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CONDITIONAL TRANSFORM WITH RESPECT TO THE GAUSSIAN PROCESS INVOLVING THE CONDITIONAL CONVOLUTION PRODUCT AND THE FIRST VARIATION

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Abstract. In this paper, we define a conditional transform with respect to the Gaussian process, the conditional convolution product and the first variation of functionals via the Gaussian process. We then examine various relationships of the conditional transform with respect to the Gaussian process, the conditional convolution product and the first variation for functionals F in S_α [5, 8].

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

A subset $\mathcal{B}$ of $C_0[0,T]$ is said to be scale-invariant measurable provided $\rho\mathcal{B}$ is $\mathcal{M}$-measurable for all $\rho>0$, and a scale-invariant measurable set $\mathcal{N}$ is said to be a scale-invariant null set provided $m(\rho\mathcal{N})=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [10].

In [5, 8], the authors introduced the concept of Banach algebra S_α. In [8], the authors studied the generalized integral transform (GIT) and the generalized convolution product (GCP) of functionals in S_α. Additionally, they established some properties for the GIT and GCP. In [5], the authors obtained the conditional integral transform and the conditional convolution product of

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functionals in S_α. They established relationships between two of the three concepts of conditional integral transform, the conditional convolution product and the first variation. Recently, in [14], the authors introduced the transform with respect to the Gaussian process and the conditional convolution product.

In this paper, we introduce a conditional transform with respect to the Gaussian process and conditional convolution product. We then establish relationships between the conditional transform with respect to the Gaussian process of the conditional convolution product and the first variation. In Section 3, we establish the conditional transform with respect to the Gaussian process involving the functional of a functional in S_α. We also obtain the conditional convolution product of functionals in S_α. In Section 4, we establish the relationships between two of the three concepts of the conditional transform, the conditional convolution product, and first variation of the functionals. In Section 5, we establish all relationships between all three of these concepts.

2. Definitions and preliminaries

In this section, we list some definitions and properties from [5, 8, 11, 12, 14].

Now we state definitions and notations which are needed to understand this paper.

(1) For $h \in L^2[0,T]$, we define the Gaussian process Z_h by

$$Z_h(x,t) = \int_0^t h(s)\tilde{d}x(t), \tag{2.1}$$

where $\int_0^t h(s)\tilde{d}x(t)$ denotes the PWZ integral. For each $v \in L^2[0,T]$, let $\langle v, x\rangle = \int_0^T v(t)\tilde{d}x(t)$. From [6], we note that

$$\langle v, Z_h(x,\cdot)\rangle = \langle vh, x\rangle$$

for $h \in L_\infty[0,T]$ and s-a.e. $x \in C_0[0,T]$. Thus, throughout this paper, we require h to be in $L_\infty[0,T]$ rather than simply in $L^2[0,T]$.

(2) For all $v \in L^2[0,T]$, let

$$B_v = \frac{1}{T}\int_0^T v(t)dt. \tag{2.2}$$

(3) Let $K_0[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_0[0,T]$; namely,

$$K_0[0,T] = \{x : [0,T] \to \mathbb{C} \mid x(0)=0,\ \mathrm{Re}(x) \in C_0[0,T] \text{ and } \mathrm{Im}(x) \in C_0[0,T]\}.$$

Thus $C_0[0,T]$ is the subspace of all real-valued functions in $K_0[0,T]$.

(4) Let $\mathbb{C}$ be the class of all complex numbers. For each $\alpha \in \mathbb{C}$, let

$$E_\alpha \equiv \{(\gamma,\beta) \in \mathbb{C}\times\mathbb{C} : \mathrm{Re}(\alpha^2\gamma^2) \le 0 \text{ and } \mathrm{Re}(\alpha^2\beta^2) \le 0\}.$$

Now, we state the definitions of the transform with respect to the Gaussian process, the conditional convolution product and the first variation.

Definition 2.1. Let F and G be functionals on $K_0[0,T]$ and let γ, β, ρ and τ be non-zero complex numbers. Then the transform with respect to the Gaussian process, the convolution product and the first variation are defined by formulas

$$(T^{h_1,h_2}_{\gamma,\beta}(F))(y) = \int_{C_0[0,T]} F\Big(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot)\Big) dm(x), \tag{2.3}$$

$$\begin{aligned}((F *_{s_1 s_2} G)_{\rho,\tau})(y) = \int_{C_0[0,T]} & F\Big(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}(x,\cdot)\Big) \\ & \cdot G\Big(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}(x,\cdot)\Big) dm(x),\end{aligned} \tag{2.4}$$

$$\delta F\Big(Z_h(x,\cdot) | Z_s(z,\cdot)\Big) = \frac{\partial}{\partial k} F\Big(Z_h(x,\cdot) + k Z_s(z,\cdot)\Big)\Big|_{k=0} \tag{2.5}$$

if they exist.

Remark 2.2. (1) When $h_1(t) = h_2(t) = 1$ on $[0,T]$, $T^{1,1}_{\gamma,\beta}$ is the integral transform used by Kim and Skoug [13]. In particular, $T^{1,1}_{1,i}(F)$ is the Fourier-Wiener transform introduced by Cameron in [1]. Also, $T^{1,1}_{\sqrt{2},i}(F)$ is the Fourier-Wiener transform used by Cameron and Martin [2].

(2) If $s_1(t) = s_2(t) = 1$ on $[0,T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{1}{\sqrt{2\lambda}}$ for $\lambda \in \tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$, then the convolution product $(F *_{11} G)_{\rho,\tau}$ coincides with convolution product $(F * G)_\lambda$ [3, 4, 7, 8]; that is to say, $(F *_{11} G)_{\rho,\tau} = (F * G)_\lambda$ for $\lambda \in \tilde{\mathbb{C}}_+$.

(3) If $h(t) = s(t) = 1$ on $[0,T]$, then the first variation of F with respect to the Gaussian process coincides with the first variation of F [5].

Let X be a $\mathbb{R}$-valued function on $C_0[0,T]$ whose probability distribution m_X is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Let F be a $\mathbb{C}$-valued m-integrable functional on $C_0[0,T]$. Then the conditional integral of F given X, denoted by $E[F\|X](\eta)$, is a Lebesgue measurable function of η, unique up to null sets in $\mathbb{R}$, satisfying the equation

$$\int_{X^{-1}(B)} F(x) dm(x) = \int_B E[F\|X](\eta) dm_X(\eta)$$

for all Borel sets B in $\mathbb{R}$.

