + l,$$

where either $l = 0$ or else l is a polynomial with monomials of total degrees at least $i + j + 1$. Thus $g_i h_j \neq 0$ is the homogeneous part of f of total degree $i + j$. But f is homogeneous of total degree m and so it has a unique homogeneous part. We conclude that $l = 0$, $i + j = m$, $i = r$, $j = s$. Thus $f = g_r h_s$ and the proof is complete. □

How the factors of a polynomial with square-free monomials can be is the statement of the next theorem.

Theorem 2.2. *Let $f, g, h \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be nonzero and $f = gh$. If all monomials of f are square-free, then all monomials of g and h are also square-free and $g \in \mathbb{C}[A]$ and $h \in \mathbb{C}[B]$, where $\{A, B\}$ is a partition of $\{x_1, x_2, \ldots, x_n\}$.*

Proof. By symmetry it suffices to show that all monomials of g are square-free and every indeterminate appearing in g does not appear in h. Let $y \in \{x_1, x_2, \ldots, x_n\}$ be an indeterminate which appears in g. Then g and h can be written as follows:

$$g = a_r y^r + \cdots + a_1 y + a_0, \quad h = b_s y^s + \cdots + b_1 y + b_0,$$

where the a_i and b_i are polynomials in which the indeterminate y does not appear, $a_r \neq 0 \neq b_s$, and $r \geq 1$ because y appears in g. Therefore,

$$f = gh = a_r b_s y^{r+s} + \cdots + (a_1 b_0 + a_0 b_1) y + a_0 b_0.$$

But all monomials of f are square-free, and so $r + s = 1$. This implies that $r = 1$ and $s = 0$, completing the proof of the theorem. □

Remark 1. It is easy to see that if all the indeterminates $x_1, x_2, \ldots, x_n$ appear in f, then there is a unique partition $\{A, B\}$ of $\{x_1, x_2, \ldots, x_n\}$ such that $g \in \mathbb{C}[A]$ and $h \in \mathbb{C}[B]$.

It is known that the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is a UFD, that is, every nonconstant polynomial can be uniquely factored as a product of irreducible polynomials. Therefore, as a consequence we obtain the following:

Corollary 2.3. *Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a nonconstant polynomial with square-free monomials and $f_1, f_2, \ldots, f_m$ be the distinct irreducible factors of f. Then all monomials of the f_i are also square-free, $f = f_1 f_2 \cdots f_m$, and $f_i \in \mathbb{C}[A_i]$, where $\{A_1, \ldots, A_m\}$ is a partition of $\{x_1, x_2, \ldots, x_n\}$.*

Remark 2. It is obvious that $m \leq \deg f$. Also, the last part of the above corollary shows that $m \leq n$, and the equality holds if and only if, after reordering if necessary, $f_i = a_i x_i + b_i$ for some $a_i, b_i \in \mathbb{C}$.

Now we summarize some properties of the polynomial $d_\chi^G(X)$ in the ring $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ in the next corollary.

Corollary 2.4. *Let $G \leq S_n$, and $\chi : G \to \mathbb{C}$ be a nonzero function. Also let $f_1, f_2, \ldots, f_m$ be the distinct irreducible factors of $d_\chi^G(X)$. Then*

i) *$d_\chi^G(X)$ is a homogeneous polynomial of total degree n with square-free monomials*;
ii) *the f_i are homogeneous polynomials with square-free monomials, $m \leq n$, and $d_\chi^G(X) = f_1 f_2 \cdots f_m$*;
iii) *$f_i \in \mathbb{C}[A_i]$, where $\{A_1, \ldots, A_m\}$ is a partition of $\{x_{11}, x_{12}, \ldots, x_{nn}\}$.*

For a given $f \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$, let B be the set of all indeterminates appearing in f, $R = \{r \mid \exists s, x_{rs} \in B\}$, and $C = \{s \mid \exists r, x_{rs} \in B\}$. In the next theorem we give detailed information about $d_\chi^G(X)$.

