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Estimation for Mean and Standard Deviation of Normal Distribution under Type II Censoring

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Abstract

In this paper, we consider maximum likelihood estimators of normal distribution based on type II censoring. Gupta (1952) and Cohen (1959, 1961) required a table for an auxiliary function to compute since they did not have an explicit form; however, we derive an explicit form for the estimators using a method to approximate the likelihood function. The derived estimators are a special case of Balakrishnan *et al.* (2003). We compare the estimators with the Gupta's linear estimators through simulation. Gupta's linear estimators are unbiased and easily calculated; subsequently, the proposed estimators have better performance for mean squared errors and variances, although they show bigger biases especially when the ratio of the complete data is small.

Keywords: Asymptotic variances, maximum likelihood estimators, normal distribution, plotting position, type II censoring.

1. Introduction

A variety of censoring observations and types occur frequently in life testing. The most common and simple censoring schemes are type I and type II censoring. If the experiment is continued for a predetermined time t, the censoring is type I with the number of items failing in t random. However, we have type II censoring if the experiment is continued until a fixed number of failures occur. The inferences in two types of censoring are nearly identical. The common distributions in survival studies are the exponential, Weibull, lognormal, and gamma distribution. Lognormal distribution is suitable for survival patterns with an initially increasing and then decreasing hazard rate. The distribution is positively skewed and the logarithm of the variable follows normal distribution; consequently, the inference for the normal would help that for the lognormal. In this paper, we consider maximum likelihood estimators (MLEs) for the mean and standard deviation of the normal distribution $N(\mu, \sigma^2)$ with the density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

when the sample is type II (right) censored. The estimation problem of the normal distribution for a complete sample is one of the simple textbook example. The MLEs of the censored data are not that simple. Gupta (1952), Cohen (1959, 1961), and others have addressed this problem. According to them, the MLEs are complicated to compute and they may need a table to compute since they do not

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have an explicit form. In this paper we consider an explicit form of the MLEs by appropriately approximating the likelihood equation and compare the MLEs with the Gupta(1952)'s linear estimators, which seem to be used more frequently than the MLEs.

The approximation method was first developed by Balakrishnan (1989) to obtain the approximate MLE for the explicit estimator of the scalar parameter in Rayleigh distribution with left and right type II censoring. Balakrishnan and Wong (1991) used the method for the half logistic distribution with type II censoring. We may obtain the approximate MLE by applying the method when the MLE does not exist in an explicit form (a usual situation for many distributions when data are censored). Numerous authors used it for many distributions under several censoring schemes.

The following authors applied it under progressively type II censoring for several distributions. Balakrishnan and Kannan (2001), Balakrishnan *et al.* (2004) studied the inference procedures of approximate MLE for logistic distribution and the extreme value distribution, respectively. Asgharzadeh (2006, 2009) did it for the generalized logistic distribution and the generalized exponential distribution, respectively. For the half logistic distribution, Balakrishnan and Asgharzadeh (2005), Kang *et al.* (2008) used the method. For the Rayleigh distribution, Seo and Kang (2007), Kim and Han (2009) discussed the procedure. Recently Sultan *et al.* (2014) applied it to the inverse Weibull distribution. Besides, many other authors applied it under other censoring schemes such as multiply type II censoring, type II hybrid censoring, and random censoring. Kim (2014) did it for generalized exponential distribution under random censoring.

Balakrishnan *et al.* (2003) derived an approximate MLE for normal distribution under progressively type II censoring. The estimators we obtained are a special case since progressively type II censoring is a generalization of the traditional type II censoring.

In Section 2, we derive MLEs based on type II censoring in an explicit form with description of Cohen's method for comparison. In Section 3, an example is presented and a simulation is carried out to compare approximate MLEs and the Gupta (1952)'s linear estimators. Section 4 ends the paper with some concluding remarks.