Throughout this paper, we will condition by the function $X : C_0[0,T] \to \mathbb{R}$ given by

$$X(x) = x(T). \tag{2.6}$$

In [15], Park and Skoug gave a simple formula for expression conditional function space integrals in terms of an ordinary function space integrals by the formula

$$E[F\|X](\eta) = \int_{C_0[0,T]} F\Big(x(\cdot) - \frac{\cdot}{T} x(T) + \frac{\cdot}{T}\eta\Big) dm(x).$$

The following Wiener integral is used several times in this paper. For each $\alpha \in \mathbb{C}$ and for $v \in L^2[0,T]$

$$\int_{C_0[0,T]} \exp\{\alpha\langle v, x\rangle\} dm(x) = \exp\left\{\frac{\alpha^2}{2}\|v\|_2^2\right\}. \tag{2.7}$$

Next, we give the definition of the conditional transform with respect to the Gaussian process.

For each partition $\tau = \{t_1, \ldots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_\tau : C_0[0,T] \to \mathbb{R}^n$ be defined by

$$X_\tau(x) = (x(t_1), \ldots, x(t_n)). \tag{2.8}$$

Define a function $[x]$ on $[0,T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1}))$$

for $t_{j-1} \le t \le t_j$. Similarly, for $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, define the function $[\vec{\eta}]$ of $\vec{\eta}$ on $[0,T]$ by

$$[\vec{\eta}](t) = \eta_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\eta_j - \eta_{j-1}),$$

where $t_{j-1} \le t \le t_j$. Then both $[x]$ and $[\vec{\eta}]$ are continuous on $[0,T]$, they are line segments on each interval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\eta}](t_j) = \eta_j$ at each t_j.

Definition 2.3. Let F be a functional defined on $K_0[0,T]$ and let X be given by equation (2.8). For each non-zero complex numbers γ and β, the conditional transform with respect to the Gaussian process $T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta)$ of F given X is given by the formula

$$T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta) = \int_{C_0[0,T]} F\Big(\gamma Z_{h_1}(x - [x] + [\vec{\eta}], \cdot) + \beta Z_{h_2}(y, \cdot)\Big) dm(x)$$

for $y \in K_0[0,T]$ and $\eta \in \mathbb{R}$, if it exists. In particular, if X is given by equation (2.6), then

$$\begin{aligned} &T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta) \\ &= \int_{C_0[0,T]} F\Big(\gamma Z_{h_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big) + \beta Z_{h_2}(y, \cdot)\Big) dm(x). \end{aligned} \tag{2.9}$$

The following simple example illustrates Definition 2.3 above.

Example 2.4. Define the functional $F : K_0[0,T] \to \mathbb{C}$ by $F(x) = \exp\{\langle v, x\rangle\}$ for $v \in L^2[0,T]$. Then

$$\begin{aligned} &T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta) \\ &= \int_{C_0[0,T]} F\Big(\gamma Z_{h_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big) + \beta Z_{h_2}(y, \cdot)\Big) dm(x) \end{aligned}$$

$$= \int_{C_0[0,T]} \exp\Big\{\gamma\Big\langle v, Z_{h_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big)\Big\rangle + \beta\langle vh_2, y\rangle\Big\} dm(x)$$

$$= \exp\{\beta\langle vh_2, y\rangle + \gamma\eta B_{vh_1}\} \int_{C_0[0,T]} \exp\{\gamma\langle vh_1 - B_{vh_1}, x\rangle\} dm(x)$$

$$= \exp\Big\{\beta\langle vh_2, y\rangle + \frac{\gamma^2}{2}\|vh_1 - B_{vh_1}\|_2^2 + \gamma\eta B_{vh_1}\Big\}.$$

Next, we state the definition of the conditional convolution product of functionals F and G on $K_0[0,T]$.

Definition 2.5. Let F and G be functionals defined on $K_0[0,T]$ and let X be given by equation (2.6). For each non-zero complex numbers ρ and τ, the conditional convolution product $((F *_{s_1 s_2} G)_{\rho,\tau}\|X)(y,\eta)$ of F and G given X is given by the formula

$$\begin{aligned}(2.10)\qquad & ((F *_{s_1 s_2} G)_{\rho,\tau}\|X)(y,\eta) \\ &= \int_{C_0[0,T]} F\Big(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big)\Big) \\ &\qquad \cdot G\Big(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big)\Big) dm(x)\end{aligned}$$

for $y \in K_0[0,T]$ and $\eta \in \mathbb{R}$, if it exists.

Remark 2.6. (1) When $h_1(t) = h_2(t) = 1$ on $[0,T]$, $T^{1,1}_{\gamma,\beta}(F\|X)(y,\eta)$ is the conditional integral transform $\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta)$ introduced by Chung, Choi and Chang [5].

(2) When $s_1(t) = s_2(t) = 1$ on $[0,T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{\gamma}{\sqrt{2}}$, $((F *_{11} G)_{\rho,\tau}\|X)(y,\eta)$ is the conditional convolution product $((F * G)_\gamma\|X)(y,\eta)$ introduced by Chung, Choi and Chang [5].

3. Conditional transform with respect to the Gaussian process on function space

In this section, we establish the conditional transform with respect to the Gaussian process for the functional F in S_α. We then obtain the conditional transform with respect to the Gaussian process involving the conditional convolution product and the first variation.

Let $\mathcal{M}(L^2[0,T])$ be the space of $\mathbb{C}$-valued, countably additive (and hence finite) Borel measures on $L^2[0,T]$ [9, pp. 126–127] and [16, p. 119]. $\mathcal{M}(L^2[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication. For each complex number α with $\mathrm{Re}(\alpha^2) \le 0$, let S_α be the class of functionals of the form

$$(3.1)\qquad F(x) = \int_{L^2[0,T]} \exp\{\alpha\langle v, x\rangle\} df(v)$$

for s-a.e. $x \in C_0[0,T]$, where $f \in \mathcal{M}(L^2[0,T])$ [5, 8].

