Communications for Statistical Applications and Methods
2014, Vol. 21, No. 6, 513–520
DOI: http://dx.doi.org/10.5351/CSAM.2014.21.6.513
Print ISSN 2287-7843 / Online ISSN 2383-4757

Numerical Iteration for Stationary Probabilities of Markov Chains

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Abstract

We study numerical methods to obtain the stationary probabilities of continuous-time Markov chains whose embedded chains are periodic. The power method is applied to the balance equations of the periodic embedded Markov chains. The power method can have the convergence speed of exponential rate that is ambiguous in its application to original continuous-time Markov chains since the embedded chains are discrete-time processes. An illustrative example is presented to investigate the numerical iteration of this paper. A numerical study shows that a rapid and stable solution for stationary probabilities can be achieved regardless of periodicity and initial conditions.

Keywords: Markov chain, embedded chain, periodicity, power method, stationary probability, numerical iteration, balance equation.

1. Introduction

The limiting stationary probabilities of Markov chains are usually given by balance equations which are a system of simultaneous linear equations. These probabilities are essential in Markov modeling since the system analysis is based on many real problems. It frequently happens that the analytic solution is difficult to obtain; consequently, numerical solutions are considered instead. In this paper we propose an iteration method based on balance equations for the calculation of the stationary probabilities of a discrete time Markov chain and provide a theoretical foundation. This result can be applied to continuous time Markov chains immediately and the calculation of the stationary probabilities becomes easy even for complex systems.

Many iterative numerical methods have been developed to find the stationary probabilities of Markov chains. In O'Leary (1993) and Stewart (2000), for example, matrix algebra approaches such as LU decomposition, compact storage scheme and Grassmann-Taksar-Heyman algorithm, and iteration methods such as the power method, Gauss-Seidel iteration, symmetric successive overrelaxation algorithm, and preconditioned power iteration are extensively discussed. Among various iterative methods, the power method is simple to understand and easy to implement, but it may suffer from the slow speed of convergence. The problem of finding the stationary probabilities of a continuous time Markov chain can be regarded as an eigenvalue problem for the transition rate matrix; consequently, the convergence rate of the power method can be very slow if the absolute values are close between two eigenvalues which are the largest in absolute magnitude. Recently, Zhao *et al.* (2012) and Nesterov and Nemirovski (2014) developed new iterative methods to solve the problem of the convergence rate. However, the power method has been used widely for real applications of the Markov modeling in that the balance equations are employed without additional transformation.

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Published 30 November 2014 / journal homepage: http://csam.or.kr

It is possible to use the balance equations iteratively based on the transition rates of continuous time Markov chains; however, it is preferred to employ the balance equations of the embedded discrete time Markov chains that are better to understand intuitively. In Ross (1996), it is proved in pages 418–420 that the n-step transition probabilities of a discrete time Markov chain converge exponentially to the limiting probabilities. The iterative method in this paper uses n-step transition probabilities of the embedded Markov chain directly, which can solve the problem of convergence speed of the power method. Therefore, we focus on the iterative calculation methods for the discrete time Markov chains. In addition, the stationary probabilities of periodic Markov chains are of primary. The iterative method in this paper has large possibility of application in real situations since the periodic embedded Markov chains frequently occur in many processes such as in birth-and-death processes. In Na (2010), a Markov queueing model for a real telecommunication system is discussed, where the analytic solution of the continuous time Markov chain for the system is difficult to obtain and a numerical solution should be an alternative. Furthermore, it can be seen that the original continuous time Markov chain in Na (2010) has a periodic embedded chain; consequently, the iterative algorithm in this paper can be a good tool for the numerical solution.

This paper is organized as follows. In Section 2, we explain the procedure of the power method for the calculation of the stationary probabilities of discrete time and continuous time Markov chains. The procedure for discrete time periodic Markov chains is investigated in detail. The proposed method is illustrated with an appropriate example in Section 3.

2. An Iterative Method for the Numerical Solution

Consider a continuous time irreducible Markov chain $X(t),\ t \geq 0$, in which the state space is $S = \{1, 2, \ldots, N\}$ and the transition rates are $\lambda_{ij},\ i, j \in S$. Let $\pi_j,\ j \in S$ be the limiting or stationary probabilities of the ergodic process $X(t)$. The stationary probabilities are nonnegative numbers satisfying the following balance equations

$$\pi_j \lambda_j = \sum_{i \in S} \pi_i \lambda_{ij}, \quad j \in S, \tag{2.1}$$

where $\lambda_j = \sum_{i \in S} \lambda_{ji}$ and $\sum_{j \in S} \pi_j = 1$. If the number of states is large, it becomes difficult directly to obtain the analytic solution of the balance equations (2.1) without the aid of some particular structural properties such as time reversibility. One may transform the balance equations (2.1) appropriately to the eigenvalue problem, null space problem, or linear system problem, where numerical solution can be determined.