2. MLE and Approximate MLE

2.1. MLE

Let $x_1, \ldots, x_n$ be an ordered sample of size n from a normal distribution $N(\mu, \sigma^2)$ and let $x_1, \ldots, x_k$ be the uncensored sample of size k. The $(n-k)$ observations $x_{k+1}, \ldots, x_n$ are censored and they are known to be greater than x_k, where k is fixed and x_k is the greatest observation among $x_1, \ldots, x_k$.

The likelihood function based on such a sample is

$$L(\mu,\sigma) = \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^k \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{k}(x_i-\mu)^2\right]\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{x_k}^{\infty}\exp\left[-\frac{1}{2\sigma^2}(t-\mu)^2\right]dt\right)^{n-k}$$

and the log-likelihood function $l(\mu,\sigma) = \ln L(\mu,\sigma)$ becomes

$$l(\mu,\sigma) = C - k\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{k}(x_i-\mu)^2 + (n-k)\ln\bar{\Phi}(\xi)$$

where C is a constant, $\xi = (x_k-\mu)/\sigma$, $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-1/2t^2}$, and $\bar{\Phi}(\xi) = \int_{\xi}^{\infty}\phi(t)dt = 1-\Phi(\xi)$.

The likelihood equations are

$$\frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{k}(x_i - \mu) + (n-k)\frac{1}{\sigma}\frac{\phi(\xi)}{\bar{\Phi}(\xi)} = 0, \tag{2.1}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{k}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{k}(x_i - \mu)^2 + (n-k)\frac{\xi}{\sigma}\frac{\phi(\xi)}{\bar{\Phi}(\xi)} = 0. \tag{2.2}$$

Gupta (1952) and Cohen (1959, 1961, 1991) considered the solution of the likelihood equations (2.1) and (2.2). Their estimates require some auxiliary functions and they tabulated them. Cohen (1959) obtained the estimates as follows. Let

$$p = \frac{k}{n}, \quad Q(\xi) = \frac{\phi(\xi)}{\bar{\Phi}(\xi)}, \quad \text{and} \quad \Omega = \Omega(p, \xi) = -\frac{1-p}{p}Q(\xi), \tag{2.3}$$

then equation (2.1) and equation (2.2) becomes

$$\bar{x} - \mu = \sigma\Omega, \tag{2.4}$$

$$(\bar{x} - \mu)^2 + s^2 = \sigma^2(1 + \xi\Omega) \tag{2.5}$$

where $\bar{x} = \sum_{i=1}^{k} x_i/k$ and $s^2 = \sum_{i=1}^{k}(x_i - \bar{x})^2/k$. From equation (2.4) and equation (2.5), we get

$$\bar{x} - x_k = \sigma(\Omega - \xi), \tag{2.6}$$

$$s^2 = \sigma^2(1 - \Omega(\Omega - \xi)), \tag{2.7}$$

and

$$\frac{s^2}{(\bar{x} - x_k)^2} = \frac{1 - \Omega(\Omega - \xi)}{(\Omega - \xi)^2} = \alpha(p, \xi). \tag{2.8}$$

Therefore the function $\alpha(p, \xi)$ in equation (2.8) can be computed from data $x_1, \ldots, x_k$. From equation (2.6), we get

$$\sigma = \frac{\bar{x} - x_k}{\Omega - \xi} \tag{2.9}$$

and equation (2.7) becomes

$$\sigma^2 = s^2 + \frac{\Omega}{\Omega - \xi}(\bar{x} - x_k)^2$$

by substituting the expression for σ given by equation (2.9) into equation (2.7). Using equation (2.4) and equation (2.9),

$$\mu = \bar{x} - \sigma\Omega = \bar{x} - \frac{\Omega}{\Omega - \xi}(\bar{x} - x_k).$$