Theorem 2.5. *Assume that the hypotheses of Corollary 2.4 hold. Let also B_i, R_i, and C_i be as above for f_i. Then*

i) *if $i \neq j$, $x_{rs} \in B_i$, $x_{pq} \in B_j$, then $r \neq p$, $s \neq q$;*
ii) *if a monomial of f_i contains distinct x_{rs}, x_{pq}, then $r \neq p$, $s \neq q$;*
iii) $|R_i| = |C_i| = \deg f_i$;
iv) *the number of monomials of f_i is at most* $(\deg f_i)!$;
v) *the number of monomials of $d_\chi^G(X)$ is at most* $\prod_{i=1}^m (\deg f_i)!$.

Proof. To prove (i), we may assume by way of contradiction that there exist $x_{rs} \in B_i$ and $x_{rq} \in B_j$ with $i \neq j$. By part (iii) of the previous corollary, we know that $x_{rs} \in A_i$ and $x_{rq} \in A_j$ and A_i is disjoint from A_j. Hence there must exist a square-free monomial, say g, in $f_1 f_2 \cdots f_m$ containing the indeterminates x_{rs} and x_{rq}. Since $d_\chi^G(X) = f_1 f_2 \cdots f_m$, thus there must be a $\sigma \in G$ such that $g = \prod_{i=1}^n x_{i\sigma(i)}$. This implies that $s = \sigma(r) = q$, saying that A_i is not disjoint from A_j, a contradiction. Similarly, if there are $x_{rs} \in B_i$ and $x_{ps} \in B_j$ with $i \neq j$, then we can get a contradiction.

The same method as above can be used to prove (ii).

To obtain (iii), first we claim that if $\chi(\sigma) \neq 0$ for some $\sigma \in G$, then $\sigma(R_i) \subseteq C_i$. Let $r \in R_i$ be arbitrary and so $x_{rs} \in B_i$ for some s. Since $\chi(\sigma) \neq 0$, hence the monomial $\prod_{i=1}^n x_{i\sigma(i)}$ must appear in $d_\chi^G(X) = f_1 f_2 \cdots f_m$. This implies that $x_{r\sigma(r)} \in B_j$ for some j. By part (i), $i = j$ and therefore $\sigma(r) \in C_i$. Now from χ being nonzero, we conclude that $\{R_i : 1 \leq i \leq m\}$ and $\{C_i : 1 \leq i \leq m\}$ are partitions of $\{1, 2, \ldots, n\}$. Therefore by the claim we deduce that $\sigma(R_i) = C_i$ for each i. Hence $|R_i| = |C_i|$. By part (ii), $\deg f_i \leq |R_i|$. But

$$n = \sum_{i=1}^m \deg f_i \leq \sum_{i=1}^m |R_i| = n,$$

and so $\deg f_i = |R_i|$ for each i.

By combining together parts (ii) and (iii), we obtain (iv).

Finally part (iv) implies (v). □

Remark 3. The first part of the above theorem says that any two indeterminates appearing in two distinct f_i do not lie in the same row or in the same column of the matrix $X = (x_{ij})$. Also the second part says that no two indeterminates of B_i which lie in the same row or in the same column of the matrix $X = (x_{ij})$ can appear in the same monomial of f_i.

As a consequence we obtain the following:

Corollary 2.6. *Let $G \leq S_n$, and $\chi : G \to \mathbb{C}$ be a nonzero function. Then*

i) *$m = n$ if and only if $d_\chi^G(X)$ is a multiple of a monomial if and only if there is a unique $\sigma \in G$ such that $\chi(\sigma) \neq 0$;*

ii) *if $G = S_n$ and χ is nonzero everywhere, then $d_\chi^G(X)$ is irreducible. In particular,* $\det(X)$ *and* $\mathrm{per}(X)$ *are irreducible.*

Proof. Part (i) can be deduced from part (iv) of the above theorem.

For part (ii), by hypothesis and part (v) of the above theorem we have

$$(\sum_{i=1}^{m} \deg f_i)! = n! = |\{\sigma \in S_n : \chi(\sigma) \neq 0\}| \leq \prod_{i=1}^{m} (\deg f_i)! \leq (\sum_{i=1}^{m} \deg f_i)!,$$

hence the equality holds in the above. But this is possible if and only if $m = 1$. This means that $d_\chi^G(X)$ must be irreducible. □

Our next goal is to give a refinement of part (ii) of the above corollary.