Remark 3.1. (1) For each complex number α with $\mathrm{Re}(\alpha^2) \le 0$, using formula (2.7) above, we have

$$\int_{C_0[0,T]} \int_{L^2[0,T]} \exp\{\alpha\langle v, x\rangle\} df(v) dm(x) = \int_{L^2[0,T]} \exp\Big\{\frac{\alpha^2}{2}\|v\|_2^2\Big\} df(v) < \infty$$

since $|\exp\{\frac{\alpha^2}{2}\|v\|_2^2\}| \le 1$. This tells us that $\int_{L^2[0,T]} \exp\{\alpha\langle v, x\rangle\} df(v)$ exists for a.e. $x \in C_0[0,T]$. Furthermore, for all real number $\rho > 0$,

$$\int_{C_0[0,T]} \int_{L^2[0,T]} \exp\{\alpha\langle v, \rho x\rangle\} df(v) dm(x) = \int_{L^2[0,T]} \exp\Big\{\frac{\rho^2\alpha^2}{2}\|v\|_2^2\Big\} df(v) < \infty$$

because $\mathrm{Re}(\rho^2\alpha^2) \le 0$. Hence the functional F given by equation (3.1) is well defined for s-a.e. $x \in C_0[0,T]$.

(2) The map $f \to F$ defined by (3.1) sets up on algebra isomorphism between $\mathcal{M}(L^2[0,T])$ and S_α. In this case, S_α becomes a Banach algebra under the norm $\|F\| = \|f\|$.

(3) Note that for $F \in S_\alpha$, the function $G : C_0[0,T] \to \mathbb{C}$ given by $G(x) = F(Z_h(x,\cdot))$ with $h \in L_\infty[0,T]$, belongs to the Banach algebra S_α, see [6].

Remark 3.2 ([5, Remark 2.3]). (1) First we could consider the following integral

$$\int_{L^2[0,T]} \exp\{\alpha\langle v, x\rangle + \xi B_v\} df(v), \quad \xi \in \mathbb{C}. \tag{3.2}$$

If we assume that

$$\int_{L^2[0,T]} \exp\Big\{|\xi| \int_0^T |v(t)|dt\Big\} |df(v)| < \infty \tag{3.3}$$

for all complex number ξ, then

$$\int_{L^2[0,T]} \exp\{\xi B_v\} df(v) < \infty \quad \text{and} \quad \int_{L^2[0,T]} \exp\{\alpha\langle v, x\rangle\} df(v)$$

exists for s-a.e. $x \in C_0[0,T]$. However, the integral (3.2) might not exist because the product of $L^1[0,T]$-functionals might not be in $L^1[0,T]$.

(2) In this paper, we need a condition for f in $\mathcal{M}(L^2[0,T])$ to show the existence of the integral in equation (3.2).

(i) If $v \in L^2[0,T]$ is a function of bounded variation, then for each $x \in C_0[0,T]$

$$|\langle v, x\rangle| \le \|x\|_\infty (|v(T)| + V_0^T(v)) < \infty,$$

where $V_0^T(v)$ is the total variation of v on $[0,T]$. Hence if we assume that

$$\int_{L^2[0,T]} \exp\Big\{|\xi| \Big[|v(T)| + |V_0^T(v)| + \int_0^T |v(t)|dt\Big]\Big\} |df(v)| < \infty, \tag{3.4}$$

then the integral (3.2) always exists.

(ii) Let v be an element of $L^2[0,T]$. Then we note that

$$|\langle v, x\rangle| = \lim_{n\to\infty} |\langle v_n, x\rangle| \le \lim_{n\to\infty} \|x\|_\infty (|v_n(T)| + V_0^T(v_n)), \tag{3.5}$$

where $v_n(t)=\sum_{k=1}^{n}(v,\phi_k)_2\phi_k(t)$, $\{\phi_k\}$ is a complete orthonormal set in $L^2[0,T]$ and $(\cdot,\cdot)_2$ is the inner product on $L^2[0,T]$. Hence if we add a condition

$$\lim_{n\to\infty}(|v_n(T)|+V_0^T(v_n)) \tag{3.6}$$

exists, then we can obtain the existence of the integral (3.2) under the condition which is similar to the condition (3.4).

(3) As mentioned above, we can give the condition (3.6) because the expression (3.5) is independent of the choice of the complete orthonormal set $\{\phi_k\}$ and the all expressions in equation (3.5) exists for s-a.e. $x\in C_0[0,T]$. Hence, we assume that for $f\in\mathcal{M}(L^2[0,T])$ which satisfies the condition (3.3) above, the integral (3.2) always exists.

In our next theorem, we obtain the conditional transform with respect to the Gaussian process of a functional in S_α.

Theorem 3.3. *Let F be an element of S_α. Let X be given by equation* (2.6). *Then for all $(\gamma,\beta)\in E_\alpha$, the conditional transform with respect to the Gaussian process $T_{\gamma,\beta}^{h_1,h_2}(F\|X)$ exists and is given by the formula*

$$\begin{aligned} T_{\gamma,\beta}^{h_1,h_2}&(F\|X)(y,\eta)\\ &=\int_{L^2[0,T]}\exp\Big\{\alpha\beta\langle vh_2,y\rangle+\frac{\gamma^2\alpha^2}{2}\|vh_1-B_{vh_1}\|_2^2+\gamma\alpha B_{vh_1}\eta\Big\}df(v)\end{aligned} \tag{3.7}$$

for s-a.e. $y\in C_0[0,T]$ and a.e. $\eta\in\mathbb{R}$, where B_{vh_1} is given by equation (2.2). *Furthermore $T_{\gamma,\beta}^{h_1,h_2}(F\|X)$ is an element of $S_{\alpha\beta}$ with associated measure ϕ_1^η defined by*

$$\phi_1^\eta(E)=\int_E\exp\Big\{\frac{\gamma^2\alpha^2}{2}\|vh_1-B_{vh_1}\|_2^2+\gamma\alpha B_{vh_1}\eta\Big\}df(v) \tag{3.8}$$

for $E\in\mathcal{B}(L^2[0,T])$.