If we define

$$\lambda = \lambda\,(p, \xi) = \lambda(p, \alpha) = \frac{\Omega}{\Omega - \xi}$$

then the final estimates are

$$\sigma^2 = s^2 + \hat{\lambda}(\bar{x} - x_k)^2, \tag{2.10}$$

$$\mu = \bar{x} - \hat{\lambda}(\bar{x} - x_k) \tag{2.11}$$

where $\hat{\lambda} = \lambda(p, \hat{\alpha})$, $\hat{\alpha} = \alpha(p, \hat{\xi}) = s^2/(\bar{x} - x_k)^2$, and the function α is defined in equation (2.8). Cohen (1959, 1961, 1991) provided the tables of the auxiliary estimating function $\lambda(p, \alpha)$. In practical applications, it is necessary to calculate $p = k/n$, $\hat{\alpha} = s^2/(\bar{x} - x_k)^2$ from the k complete observations and find the value $\hat{\lambda}$ from his tables by interpolation, if necessary. In his tables, h is $h = 1 - p$.

2.2. Approximate MLE

In this subsection, we consider approximate MLEs that do not need any tables and have an explicit form. From equation (2.1), we have

$$(n-k)\frac{Q}{\sigma} = -\frac{1}{\sigma^2}\sum_{i=1}^{k}(x_i - \mu)$$

and substituting it into equation (2.2) gives

$$-\sigma^2 + s^2 + (\bar{x} - \mu)^2 - (x_k - \mu)(\bar{x} - \mu) = 0,$$

$$\mu = \bar{x} + \frac{\sigma^2 - s^2}{x_k - \bar{x}}. \tag{2.12}$$

If we approximate $Q(\xi) = \phi(\xi)/\bar{\Phi}(\xi)$ by $Q(\xi) \cong a + b\xi$ expanding $Q(\xi)$ in the Taylor series around a suitable point which will be mentioned later, equation (2.1) becomes

$$\frac{1}{\sigma^2}\sum_{i=1}^{k}(x_i - \mu) + (n-k)\frac{1}{\sigma}(a + b\xi) = 0$$

where the coefficients a and b are defined later. Substituting $\xi = (x_k - \mu)/\sigma$ and doing some algebraic computations, we get

$$\mu = \frac{k\bar{x} + (n-k)(a\sigma + bx_k)}{k + (n-k)b}. \tag{2.13}$$

By equating equation (2.12) and equation (2.13), we have a quadratic equation in σ as follows,

$$\sigma^2 - \frac{(n-k)a(x_k - \bar{x})}{k + (n-k)b}\sigma + \bar{x}(x_k - \bar{x}) - s^2 - \frac{(k\bar{x} + (n-k)bx_k)(x_k - \bar{x})}{k + (n-k)b} = 0.$$

Simplifying the constant term, we have

$$\sigma^2 - \frac{(n-k)a(x_k - \bar{x})}{k + (n-k)b}\sigma - s^2 - \frac{(n-k)b(x_k - \bar{x})^2}{k + (n-k)b} = 0, \tag{2.14}$$

$$\sigma^2 - A\sigma - B = 0.$$

Therefore the positive root of σ is

$$\hat{\sigma}_{\mathrm{AM}} = \frac{A + \sqrt{A^2 + 4B}}{2} \tag{2.15}$$

where

$$A = \frac{(n-k)a(x_k - \bar{x})}{k + (n-k)b}, \tag{2.16}$$

$$B = s^2 + \frac{(n-k)b(x_k - \bar{x})^2}{k + (n-k)b}. \tag{2.17}$$

By substituting equation (2.15) into equation (2.12), we get

$$\hat{\mu}_{\mathrm{AM}} = \bar{x} + \frac{\hat{\sigma}^2_{\mathrm{AM}} - s^2}{x_k - \bar{x}}. \tag{2.18}$$