Theorem 2.7. *Let G be the alternating group A_n for $n \geq 4$ or the symmetric group S_n for $n \geq 3$. Let also $\chi : G \to \mathbb{C}$ be a function which is nonzero on every element of some nontrivial conjugacy class of G. Then $d_\chi^G(X)$ is irreducible.*

Proof. Suppose that $1 \neq \sigma \in G$ and K is the conjugacy class of σ in G such that χ is nonzero on every element of K. Suppose also that $\sigma = \sigma_1 \cdots \sigma_r$ is the decomposition of σ into the nontrivial disjoint cycles and $\sigma_1 = (i_1 i_2 \cdots i_s)$. Let $\{i_{s+1}, \ldots, i_n\}$ be the complement of $\{i_1, \ldots, i_s\}$ in $\{1, 2, \ldots, n\}$. First assume that $n \geq 4$. Since $\chi(\sigma) \neq 0$, so by notation of Theorem 2.5 the monomial $\prod_{i=1}^{n} x_{i\sigma(i)}$, containing the indeterminate $x_{i_1 i_2}$, must appear in $d_\chi^G(X) = f_1 f_2 \cdots f_m$. Therefore, without loss of generality, we may assume that $x_{i_1 i_2}$ appears in f_1 and hence $x_{i_1 i_2} \in B_1$. Now let $i_k \neq i_l$ be arbitrary such that $\{i_1, i_2\} \cap \{i_k, i_l\} = \emptyset$. Consider the elements $\tau = (i_2 i_k i_l)$, $\nu = (i_1 i_2 i_l)$ and $\mu = (i_1 i_2)(i_k i_l)$ from G. Now $\tau^{-1}\sigma\tau, \mu^{-1}\sigma\mu, \nu^{-1}\sigma\nu \in K$ and by hypothesis neither of $\chi(\tau^{-1}\sigma\tau)$, $\chi(\nu^{-1}\sigma\nu)$, $\chi(\mu^{-1}\sigma\mu)$ is zero. But $(\tau^{-1}\sigma\tau)(i_1) = i_k$, $(\nu^{-1}\sigma\nu)(i_2) = i_l$, $(\mu^{-1}\sigma\mu)(i_2) = i_1$, and so in a similar manner as above, we deduce that $x_{i_1 i_k}$, $x_{i_2 i_l}$, $x_{i_2 i_1}$ all must appear in $d_\chi^G(X)$. Since $x_{i_1 i_2} \in B_1$, hence by part (i) of Theorem 2.5, we have $x_{i_1 i_k} \in B_1$, which in turn implies that $\{i_2, \ldots, i_n\} \subseteq C_1$. In particular, $x_{i_1 i_l} \in B_1$, and again by part (i) of Theorem 2.5 we have $x_{i_2 i_l} \in B_1$. Finally by the same reason $x_{i_2 i_1} \in B_1$. This means that $i_1 \in C_1$ and so $\deg f_1 = |C_1| = n$, showing that $d_\chi^G(X) = f_1$ and the proof is complete in this case. The assertion can be proved similarly when $G = S_3$. □

Remark 4. The above theorem may be false if G is different from S_n or A_n. For example, if we let $G = \{1, (12), (34), (12)(34)\} \leq S_4$ and $\chi = 1_G$ be the principal character of G, then $d_\chi^G(X)$ is reducible since

$$d_\chi^G(X) = (x_{11}x_{22} + x_{12}x_{21})(x_{33}x_{44} + x_{34}x_{43}).$$

The following result is a consequence of the above theorem.

Corollary 2.8. *Let $G = A_n$ for $n \neq 2$ or $G = S_n$ and χ be an irreducible character of G. Then $d_\chi^G(X)$ is irreducible.*

3. On χ-singular and χ-nonsingular matrices

From linear algebra we know that a matrix $A \in M_n(\mathbb{C})$ is called singular if $\det(A) = 0$ and nonsingular if $\det(A) \neq 0$, that is, $d_\varepsilon^{S_n}(A) = 0$ and $d_\varepsilon^{S_n}(A) \neq 0$, respectively. Motivated by this, we introduce a more general concept. Let $G \leq S_n$ and $\chi : G \to \mathbb{C}$ be a function. A matrix $A \in M_n(\mathbb{C})$ is said to be χ-singular if $d_\chi^G(A) = 0$ and χ-nonsingular if $d_\chi^G(A) \neq 0$. So an ordinary singular (nonsingular) matrix is in fact an ε-singular (ε-nonsingular) matrix. Also a matrix $A \in M_n(\mathbb{C})$ is 1_{S_n}-singular if and only if $\text{per}(A) = d_{1_{S_n}}^{S_n}(A) = 0$. The following is Theorem 2.1 in [3].