Proof. Using equations (2.9) and (3.1), we have

$$\begin{aligned}&T_{\gamma,\beta}^{h_1,h_2}(F\|X)(y,\eta)\\ &=\int_{C_0[0,T]}\int_{L^2[0,T]}\exp\Big\{\alpha\gamma\langle vh_1-B_{vh_1},x\rangle+\alpha\gamma\eta B_{vh_1}+\alpha\beta\langle vh_2,y\rangle\Big\}df(v)dm(x).\end{aligned}$$

Applying the Fubini theorem and equation (2.7) to the equation above, it follows that

$$\begin{aligned}&T_{\gamma,\beta}^{h_1,h_2}(F\|X)(y,\eta)\\ &=\int_{L^2[0,T]}\exp\Big\{\alpha\beta\langle vh_2,y\rangle+\frac{\gamma^2\alpha^2}{2}\|vh_1-B_{vh_1}\|_2^2+\gamma\alpha B_{vh_1}\eta\Big\}df(v).\end{aligned}$$

Since $(\gamma,\beta)\in E_\alpha$ and hypothesis of Remark 3.2, the last expression above exists. Thus the equation (3.7) is established. Also, the last expression above becomes

$$T_{\gamma,\beta}^{h_1,h_2}(F\|X)(y,\eta) = \int_{L^2[0,T]} \exp\Big\{\alpha\beta\langle vh_2, y\rangle\Big\} d\phi_1^\eta(v).$$

Hence $T_{\gamma,\beta}^{h_1,h_2}(F\|X)$ is an element of $S_{\alpha\beta}$ because ϕ_1^n is an element of $\mathcal{M}(L^2[0,T])$. □

Remark 3.4. The main result in [5, Theorem 3.1] follows immediately from Theorem 3.3 above by choosing $h_1(t) = h_2(t) = 1$ on $[0,T]$.

In our next theorem, we establish the double conditional transform involving the functional of a functional in S_α.

Theorem 3.5. *Let F and X be as in Theorem 3.3. Assume that $h_3(t) = h_1(t)h_4(t)$ on $[0,T]$ and $\gamma_2 = \gamma_1\beta_2$. Then for all $(\gamma_2, \beta_2) \in E_\alpha$, $(\gamma_1, \beta_1) \in E_{\alpha\beta_2}$ and $(\gamma_2, \beta_1\beta_2) \in E_\alpha$,*

$$T_{\gamma_1,\beta_1}^{h_1,h_2}(T_{\gamma_2,\beta_2}^{h_3,h_4}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2) = T_{\sqrt{2}\gamma_2,\beta_1\beta_2}^{h_3,h_2h_4}(F\|X)\Big(y, \tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big) \tag{3.9}$$

for s-a.e. $y \in C_0[0,T]$ and a.e. $\eta_1, \eta_2 \in \mathbb{R}$. Also, both sides of the expressions in equation (3.9) are given by the formula

$$\int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle + \alpha^2\gamma_2^2\|vh_3 - B_{vh_3}\|_2^2 + \alpha\gamma_2(\eta_2+\eta_1)B_{vh_3}\Big\} df(v). \tag{3.10}$$

Furthermore, $T_{\sqrt{2}\gamma_2,\beta_1\beta_2}^{h_3,h_2h_4}(F\|X)$ is an element of $S_{\alpha\beta_1\beta_2}$ with associated measure ϕ_2^η defined by

$$\phi_2^\eta(E) = \int_E \exp\Big\{\alpha^2\gamma_2^2\|vh_3 - B_{vh_3}\|_2^2 + \alpha\gamma_2(\eta_2+\eta_1)B_{vh_3}\Big\} df(v)$$

for $E \in \mathcal{B}(L^2[0,T])$.

Proof. By using equations (2.9) and (3.1), we have

$$\begin{aligned}
&T_{\gamma_1,\beta_1}^{h_1,h_2}(T_{\gamma_2,\beta_2}^{h_3,h_4}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&= \int_{C_0[0,T]}\int_{C_0[0,T]}\Big[\int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle + \alpha\gamma_2\langle vh_3 - B_{vh_3}, w\rangle\\
&\quad + \alpha\gamma_2\eta_1 B_{vh_3} + \alpha\gamma_1\beta_2\langle vh_1h_4 - B_{vh_1h_4}, x\rangle\\
&\quad + \alpha\gamma_1\beta_2\eta_2 B_{vh_1h_4}\Big\} df(v)\Big] dm(x)dm(w)\\
&= \int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle + \alpha\gamma_2\eta_1 B_{vh_3} + \alpha\gamma_1\beta_2\eta_2 B_{vh_1h_4}\Big\}\\
&\quad \cdot \Big[\int_{C_0[0,T]} \exp\Big\{\alpha\gamma_1\beta_2\langle vh_1h_4 - B_{vh_1h_4}, x\rangle\Big\} dm(x)\Big]\\
&\quad \cdot \Big[\int_{C_0[0,T]} \exp\Big\{\alpha\gamma_2\langle vh_3 - B_{vh_3}, w\rangle\Big\} dm(w)\Big] df(v)
\end{aligned}$$

$$= \int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle + \alpha\gamma_2\eta_1 B_{vh_3} + \alpha\gamma_1\beta_2\eta_2 B_{vh_1h_4} + \frac{\alpha^2\gamma_1^2\beta_2^2}{2}\|vh_1h_4 - B_{vh_1h_4}\|_2^2 + \frac{\alpha^2\gamma_2^2}{2}\|vh_3 - B_{vh_3}\|_2^2\Big\}.$$

Since $h_3(t) = h_1(t)h_4(t)$ on $[0,T]$ and $\gamma_2 = \gamma_1\beta_2$, the last expression equation above is equal to

$$\int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle + \alpha^2\gamma_2^2\|vh_3 - B_{vh_3}\|_2^2 + \alpha\gamma_2(\eta_2+\eta_1)B_{vh_3}\Big\}df(v).$$

Hence by Definition 2.3, we have

$$T_{\gamma_1,\beta_1}^{h_1,h_2}(T_{\gamma_2,\beta_2}^{h_3,h_4}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2) = T_{\sqrt{2}\gamma_2,\beta_1\beta_2}^{h_3,h_2h_4}(F\|X)\Big(y, \tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big).$$

Thus the equation (3.9) is established. Using the similar method in the proof of Theorem 3.3, both sides of the equation (3.5) exists. Also, equation (3.10) becomes

$$\int_{L^2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle vh_2h_4, y\rangle\Big\}d\phi_2^{\eta}(v).$$

Hence equation (3.9) above is an element of $S_{\alpha\beta_1\beta_2}$ because ϕ_2^{η} is an element of $\mathcal{M}(L^2[0,T])$. □

In our next theorem, we obtain the conditional convolution product of functionals in S_α.