Now let us consider the concrete form of a and b in $Q(\xi) \cong a + b\xi$. Let

$$\xi_0 = \Phi^{-1}(p_k)$$

where Φ^{-1} is the inverse of Φ and p_k is referred to as a plotting position or a quantile probability,

$$p_k = \frac{k - c}{n - 2c + 1} \tag{2.19}$$

for some $0 \le c \le 1$. In this paper we use $c = 3/8 = 0.375$ and

$$p_k = \frac{k - 0.375}{n + 0.25}. \tag{2.20}$$

The plotting position in equation (2.20) is suggested by Blom (1958) as the approximation of $E(Z_{(k)})$, where $Z_{(k)}$ is the k-th order statistic from the standard normal distribution $N(0, 1)$. By expanding $Q(\xi) = \phi(\xi)/\bar{\Phi}(\xi)$ around ξ_0 in the Taylor series keeping only the first two terms, the function is approximated by

$$Q(\xi) \cong Q(\xi_0) + Q'(\xi_0)(\xi - \xi_0) \equiv a + b\xi$$

where

$$a = Q(\xi_0) - Q'(\xi_0)\xi_0, \tag{2.21}$$

$$b = Q'(\xi_0) = Q(\xi_0)(Q(\xi_0) - \xi_0). \tag{2.22}$$

Remark 1. Using $\phi'(\xi) = -\xi\phi(\xi)$, $\bar{\Phi}'(\xi) = -\phi(\xi)$, we can easily show the second equality in b,

$$Q'(\xi) = \frac{\phi(\xi)(\phi(\xi) - \xi\bar{\Phi}(\xi))}{(\bar{\Phi}(\xi))^2} = Q(\xi)(Q(\xi) - \xi).$$

Let $g(\xi) = \phi(\xi) - \xi\bar{\Phi}(\xi)$, then $g(\xi)$ is decreasing and positive, since $g'(\xi) = -\bar{\Phi}(\xi) < 0$, $g(\xi) \to \infty$ as $\xi \to -\infty$ and $g(\xi) \to 0$ as $\xi \to \infty$. Therefore $b = Q'(\xi_0)$ in equation (2.22) is positive and B in equation (2.17) is also positive. Hence the other root of the quadratic equation (2.14) in σ has a negative sign and cannot be a solution of σ.

Remark 2. As we mentioned in Section 1, Balakrishnan *et al.* (2003) derived the approximate MLEs for the normal distribution under progressively type II censoring. Since the conventional type II censoring is a special case of progressively type II censoring, we can show that their estimator for σ reduce to $\hat{\sigma}_{\mathrm{AM}}$ in equation (2.15) under type II censoring through some algebraic computations. Their plotting position in equation (2.19) is $p_k = k/(n + 1)$ with $c = 0$, which is the most commonly used Weibull (1939)'s position.

Remark 3. For a complete sample, we have $k = n$, getting $A = 0$, $B = s^2$ in equation (2.16), (2.17). Hence we obtain $\hat{\sigma}_{\mathrm{AM}} = s$, $\hat{\mu}_{\mathrm{AM}} = \bar{x}$ which are the MLEs of σ and μ.

2.3. Asymptotic variances of the estimates

The asymptotic variance-covariance matrix of the MLEs is given by inverting the Fisher information matrix whose elements are negatives of expected values of the second order partial derivatives of the loglikelihood function with respect to the parameters. According to Cohen (1959, 1991), the Fisher information matrix becomes

$$\frac{n}{\sigma^2}\begin{bmatrix} \psi_{11} & -\psi_{12} \\ -\psi_{12} & \psi_{22} \end{bmatrix} \equiv \frac{n}{\sigma^2}\begin{bmatrix} p+(1-p)Q(Q-\xi) & -(1-p)Q(1-\xi(Q-\xi)) \\ -(1-p)Q(1-\xi(Q-\xi)) & 2p-\xi\psi_{12} \end{bmatrix}$$