Theorem 3.1. *Let $G \leq S_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then the following are equivalent*:

i) $d_\chi^G(A) \neq 0$ *for all ε-nonsingular matrices A*;
ii) $d_\chi^G(A) = 0$ *for all ε-singular matrices A*;
iii) $G = S_n$ *and* $\chi = \chi(1)\varepsilon$.

The above theorem as well as all other results of [3] with slight changes in proofs remain true if one substitutes 1_{S_n} for ε. Our next goal is to generalize the above theorem. Unfortunately, the proof given in [3] does not work in the general case and so we have to take a totally different approach.

Before stating the next theorem, we recall a few notation, definitions, and Hilbert's Nullstellensatz from [2].

Let R be a commutative ring with identity and I an ideal of R. The radical of I, denoted by $\sqrt{I}$ or $\text{rad}(I)$, is defined as follows:

$$\sqrt{I} = \{r \in R \,|\, r^n \in I \text{ for some } n \in \mathbb{N}\},$$

which is obviously an ideal of R containing I.

For a subset T of $\mathbb{C}^n$ we define $\mathcal{I}(T)$ as follows:

$$\mathcal{I}(T) = \{f \in \mathbb{C}[x_1, x_2, \ldots, x_n] \,|\, f(a_1, a_2, \ldots, a_n) = 0 \text{ for all } (a_1, a_2, \ldots, a_n) \in T\},$$

which is certainly an ideal of $\mathbb{C}[x_1, x_2, \ldots, x_n]$.

Also for a subset S of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ we define $\mathcal{Z}(S)$ as follows:

$$\mathcal{Z}(S) = \{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \,|\, f(a_1, a_2, \ldots, a_n) = 0 \text{ for all } f \in S\}.$$

Now Hilbert's Nullstellensatz states that if I is an ideal of $\mathbb{C}[x_1, x_2, \ldots, x_n]$, then $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

We are now ready to prove the next theorem from which our generalization of the above theorem will be derived. Though the following theorem has been proved in [1], but our proof, which uses Hilbert's Nullstellensatz and results of the previous section, is entirely different than theirs.

Theorem 3.2. *Let $\chi, \varphi : S_n \to \mathbb{C}$ be two functions. If every χ-singular matrix of $M_n(\mathbb{C})$ is a φ-singular matrix, then there is a $c \in \mathbb{C}$ such that $\varphi = c\chi$.*

Proof. If $\chi = 0$, then all matrices in $M_n(\mathbb{C})$ are χ-singular and so, by hypothesis, are φ-singular. Hence for an arbitrary $\sigma \in S_n$, we have $\varphi(\sigma) = d_\varphi^{S_n}(A_\sigma) = 0$, where A_σ is the permutation matrix induced by σ. Thus $\varphi = 0$ and the result follows in this case. Also, if $\varphi = 0$, then by taking $c = 0$ the result follows too.

Assume now that $\chi \neq 0 \neq \varphi$. Therefore $d_\chi^{S_n}(X)$ and $d_\varphi^{S_n}(X)$ are nonzero homogeneous polynomials with the same total degree n. Let I and J be the ideals of the ring $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ generated by the polynomials $d_\chi^{S_n}(X)$ and $d_\varphi^{S_n}(X)$, respectively. By hypothesis we have $\mathcal{Z}(I) \subseteq \mathcal{Z}(J)$ and hence $\mathcal{I}(\mathcal{Z}(J)) \subseteq \mathcal{I}(\mathcal{Z}(I))$. Therefore $\sqrt{J} \subseteq \sqrt{I}$ by Hilbert's Nullstellensatz, and since $d_\varphi^{S_n}(X) \in J \subseteq \sqrt{J}$ we obtain $(d_\varphi^{S_n}(X))^k \in I$ for some $k \in \mathbb{N}$. This implies that $(d_\varphi^{S_n}(X))^k$ is divisible by $d_\chi^{S_n}(X)$. Therefore, $d_\varphi^{S_n}(X)$ is divisible by $d_\chi^{S_n}(X)$ by part (ii) of Corollary 2.4. So there exists a $c \in \mathbb{C}$ such that $d_\varphi^{S_n}(X) = c d_\chi^{S_n}(X)$. This means that $\varphi = c\chi$ and the proof is complete. □