Theorem 3.6. *Let F and G be elements of S_α. Let X be given by equation (2.6). Then for all $(\rho,\tau) \in E_\alpha$, the conditional convolution product $((F *_{s_1s_2} G)_{\rho,\tau}\|X)$ exists and is given by the formula*

$$(3.11)\quad ((F *_{s_1s_2} G)_{\rho,\tau}\|X)(y,\eta) = \int_{L^2[0,T]}\int_{L^2[0,T]} \exp\Big\{\alpha\tau\langle (v+u)s_2, y\rangle + \frac{\alpha^2\rho^2}{2}\|(v-u)s_1 - B_{(v-u)s_1}\|_2^2 + \alpha\rho\eta B_{(v-u)s_1}\Big\}df(v)dg(u)$$

for s-a.e. $y \in C_0[0,T]$ and a.e. $\eta \in \mathbb{R}$. Furthermore $((F *_{s_1s_2} G)_{\rho,\tau}\|X)$ is an element of $S_{\alpha\tau}$.

Proof. By using equations (2.7) and (2.10), we have

$$((F *_{s_1s_2} G)_{\rho,\tau}\|X)(y,\eta)$$
$$= \int_{C_0[0,T]} F\Big(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big) \cdot G\Big(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big)dm(x)$$
$$= \int_{L^2[0,T]}\int_{L^2[0,T]} \exp\Big\{\alpha\tau\langle (v+u)s_2, y\rangle + \alpha\rho\eta(B_{vs_1} - B_{us_1})\Big\}$$

$$\cdot \Big[\int_{C_0[0,T]} \exp\Big\{ \alpha\rho\langle (v-u)s_1 - (B_{vs_1} - B_{us_1}), x\rangle \Big\} dm(x) \Big] df(v)dg(u)$$
$$= \int_{L^2[0,T]} \int_{L^2[0,T]} \exp\Big\{ \alpha\tau\langle (v+u)s_2, y\rangle + \alpha\rho\eta B_{(v-u)s_1}$$
$$+ \frac{\alpha^2\rho^2}{2} \|(v-u)s_1 - B_{(v-u)s_1}\|_2^2 \Big\} df(v)dg(u).$$

Let a set function $\phi_3^\eta : \mathcal{B}(L^2[0,T] \times L^2[0,T]) \to \mathbb{C}$ be defined by

$$\phi_3^\eta(E) = \int_E \exp\Big\{ \frac{\alpha^2\rho^2}{2} \|(v-u)s_1 - B_{(v-u)s_1}\|_2^2 + \alpha\rho\eta B_{(v-u)s_1} \Big\} df(v)dg(u)$$

for $E \in \mathcal{B}(L^2[0,T] \times L^2[0,T])$. Then ϕ_3^η is a complex Borel measure on $\mathcal{B}(L^2[0,T] \times L^2[0,T])$. Now we define a function $\varphi : L^2[0,T] \times L^2[0,T] \to L^2[0,T]$ by $\varphi(u,v) = u + v$. Let $\tilde{\phi} = \phi_3^\eta \circ \varphi^{-1}$. Then $\tilde{\phi}$ belongs to $\mathcal{M}(L^2[0,T])$ since $(\rho,\tau) \in E_\alpha$. So

$$((F *_{s_1 s_2} G)_{\rho,\tau} \| X)(y,\eta) = \int_{L^2[0,T]} \exp\{\alpha\tau\langle ws_2, y\rangle\} d\tilde{\phi}(w).$$

Hence $((F *_{s_1 s_2} G)_{\rho,\tau} \| X)$ exists and is given by (3.11) and it belongs to $S_{\alpha\tau}$. □

Remark 3.7. The main result in [5, Theorem 3.3] follows immediately from Theorem 3.6 above by choosing $s_1(t) = s_2(t) = 1$ on $[0,T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{\gamma}{\sqrt{2}}$.

4. Relationships between two concepts

In Section 3, we introduced the conditional transform with respect to the Gaussian process and the conditional convolution product. In this section, we establish the relationships between exactly two of three concepts of the conditional transform, conditional convolution product and the first variation for functionals.

In our next theorem, we obtain the conditional transform with respect to the Gaussian process involving the conditional convolution product of functionals in S_α.

Theorem 4.1. *Let F, G and X be as in Theorem 3.3. Assume that $\tau\gamma = \rho$ and $h_1(t)s_2(t) = s_1(t)$ on $[0,T]$. Then for all $(\rho,\tau) \in E_\alpha$, $(\gamma,\beta) \in E_{\alpha\tau}$ and $(\rho,\beta\tau) \in E_\alpha$,*

$$T_{\gamma,\beta}^{h_1,h_2}(((F *_{s_1 s_2} G)_{\rho,\tau} \| X)(\cdot,\eta_1) \| X)(y,\eta_2) \tag{4.1}$$
$$= T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2 s_2}(F\|X)\Big(y, \tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big) T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2 s_2}(G\|X)\Big(y, \tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_0[0,T]$ and a.e. $\eta_1, \eta_2 \in \mathbb{R}$. Also, both sides of the expression in equation (4.1) are given by the formula

$$\int_{L^2[0,T]} \int_{L^2[0,T]} \exp\Big\{ \alpha\beta\tau\langle (v+u)h_2 s_2, y\rangle + \alpha^2\rho^2 (\|vs_1 - B_{vs_1}\|_2^2 + \|us_1 - B_{us_1}\|_2^2)$$

$$+ \alpha\rho(\eta_2 B_{(v+u)s_1} + \eta_1 B_{(v-u)s_1})\Big\} df(v)dg(u).$$

Proof. First by using equations (2.9), (2.10) and the Fubini theorem, we obtain that