with p and $Q = Q(\xi)$ in equation (2.3). By inverting the matrix, we obtain

$$\mathrm{Var}(\hat{\mu}) = \frac{\sigma^2}{n}\sigma_{11}, \quad \mathrm{Var}(\hat{\sigma}) = \frac{\sigma^2}{n}\sigma_{22}, \quad \mathrm{Cov}(\hat{\mu},\hat{\sigma}) = \frac{\sigma^2}{n}\sigma_{12} \tag{2.23}$$

with

$$\sigma_{11} = \frac{\psi_{22}}{\psi_{11}\psi_{22}-\psi_{12}^2}, \quad \sigma_{22} = \frac{\psi_{11}}{\psi_{11}\psi_{22}-\psi_{12}^2}, \quad \sigma_{12} = \frac{\psi_{12}}{\psi_{11}\psi_{22}-\psi_{12}^2}. \tag{2.24}$$

The equations in equation (2.23) may be used as the asymptotic variances and covariance of the approximate MLEs in equation (2.15), (2.18).

Gupta (1952) suggested linear estimates of μ and σ. These are linear combinations of the k order statistics. The best linear estimates are obtained by generalized least squares when x_i is regressed against $E(Z_{(i)})$, $i = 1,\ldots,k$. Gupta (1952) gave relevant coefficients for $n \le 10$, $k \le 10$. Sarhan and Greenberg (1956, 1958, 1962) also tabulated coefficients for all possible conditions of censoring. The coefficients of the best linear estimates are tedious to calculate and they depend not only on the expectations $E(Z_{(i)})$ but also on the variances and covariances of the order statistics from $N(0,1)$, which are not readily available for larger samples; therefore, Gupta (1952) proposed alternative linear estimates for sample sizes greater than 10. The coefficients of the estimates could be easily calculated since the estimates are obtained based on the assumption that the variance-covariance matrix of the order statistics from $N(0,1)$ is a unit matrix. The estimates are

$$\hat{\mu}_G = \sum_{i=1}^{k} b_i x_i, \quad \hat{\sigma}_G = \sum_{i=1}^{k} c_i x_i, \tag{2.25}$$

$$b_i = \frac{1}{k} - \frac{\bar{m}(m_i-\bar{m})}{\sum_{i=1}^{k}(m_i-\bar{m})^2},$$

$$c_i = \frac{m_i-\bar{m}}{\sum_{i=1}^{k}(m_i-\bar{m})^2}$$

where $m_i = E(Z_{(i)})$ and $\bar{m} = \sum_{i=1}^{k} m_i/k$. Values of m_i are tabulated in Harter (1961). Otherwise they can be well approximated by $m_i = \Phi^{-1}((i-0.375)/(n+0.25))$ using Blom's plotting position in equation (2.20) as it is mentioned before. The variances of the estimates can be written as

$$\mathrm{Var}(\hat{\mu}_G) = \sigma^2\mathbf{b}'\mathbf{V}\mathbf{b}, \quad \mathrm{Var}(\hat{\sigma}_G) = \sigma^2\mathbf{c}'\mathbf{V}\mathbf{c} \tag{2.26}$$

where $\mathbf{b}' = (b_1,\ldots,b_k)$, $\mathbf{c}' = (c_1,\ldots,c_k)$, $\mathbf{b},\mathbf{c}$ are the transposes of $\mathbf{b}'$, $\mathbf{c}'$ respectively, and $\mathbf{V} = (v_{ij})$ is the variance-covariance matrix of the order statistics from $N(0,1)$. According to Hastings *et al.* (1947), the asymptotic value of v_{ij} is

$$\frac{j(n-i+1)}{n(n+1)^2\phi\left(\Phi^{-1}\left(\frac{i}{n+1}\right)\right)\phi\left(\Phi^{-1}\left(\frac{j}{n+1}\right)\right)}, \quad j \le i,$$

Table 1: Gupta (1952)'s data

y	41	44	46	54	55	58	60
$x = \log_{10} y$	1.6128	1.6435	1.6628	1.7324	1.7404	1.7634	1.7782

and $v_{ij} = v_{n-i+1,n-j+1}$. In Section 3, the above formula is used to find variances of Gupta's estimates.