Before giving the next results, we introduce a notation. For $G \leq S_n$ and χ a complex valued function defined on G, let $\hat{\chi}$ be an extension of χ to S_n which vanishes outside of G. It is obvious that $d_\chi^G = d_{\hat{\chi}}^{S_n}$. Hence a matrix $A \in M_n(\mathbb{C})$ is χ-singular if and only if it is $\hat{\chi}$-singular. The next consequence is a generalization of Theorem 3.1.

Corollary 3.3. *Let H and K be two subgroups of S_n, and let $\chi : H \to \mathbb{C}$ and $\varphi : K \to \mathbb{C}$ be two nonzero functions. Then the following are equivalent*:

i) *every φ-nonsingular matrix of $M_n(\mathbb{C})$ is a χ-nonsingular matrix*;
ii) *every χ-singular matrix of $M_n(\mathbb{C})$ is a φ-singular matrix*;
iii) *χ and φ vanish outside of $H \cap K$, and there exists a $c \in \mathbb{C}$ such that $\varphi_{H\cap K} = c\chi_{H\cap K}$*;
iv) *χ and φ vanish outside of $H \cap K$, and there exists a $c \in \mathbb{C}$ such that $\chi_{H\cap K} = c\varphi_{H\cap K}$*;
v) *every φ-singular matrix of $M_n(\mathbb{C})$ is a χ-singular matrix*;
vi) *every χ-nonsingular matrix of $M_n(\mathbb{C})$ is a φ-nonsingular matrix.*

Proof. Let $\hat{\chi}$ and $\hat{\varphi}$ be as above. Since a matrix $A \in M_n(\mathbb{C})$ is χ-singular (φ-singular) if and only if it is $\hat{\chi}$-singular ($\hat{\varphi}$-singular), the result follows from Theorem 3.2. □

The following theorem can be viewed as a generalization of Theorem 2.2 of [3].

Theorem 3.4. *Let $G, H, K \leq S_n$, and χ, φ, ψ be complex valued functions defined on G, H, K, respectively, with $\chi(1) \neq 0$. Then the following are equivalent*:

i) $d_\varphi^H(A) = d_\psi^K(A)$ *for all χ-nonsingular matrices A*;
ii) $d_\varphi^H(A) = d_\psi^K(A)$ *for all χ-singular matrices A, and $\varphi(1) = \psi(1)$*;
iii) *φ and ψ vanish outside of $H \cap K$, and $\varphi_{H\cap K} = \psi_{H\cap K}$.*

Proof. It is trivial that (iii) implies (i) and (ii).

We now show that (i) implies (iii). It is sufficient to show the claim $\hat{\varphi}(\sigma) = \hat{\psi}(\sigma)$ for all $\sigma \in S_n$. Let $\sigma = \sigma_1 \cdots \sigma_r$ be the decomposition of σ into the nontrivial disjoint cycles. We prove the above claim by induction on r. If $r = 0$ then $\sigma = 1$, and since I_n is $\hat{\chi}$-nonsingular by hypothesis, so

$$\hat{\varphi}(1) = d_{\hat{\varphi}}^{S_n}(I_n) = d_{\hat{\psi}}^{S_n}(I_n) = \hat{\psi}(1).$$

Assume now that $r > 0$. If $\hat{\chi}(\sigma) \neq 0$, then A_σ, the permutation matrix induced by σ, is $\hat{\chi}$-nonsingular and so

$$\hat{\varphi}(\sigma) = d_{\hat{\varphi}}^{S_n}(A_\sigma) = d_{\hat{\psi}}^{S_n}(A_\sigma) = \hat{\psi}(\sigma).$$