(4.2)

$$\begin{aligned}
&T^{h_1,h_2}_{\gamma,\beta}(((F *_{s_1 s_2} G)_{\rho,\tau} \| X)(\cdot, \eta_1) \| X)(y, \eta_2) \\
&= \int_{C_0[0,T]} \int_{C_0[0,T]} F\Big(\tau\gamma Z_{h_1 s_2}(x, \cdot) - \tau\gamma B_{h_1 s_2} x(T) + \tau\gamma\eta_2 B_{h_1 s_2} \\
&\qquad + \tau\beta Z_{h_2 s_2}(y, \cdot) + \rho Z_{s_1}(w, \cdot) - \rho B_{s_1} w(T) + \rho\eta_1 B_{s_1}\Big) \\
&\quad \cdot G\Big(\tau\gamma Z_{h_1 s_2}(x, \cdot) - \tau\gamma B_{h_1 s_2} x(T) + \tau\gamma\eta_2 B_{h_1 s_2} \\
&\qquad + \tau\beta Z_{h_2 s_2}(y, \cdot) - \rho Z_{s_1}(w, \cdot) + \rho B_{s_1} w(T) - \rho\eta_1 B_{s_1}\Big) dm(w)dm(x) \\
&= \int_{L^2[0,T]} \int_{L^2[0,T]} \exp\Big\{\alpha\tau\beta\langle (v+u)h_2 s_2, y\rangle + \alpha\rho\eta_1 B_{(v-u)s_1} + \alpha\tau\gamma\eta_2 B_{(v+u)h_1 s_2}\Big\} \\
&\quad \cdot \Big[\int_{C_0[0,T]} \exp\Big\{\alpha\tau\gamma\langle (v+u)h_1 s_2 - B_{(v+u)h_1 s_2}, x\rangle\Big\} dm(x)\Big] \\
&\quad \cdot \Big[\int_{C_0[0,T]} \exp\Big\{\alpha\rho\gamma\langle (v-u)s_1 - B_{(v-u)s_1}, w\rangle\Big\} dm(w)\Big] df(v)dg(u) \\
&= \int_{L^2[0,T]} \int_{L^2[0,T]} \exp\Big\{\alpha\beta\tau\langle (v+u)h_2 s_2, y\rangle + \alpha^2\rho^2(\|vs_1 - B_{vs_1}\|_2^2 \\
&\qquad + \|us_1 - B_{us_1}\|_2^2) + \alpha\rho(\eta_2 B_{(v+u)s_1} + \eta_1 B_{(v-u)s_1})\Big\} df(v)dg(u).
\end{aligned}$$

Next, using equation (3.7), we have

(4.3)

$$\begin{aligned}
&T^{s_1,h_2 s_2}_{\sqrt{2}\rho,\beta\tau}(F\|X)\Big(y, \tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big) \\
&= \int_{L^2[0,T]} \exp\Big\{\alpha\beta\tau\langle vh_2 s_2, y\rangle + \alpha^2\rho^2\|vs_1 - B_{vs_1}\|_2^2 + \alpha\rho(\eta_2 + \eta_1)B_{vs_1}\Big\} df(v)
\end{aligned}$$

and

(4.4)

$$\begin{aligned}
&T^{s_1,h_2 s_2}_{\sqrt{2}\rho,\beta\tau}(G\|X)\Big(y, \tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big) \\
&= \int_{L^2[0,T]} \exp\Big\{\alpha\beta\tau\langle uh_2 s_2, y\rangle + \alpha^2\rho^2\|us_1 - B_{us_1}\|_2^2 + \alpha\rho(\eta_2 - \eta_1)B_{us_1}\Big\} dg(u).
\end{aligned}$$

Thus, equation (4.1) follows from equations (4.2)–(4.4). Since $(\rho,\tau) \in E_\alpha$, $(\rho,\beta\tau) \in E_\alpha$ and hypothesis of Remark 3.2, the both sides of the expressions in equation (4.1) exists. □

Remark 4.2. The main result in [5, Theorem 3.3] follows immediately from Theorem 4.1 above by choosing $h_j(t) = s_j(t) = 1 (j = 1, 2)$ on $[0, T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{\gamma}{\sqrt{2}}$.

The following theorem follows immediately from Theorems 3.5 and 4.1.

Theorem 4.3. *Let F and X be as in Theorem 4.1. Assume that $\tau\gamma = \rho$, $\gamma = \sqrt{2}\rho\beta$, $s_1(t) = h_1(t)s_2(t)$ and $h_1(t) = s_1(t)h_2(t)$ on $[0, T]$. Then for all $(\gamma, \beta) \in E_\alpha \cap E_{\alpha\beta\tau}$, $(\rho, \tau) \in E_{\alpha\beta}$, and $(\gamma, \beta^2\tau) \in E_\alpha$,*

(4.5)

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}(((T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1) *_{s_1s_2} T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_2))_{\rho,\tau}\|X)(\cdot,\eta_3)\|X)(y,\eta_4)\\ &= T^{h_1,h_2^2s_2}_{\sqrt{2}\gamma,\beta^2\tau}(F\|X)\Big(y, \tfrac{\eta_4+\eta_3+\sqrt{2}\eta_1}{2}\Big)T^{h_1,h_2^2s_2}_{\sqrt{2}\gamma,\beta^2\tau}(G\|X)\Big(y, \tfrac{\eta_4-\eta_3+\sqrt{2}\eta_2}{2}\Big)\end{aligned}$$

for s-a.e. $y \in C_0[0, T]$ and a.e. $\eta_j (j = 1, 2, 3, 4) \in \mathbb{R}$. Also, both sides of the expression in equation (4.5) are given by the formula

$$\begin{aligned}&\int_{L^2[0,T]}\int_{L^2[0,T]} \exp\Big\{\alpha\beta^2\tau\langle(v + u)h_2^2s_2, y\rangle\\ &+ \alpha^2\gamma^2(\|vh_1 - B_{vh_1}\|_2^2 + \|uh_1 - B_{uh_1}\|_2^2)\\ &+ \frac{\sqrt{2}\alpha\gamma(\eta_4 + \eta_3 + \sqrt{2}\eta_1)}{2}B_{vh_1} + \frac{\sqrt{2}\alpha\gamma(\eta_4 - \eta_3 + \sqrt{2}\eta_1)}{2}B_{uh_1}\Big\}df(v)dg(u).\end{aligned}$$

Proof. By using equation (4.1), it follows that

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}((T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1) *_{s_1s_2} T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_2))_{\rho,\tau}(\cdot,\eta_3)\|X)(y,\eta_4)\\ &= T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}\Big(T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)\|X\Big)\Big(y, \tfrac{\eta_4+\eta_3}{\sqrt{2}}\Big)\\ &\quad\cdot T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}\Big(T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_1)\|X\Big)\Big(y, \tfrac{\eta_4-\eta_3}{\sqrt{2}}\Big).\end{aligned}$$