3. An Illustrative Example and Simulation Results

3.1. An example

Gupta (1952) gave the data of a life test on 10 laboratory mice sacrificed after being inoculated with a uniform culture of human tuberculosis. The test was terminated with the death of the seventh specimen. Since the reaction time is not likely to be normally distributed, Gupta (1952) assumed logarithms to the base 10 were distributed $N(\mu, \sigma^2)$. The data are as follows. Here y's are survival times in days from inoculation to death and x's are logarithms (Table 1).

(i) MLE The data are summarized as follows, $n = 10$, $k = 7$, $p = 0.7$, $x_7 = 1.7782$, $\bar{x} = 1.70479$, $s^2 = 0.003548$, and $\hat{\alpha} = s^2/(\bar{x} - x_7)^2 = 0.6583$. Using Table 2 in Cohen (1959, 1961) or Table 2.3 in Cohen (1991), we get $\hat{\lambda} = 0.51247$ by interpolation. Substituting these values in equation (2.10), (2.11), we obtain $\hat{\sigma}^2 = 0.0063101$, $\hat{\sigma} = 0.079436$, and $\hat{\mu} = 1.74241$. For sampling errors of the estimates, calculating equation (2.24) gives $\sigma_{11} = 1.13826$, $\sigma_{22} = 0.81975$, $\sigma_{12} = 0.20657$ with $\xi = \hat{\xi} = \Phi^{-1}(0.7)$. Substituting these values into equation (2.23), we have

$$\begin{aligned}
\mathrm{Var}\,(\hat{\mu}) &= \left(\frac{0.0063101}{10}\right)(1.13826) = 0.00071825,\\
\mathrm{Var}\,(\hat{\sigma}) &= \left(\frac{0.0063101}{10}\right)(0.81975) = 0.00051727,\\
\mathrm{Cov}\,(\hat{\mu}, \hat{\sigma}) &= \left(\frac{0.0063101}{10}\right)(0.20657) = 0.00013035
\end{aligned}$$

and $\mathrm{SE}(\hat{\mu}) = 0.0268$, $\mathrm{SE}(\hat{\sigma}) = 0.022744$. Since $p = k/n$ tends to $\Phi(\xi)$ as n tends to ∞, $\hat{\xi} = \Phi^{-1}(p)$ is used.

(ii) Approximate MLE By plugging $\xi_0 = 0.37546$, we get $a = 0.78452$, $b = 0.71046$ in equation (2.21), (2.22), $A = 0.018922$, $B = 0.0048061$ in equation (2.16), (2.17), and $\hat{\sigma}_{\mathrm{AM}} = 0.07942$, $\hat{\mu}_{\mathrm{AM}} = 1.74239$ in equation (2.15), (2.18). Standard errors may be calculated in the same way as MLE. If we use $\hat{\sigma}_{\mathrm{AM}}$ instead of $\hat{\sigma}$, we have $\mathrm{Var}(\hat{\mu}) = 0.00071277$, $\mathrm{Var}(\hat{\sigma}) = 0.00051333$, $\mathrm{Cov}(\hat{\mu}, \hat{\sigma}) = 0.00012935$, and $\mathrm{SE}(\hat{\mu}) = 0.026698$, $\mathrm{SE}(\hat{\sigma}) = 0.022657$.

(iii) Gupta's linear estimate Gupta (1952) calculated the linear estimates in equation (2.25) for this data to be $\hat{\sigma}_G = 0.094$, $\hat{\mu}_G = 1.748$, and the standard errors to be $\mathrm{SE}(\hat{\mu}_G) = 0.033$, $\mathrm{SE}(\hat{\sigma}_G) = 0.031$.