Therefore we may assume that $\hat{\chi}(\sigma) = 0$. Since $\chi(1) \neq 0$, so there exists a permutation τ so that it can be written as a product of s distinct elements of the set $\{\sigma_1, \ldots, \sigma_r\}$ with $\hat{\chi}(\tau) \neq 0$ and s as maximum as possible. Obviously $0 \leq s < r$. Without loss of generality, we may assume that $\tau = \sigma_1 \cdots \sigma_s$. Let $\Gamma_0 = \{1\}$ and for $1 \leq k \leq r - s$, let Γ_k be the set of all permutations that can be written as a product of k distinct elements of the set $\{\sigma_{s+1}, \ldots, \sigma_r\}$. We define the diagonal matrix $B = (b_{ij}) \in M_n(\mathbb{C})$ as follows:

$$b_{ij} = \begin{cases} \delta_{ij} & i \in \mathrm{Fix}(\tau) - \mathrm{Fix}(\sigma) \\ 0 & i \notin \mathrm{Fix}(\tau) - \mathrm{Fix}(\sigma). \end{cases}$$

If $A = A_\sigma + B$, then by the maximality of s we have

$$d_{\hat{\chi}}^{S_n}(A) = \sum_{k=0}^{r-s} \sum_{\lambda \in \Gamma_k} \hat{\chi}(\tau\lambda) = \hat{\chi}(\tau) \neq 0.$$

This implies that the matrix A is $\hat{\chi}$-nonsingular and so by hypothesis we obtain

$$\hat{\varphi}(\sigma) + \sum_{k=0}^{r-s-1} \sum_{\lambda \in \Gamma_k} \hat{\varphi}(\tau\lambda) = d_{\hat{\varphi}}^{S_n}(A) = d_{\hat{\psi}}^{S_n}(A) = \hat{\psi}(\sigma) + \sum_{k=0}^{r-s-1} \sum_{\lambda \in \Gamma_k} \hat{\psi}(\tau\lambda).$$

But by induction we have

$$\sum_{k=0}^{r-s-1} \sum_{\lambda \in \Gamma_k} \hat{\varphi}(\tau\lambda) = \sum_{k=0}^{r-s-1} \sum_{\lambda \in \Gamma_k} \hat{\psi}(\tau\lambda).$$

Therefore $\hat{\varphi}(\sigma) = \hat{\psi}(\sigma)$, and the result follows.

Finally we show that (ii) implies (iii). By hypothesis, for all $\hat{\chi}$-singular matrices A, we have

$$d_{\hat{\varphi}}^{S_n}(A) = d_{\varphi}^{H}(A) = d_{\psi}^{K}(A) = d_{\hat{\psi}}^{S_n}(A),$$

and so $d_{\hat{\varphi}-\hat{\psi}}^{S_n}(A) = 0$. Thus by Theorem 3.2, $\hat{\varphi} - \hat{\psi} = c\hat{\chi}$ for some $c \in \mathbb{C}$. Since $\chi(1) \neq 0$ and $\varphi(1) = \psi(1)$, so $c = 0$. This implies that $\hat{\varphi} = \hat{\psi}$, and the proof is complete. □

Remark 5. First note that if χ is a character of G, then automatically $\chi(1) \neq 0$. Also, the condition $\varphi(1) = \psi(1)$ in case (ii) of the above theorem is essential. This is because if φ is any function of S_n, then $d_{\varphi}^{S_n}(A) = d_{\varphi+\varepsilon}^{S_n}(A)$ for all ε-singular matrices A. Of course, the authors in Theorem 2.2 of [3] showed that the condition $\varphi(1) = \psi(1)$ can be removed if $G = S_n$, $\chi = \varepsilon$ or 1_{S_n}, and φ and ψ are irreducible characters. But, in general, we cannot expect to get a similar theorem here. For example, let $G = H = A_3$ and $K = S_3$. Also let χ, φ be the two distinct nonprincipal linear characters of A_3 and ψ be the nonlinear irreducible character of S_3. It is obvious that $\psi = \hat{\chi} + \hat{\varphi}$ and therefore for all χ-singular matrices A we have

$$d_{\psi}^{K}(A) = d_{\hat{\chi}+\hat{\varphi}}^{S_3}(A) = d_{\hat{\chi}}^{S_3}(A) + d_{\hat{\varphi}}^{S_3}(A) = d_{\varphi}^{H}(A).$$

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