Also, applying equation (3.9) to the last expression above, we obtain

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}((T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1) *_{s_1s_2} T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_2))_{\rho,\tau}(\cdot,\eta_3)\|X)(y,\eta_4)\\ &= T^{h_1,h_2^2s_2}_{\sqrt{2}\gamma,\beta^2\tau}(F\|X)\Big(y, \tfrac{\eta_4+\eta_3+\sqrt{2}\eta_1}{2}\Big)T^{h_1,h_2^2s_2}_{\sqrt{2}\gamma,\beta^2\tau}(G\|X)\Big(y, \tfrac{\eta_4-\eta_3+\sqrt{2}\eta_2}{2}\Big).\end{aligned}$$

This proves the desired result. □

Remark 4.4. Let $A \equiv \{w \in C_0[0, T] : w(t) = \int_0^t u(s)ds$ for some $u \in L^2[0, T]\}$. Note that for all $u, v \in L^2[0, T]$, we have that $|(u, v)_2| \le \|u\|_2\|v\|_2$. In addition, for $w \in A$ and $v \in L^2[0, T]$, the PWZ integral $\langle v, w\rangle$ exists and is given by the formula

$$\langle v, w\rangle = \int_0^T v(s)dw(s) = \int_0^T v(s)u(s)ds$$

and so $|\langle v, w\rangle| \le \|v\|_2\|u\|_2$.

The following observation below will be very useful in the development of our theorems. Let $w \in A$ and let F be an element of S_α whose associated measure $f \in \mathcal{M}(L^2[0,T])$ satisfies the inequality

$$\int_{L^2[0,T]} |\alpha| \|v\|_2 |df(v)| < \infty. \tag{4.6}$$

Then δF is an element of S_α. Hence in this paper, we always assume that the associated measure $f \in \mathcal{M}(L^2[0,T])$ of F satisfies the condition (4.6).

In our next theorem, we establish that the conditional transform involving the first variation equals the first variation of the conditional transform with respect to the Gaussian process.

Theorem 4.5. *Let F and X be as in Theorem 4.1. Assume that F satisfies the hypothesis of Remark 4.4. Let h, s, l, m and $h_j (j = 1,2,3,4)$ satisfy the following conditions*:

(1) $h_3(t) = h(t)h_1(t)$,
(2) $l(t)h_4(t) = h(t)h_2(t)$,
(3) $m(t)h_4(t) = s(t)$ *on* $[0,T]$.

Then for all $(\gamma,\beta) \in E_\alpha$,

(4.7)
$$T^{h_1,h_2}_{\gamma,\beta}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)\|X\Big)(y,\eta) = \frac{1}{\beta}\delta T^{h_3,h_4}_{\gamma,\beta}(F\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\eta\Big)$$

for s-a.e. $y \in C_0[0,T]$ *and a.e.* $\eta \in \mathbb{R}$. *Also, both sides of the expression in equation* (4.7) *are given by the formula*

$$\int_{L^2[0,T]} \alpha\langle vs, z\rangle \exp\Big\{\alpha\beta\langle vhh_2, y\rangle + \frac{\alpha^2\gamma^2}{2}\|vhh_1 - B_{vhh_1}\|_2^2 + \alpha\gamma\eta B_{vhh_1}\Big\} df(v).$$

Furthermore equation (4.7) *is an element of* $S_{\alpha\beta}$ *with associated measure* ϕ_4^η *defined by*

$$\phi_4^\eta(E) = \int_E \alpha\langle vs, z\rangle \exp\Big\{\frac{\alpha^2\gamma^2}{2}\|vhh_1 - B_{vhh_1}\|_2^2 + \alpha\gamma\eta B_{vhh_1}\Big\} df(v)$$

for $E \in \mathcal{B}(L^2[0,T])$.

Proof. First, using equations (2.5) and (2.9), we obtain that

(4.8)
$$\begin{aligned}
&T^{h_1,h_2}_{\gamma,\beta}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)\|X\Big)(y,\eta) \\
&= \int_{C_0[0,T]} \delta F\Big(Z_h\Big(\gamma Z_{h_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) + \beta Z_{h_2}(y,\cdot),\cdot\Big)|Z_s(z,\cdot)\Big) dm(x) \\
&= \int_{C_0[0,T]} \frac{\partial}{\partial k}\Big[\int_{L^2[0,T]} \exp\Big\{\alpha\beta\langle vhh_2, y\rangle + \alpha\gamma\langle vhh_1 - B_{vh_1}, x\rangle
\end{aligned}$$

$$+ \alpha\gamma\eta B_{vhh_1} + \alpha k\langle vs, z\rangle\Big\}df(v)\Big]\Big|_{k=0} dm(x)$$

$$= \int_{L^2[0,T]} \alpha\langle vs, z\rangle \exp\Big\{\alpha\beta\langle vhh_2, y\rangle + \alpha\gamma\eta B_{vhh_1}\Big\}$$

$$\cdot \Big[\int_{C_0[0,T]} \exp\Big\{\alpha\gamma\langle vhh_1 - B_{vh_1}, x\rangle\Big\}dm(x)\Big]df(v)$$

$$= \int_{L^2[0,T]} \alpha\langle vs, z\rangle \exp\Big\{\alpha\beta\langle vhh_2, y\rangle + \frac{\alpha^2\gamma^2}{2}\|vhh_1 - B_{vhh_1}\|_2^2 + \alpha\gamma\eta B_{vhh_1}\Big\}df(v).$$

Next, using equations (2.5) and (3.3), it follows that

(4.9)

$$\delta T^{h_3,h_4}_{\gamma,\beta}(F\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\eta\Big)$$

$$= \frac{\partial}{\partial k}\Big[T^{h_3,h_4}_{\gamma,\beta}(F\|X)\Big(Z_l(y,\cdot) + kZ_m(z,\cdot),\eta\Big)\Big]\Big|_{k=0}$$

$$= \frac{\partial}{\partial k}\Big[\int_{L^2[0,T]} \exp\Big\{\alpha\beta\langle vlh_4, y\rangle + \frac{\alpha^2\gamma^2}{2}\|vh_3 - B_{vh_3}\|_2^2$$

$$+ \alpha\gamma\eta B_{vh_3} + \alpha\beta k\langle vh_4 m, z\rangle\Big\}df(v)\Big]\Big|_{k=0}$$

$$= \int_{L^2[0,T]} \alpha\beta\langle vh_4 m, z\rangle \exp\Big\{\alpha\beta\langle vlh_4, y\rangle + \frac{\alpha^2\gamma^2}{2}\|vh_3 - B_{vh_3}\|_2^2 + \alpha\gamma\eta B_{vh_3}\Big\}df(v).$$

Thus, equation (4.7) follows from equations (4.8) and (4.4). □

Remark 4.6. The main result in [5, Theorem 4.2] follows immediately from Theorem 4.5 above by choosing $h_1 = h_2 = 1$.