As we see in this example, the results of two estimates in (i) and (ii) show little difference. To compare the MLEs with the approximate MLEs, we generated samples with 2,000 replications from standard normal distribution $N(0, 1)$ with $n = 10$, $k = 7$ as in the example and computed these two estimates for each sample. Table 2 indicates that the averages of the differences of two estimates are quite small and the mean squared errors (MSE) are almost identical.

Table 2: Comparison between MLEs and approximate MLEs

$\hat{\mu} - \hat{\mu}_{\mathrm{AM}}$	$\hat{\sigma} - \hat{\sigma}_{\mathrm{AM}}$	MSE($\hat{\mu}$)	MSE($\hat{\mu}_{\mathrm{AM}}$)	MSE($\hat{\sigma}$)	MSE($\hat{\sigma}_{\mathrm{AM}}$)
0.000215	0.000146	0.115	0.115	0.0883	0.0883

Table 3: Averages, MSE, and the asymptotic variances, covariances of the approximate MLEs

n	k	$\hat{\mu}_{\mathrm{AM}}$	$\hat{\sigma}_{\mathrm{AM}}$	MSE($\hat{\mu}_{\mathrm{AM}}$)	MSE($\hat{\sigma}_{\mathrm{AM}}$)	Var($\hat{\mu}_{\mathrm{AM}}$)	Var($\hat{\sigma}_{\mathrm{AM}}$)	Cov($\hat{\mu}_{\mathrm{AM}}, \hat{\sigma}_{\mathrm{AM}}$)
	2	−0.562	0.499	0.753	0.428	0.578	0.354	0.372
	3	−0.274	0.696	0.357	0.267	0.302	0.225	0.183
	4	−0.186	0.758	0.222	0.187	0.199	0.162	0.103
	5	−0.099	0.813	0.156	0.142	0.152	0.124	0.061
10	6	−0.059	0.858	0.132	0.111	0.127	0.100	0.036
	7	−0.035	0.871	0.118	0.092	0.114	0.082	0.021
	8	0.002	0.894	0.107	0.076	0.106	0.069	0.011
	9	0.005	0.907	0.104	0.064	0.102	0.059	0.004
	10	0.009	0.931	0.098	0.053	0.100	0.050	0.000
	5	−0.210	0.815	0.247	0.150	0.231	0.142	0.149
	10	−0.057	0.909	0.083	0.068	0.080	0.065	0.041
25	15	−0.025	0.942	0.051	0.043	0.051	0.040	0.014
	20	−0.009	0.959	0.041	0.029	0.043	0.028	0.004
	25	−0.008	0.963	0.040	0.022	0.040	0.020	0.000
	10	−0.109	0.912	0.117	0.071	0.116	0.070	0.074
	20	−0.034	0.955	0.041	0.034	0.040	0.032	0.021
50	30	−0.013	0.969	0.026	0.020	0.026	0.020	0.007
	40	−0.004	0.979	0.019	0.014	0.021	0.014	0.002
	50	0.003	0.985	0.020	0.010	0.020	0.010	0.000

3.2. Simulation results

A simulation study is conducted to compare the performance of the approximate MLEs with the Gupta's linear estimates. The MLEs are inconvenient to simulate since they are complicated to calculate without the auxiliary function. Therefore the approximate MLEs we propose are compared instead. We guess MLEs and approximate MLEs would have similar performances, according to the results in subsection 3.1. The simulation is carried out for sample sizes $n = 10$ with $k = 1, 2, \ldots, 10$, $n = 25$ with $k = 5, 10, 15, 20, 25$, and $n = 50$ with $k = 10, 20, 30, 40, 50$. Table 3 and Table 4 provide averaged values, MSE, and the asymptotic variances of the approximate MLEs and the Gupta's linear estimates, respectively. MSE are calculated as the average of the mean squared deviations and the asymptotic variances are by equation (2.23) and equation (2.26), respectively. In Table 4, the approximate relative efficiency (ARE) is included, defining as