The following theorem follows immediately from Theorems 3.5 and 4.5.

Theorem 4.7. *Let F and X be as in Theorem 4.5. Let h, s, l, m and $h_j (j = 1, 2, 3, 4, 5, 6)$ satisfy the following conditions*:

(1) $h_5(t) = h(t)h_3(t)$,
(2) $l(t)h_6(t) = h(t)h_2(t)h_4(t)$,
(3) $m(t)h_6(t) = s(t)$ *on* $[0, T]$.

Then for all $(\gamma_2, \beta_2) \in E_\alpha$ and $(\gamma_2, \beta_1\beta_2) \in E_\alpha$,

$$T^{h_1,h_2}_{\gamma_1,\beta_1}\Big(T^{h_3,h_4}_{\gamma_2,\beta_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2) = \frac{1}{\beta_1\beta_2}\delta T^{h_5,h_6}_{\sqrt{2}\gamma_2,\beta_1\beta_2}(F\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big) \tag{4.10}$$

for s-a.e. $y \in C_0[0,T]$ and a.e. $\eta_1, \eta_2 \in \mathbb{R}$. Also, both sides of the expression in equation (4.10) *are given by the formula*

$$\int_{L^2[0,T]} \alpha\langle vs, z\rangle \exp\Big\{\alpha\beta_1\beta_2\langle vhh_2h_4, y\rangle + \alpha^2\gamma_2^2\|vhh_3 - B_{vhh_3}\|_2^2$$

$$+\alpha\gamma_2(\eta_2+\eta_1)B_{vhh_3}\Big\}df(v).$$

5. Relationships between three concepts

In Section 4, we obtained relationships between exactly two of the three concepts of conditional transform, conditional convolution product and first variation of functionals on S_α. In this section, we establish all possible relationships between all three of these concepts.

In this section, to simplify the expressions, we will only state the formulas without conditions for existences.

Formula 5.1. Let F and X be as in Theorem 4.5. Let h, s, l, m and h_j $(j=1,2,3,4)$ satisfy the following conditions:

(1) $\tau\gamma=\rho$,
(2) $h_1(t)s_2(t)=l(t)h_3(t)=s_1(t)$,
(3) $h(t)h_2(t)s_2(t)=l(t)h_4(t)$,
(4) $m(t)=h_2(t)s_2(t)s(t)$ on $[0,T]$.

Then

$$\begin{aligned}&\delta T^{h_1,h_2}_{\gamma,\beta}(((F*_{s_1s_2}G)_{\rho,\tau}\|X)(\cdot,\eta_1)\|X)\Big(Z_h(y,\cdot)|Z_s(z,\cdot),\eta_2\Big)\\&=T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(F\|X)\Big(Z_h(y,\cdot),\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)\tau\beta T^{h_3,h_4}_{\sqrt{2}\rho,\tau\beta}\Big(\delta G\Big(Z_l(\cdot,\cdot)|Z_m(z,\cdot)\Big)\|X\Big)\Big(y,\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big)\\&\quad+\tau\beta T^{h_3,h_4}_{\sqrt{2}\rho,\tau\beta}\Big(\delta F\Big(Z_l(\cdot,\cdot)|Z_m(z,\cdot)\Big)\|X\Big)\Big(y,\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(G\|X)\Big(Z_h(y,\cdot),\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big).\end{aligned}$$

Formula 5.2. Let F and X be as in Theorem 4.5. Let h, s, l, m and h_j $(j=1,2,3,4)$ satisfy the following conditions:

(1) $\tau\gamma=\rho$,
(2) $h_1(t)s_2(t)=s_1(t)$,
(3) $h_3(t)=h(t)s_1(t)$,
(4) $l(t)h_4(t)=h(t)h_2(t)s_2(t)$,
(5) $m(t)h_4(t)=s(t)$ on $[0,T]$.

Then

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}\Big(\Big(\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)*_{s_1s_2}\delta G\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)\Big)_{\rho,\tau}\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2)\\&=\frac{1}{\beta^2}\delta T^{h_3,h_4}_{\sqrt{2}\rho,\tau\beta}(F\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)\delta T^{h_3,h_4}_{\sqrt{2}\rho,\tau\beta}(G\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big).\end{aligned}$$

Formula 5.3. Let F and X be as in Theorem 4.5. Let h, s, l, m and h_j $(j=1,2,3,4)$ satisfy the following conditions:

(1) $\tau\gamma=\rho$,
(2) $h_3(t)s_2(t)=s_1(t)$,
(3) $h_3(t)=h(t)h_1(t)$,
(4) $l(t)h_4(t)=h(t)h_2(t)$,
(5) $m(t)h_4(t)=s(t)$ on $[0,T]$.

Then

$$
\begin{aligned}
&T_{\gamma,\beta}^{h_1,h_2}\Big(\delta\Big(\Big((F *_{s_1 s_2} G)_{\rho,\tau}\|X\Big)(\cdot,\eta_1)\Big)\Big(Z_h(\cdot,\cdot)|Z_s(z,\cdot)\Big)\|X\Big)(y,\eta_2)\\
&= T_{\sqrt{2}\rho,\gamma\beta}^{s_1,h_2 s_2}(F\|X)\Big(Z_l(y,\cdot),\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)\frac{1}{\beta}\delta T_{\sqrt{2}\rho,\gamma\beta}^{s_1,h_2 s_2}(G\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big)\\
&\quad+\frac{1}{\beta}\delta T_{\sqrt{2}\rho,\gamma\beta}^{s_1,h_2 s_2}(F\|X)\Big(Z_l(y,\cdot)|Z_m(z,\cdot),\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)T_{\sqrt{2}\rho,\gamma\beta}^{s_1,h_2 s_2}(G\|X)\Big(Z_l(y,\cdot),\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big).
\end{aligned}
$$

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