$$\mathrm{ARE}\left(\hat{\mu}_{\mathrm{AM}}, \hat{\mu}_{\mathrm{G}}\right) = \frac{\mathrm{MSE}\left(\hat{\mu}_{\mathrm{AM}}\right)}{\mathrm{MSE}\left(\hat{\mu}_{\mathrm{G}}\right)} \times 100, \quad \mathrm{ARE}\left(\hat{\sigma}_{\mathrm{AM}}, \hat{\sigma}_{\mathrm{G}}\right) = \frac{\mathrm{MSE}\left(\hat{\sigma}_{\mathrm{AM}}\right)}{\mathrm{MSE}\left(\hat{\sigma}_{\mathrm{G}}\right)} \times 100.$$

From Table 3 and Table 4, we observe that the approximate MLEs have smaller MSE than the Gupta's estimates for all cases considered. ARE decreases as k becomes smaller; in addition, approximate MLEs are more efficient in terms of ARE when k is smaller for a fixed sample size n. However, approximate MLEs have signiticant biases when the ratio k/n is small and it seems the small MSE come from small variances. Theoretically the MLEs are biased for small n and the Gupta's estimates are unbiased (Table 3 and Table 4). Therefore the approximate MLEs should be carefully used when the ratio of the complete data is small despite the better efficiency.

Table 4: Averages, MSE, the asymptotic variances of Gupta's linear estimates, and ARE

n	k	$\hat{\mu}_G$	$\hat{\sigma}_G$	MSE($\hat{\mu}_G$)	MSE($\hat{\sigma}_G$)	Var($\hat{\mu}_G$)	Var($\hat{\sigma}_G$)	ARE($\hat{\mu}_{AM}, \hat{\mu}_G$)	ARE($\hat{\sigma}_{AM}, \hat{\sigma}_G$)
10	2	−0.007	0.977	1.092	0.677	0.984	0.616	69.004	63.161
	3	0.023	1.023	0.479	0.406	0.424	0.322	74.584	65.822
	4	−0.017	0.993	0.254	0.240	0.258	0.216	87.280	77.877
	5	−0.002	0.990	0.175	0.178	0.185	0.160	89.035	79.872
	6	−0.003	1.000	0.144	0.138	0.148	0.125	91.547	79.907
	7	0.001	0.992	0.122	0.108	0.127	0.100	96.210	85.698
	8	0.023	1.000	0.109	0.086	0.115	0.082	98.554	88.121
	9	0.012	0.994	0.105	0.069	0.108	0.067	99.172	93.382
	10	0.009	1.008	0.098	0.057	0.104	0.054	100	92.906
25	5	0.014	1.009	0.347	0.212	0.305	0.171	71.073	71.067
	10	0.010	1.003	0.105	0.087	0.100	0.077	79.398	77.267
	15	0.002	1.003	0.056	0.051	0.058	0.046	91.634	84.048
	20	−0.002	1.000	0.043	0.034	0.045	0.031	96.309	87.371
	25	−0.008	0.994	0.040	0.022	0.041	0.020	100	99.202
50	10	0.001	1.008	0.172	0.101	0.153	0.083	67.768	69.855
	20	−0.000	1.003	0.053	0.044	0.050	0.038	77.492	78.092
	30	−0.004	0.995	0.030	0.025	0.029	0.023	88.517	82.448
	40	0.001	1.001	0.020	0.016	0.022	0.015	96.949	87.024
	50	0.003	1.000	0.020	0.011	0.020	0.010	100	98.797

4. Concluding Remarks

In this paper, we study the MLEs of the normal distribution based on type II censoring. We consider the approximate MLEs since MLEs calculations require an auxiliary function table. Through a simulation study, the MLEs and the approximate MLEs might be almost identical in terms of differences, MSE, and variances. Hence the approximate MLEs are compared with classical Gupta's linear estimates. The results suggest that the approximate MLEs have better efficiency in terms of ARE; however, there exists large biases when the ratio of the complete data is small.

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