The iterative numerical solution becomes possible through a transformation of the balance equations in continuous time Markov chains; however, balance equations can be used directly in discrete time cases. Thus, the discrete time embedded Markov chain, which is defined to be the state at the transition times of a continuous time Markov chain, can play an important role in the analysis of the continuous time chain. Let $\tilde{X}_n,\ n = 0, 1, \ldots$ be the embedded Markov chain of the continuous time Markov chain $X(t),\ t \geq 0$. The transition probabilities of $\tilde{X}_n$ are given by $P_{ij} = \lambda_{ij}/\lambda_i,\ i, j \in S$. The limiting or stationary probabilities $\tilde{\pi}_j,\ j \in S$ of $\tilde{X}_n$ satisfy the balance equations

$$\tilde{\pi}_j = \sum_{i \in S} \tilde{\pi}_i P_{ij}, \quad j \in S, \tag{2.2}$$

where $\sum_{j \in S} \tilde{\pi}_j = 1$.

We consider the power method for stationary probabilities. The embedded Markov chain $\tilde{X}_n$ is assumed to be aperiodic. Then, $\tilde{X}_n$ is ergodic and the stationary probabilities $\tilde{\pi}_j,\ j \in S$ are also limiting probabilities. We write $\tilde{\pi}_j^{(n)},\ j \in S,\ n = 0, 1, \ldots$ for the iterative approximation values of $\tilde{\pi}_j,\ j \in S$.

Step 1. The initial values are $\tilde{\pi}_j^{(0)},\ j \in S$.

Step 2. For $n \geq 1$, iterate the equation $\tilde{\pi}_j^{(n)} = \sum_{i \in S} \tilde{\pi}_i^{(n-1)} P_{ij},\ j \in S$.

Step 3. Stop the iteration Step 2 using a stopping rule and determine the iterative numerical solution $\tilde{\pi}_j^*$ for $\tilde{\pi}_j$ in (2.2) as the $\tilde{\pi}_j^{(n)}$ at the stopped instance.

The initial value in Step 1 can be $\tilde{\pi}_j^{(0)} \equiv N^{-1}$ or $(\tilde{\pi}_1^{(0)}, \ldots, \tilde{\pi}_N^{(0)}) = (1, 0, \ldots, 0)$. Note that $\tilde{\pi}_j^{(n)},\ j \in S$ is the probability distribution of $\tilde{X}_n$ in Step 2 where the balance equations (2.2) are applied iteratively. That is, if the initial distribution in Step 1 is assumed for $\tilde{X}_0$, then $P(\tilde{X}_n = j) = \tilde{\pi}_j^{(n)},\ j \in S$. The ergodicity of the embedded Markov chain implies that $\tilde{\pi}_j^{(n)}$ converges to $\tilde{\pi}_j$ as n goes to infinity. The speed of convergence is expected to be fast because of the exponential convergence in Markov chains. In Step 3, a stopping rule of $|\tilde{\pi}_j^{(n)} - \tilde{\pi}_j^{(n-1)}| < \epsilon,\ j \in S$ is possible, where the precision $\epsilon = 10^{-4}$ can be considered.

We can determine the stationary probabilities of a continuous time Markov chain with the aid of the result of its discrete time embedded Markov chain. Note that the relation $\pi_j \propto \tilde{\pi}_j \lambda_j^{-1}$ holds since a continuous time Markov chain is a semi-Markov chain, which implies the next step to determine the stationary probabilities.

Step 4. Determine the numerical solution π_j^* for π_j in (2.1) by $\pi_j^* = \tilde{\pi}_j^* \lambda_j^{-1} / (\sum_{i \in S} \tilde{\pi}_i^* \lambda_i^{-1})^{-1},\ j \in S$.

The periodicity of Markov chains may cause problems in applying the previous Steps 1-4 to real models. Continuous time Markov chains, where each sojourn time has an exponential distribution, have no problem about periodicity, but the embedded Markov chains may have periods. Note that when a discrete time irreducible Markov chain has a period, its stationary probabilities $\tilde{\pi}_j,\ j \in S$ satisfying the balance equations (2.2) are not limiting ones any longer. The stationary probability $\tilde{\pi}_j$ is interpreted as the long-run proportion that the discrete time Markov chain stays in state j. Note also that the relation $\pi_j \propto \tilde{\pi}_j \lambda_j^{-1}$ between π_j of (2.1) and $\tilde{\pi}_j$ of (2.2) is still valid, regardless of the periodicity of the embedded Markov chain.

We consider the procedure to obtain the iterative numerical solution when the embedded Markov chain $\tilde{X}_n$ is assumed to have a period $d \geq 2$. If $P_{ij}^{(n)},\ i, j \in S$ represent the n-step transition probabilities of $\tilde{X}_n$, then the equation

$$\begin{aligned} \tilde{\pi}_j^{(n)} &= \sum_{i \in S} \tilde{\pi}_i^{(n-1)} P_{ij} \\ &= \sum_{i \in S} \tilde{\pi}_i^{(0)} P_{ij}^{(n)} \end{aligned} \tag{2.3}$$

holds for $\tilde{\pi}_j^{(n)}$ in Step 2. The periodicity of a Markov chain implies the following theorem.

Theorem 1. *Suppose that a finite state irreducible discrete time Markov chain $\tilde{X}_n,\ n = 0, 1, \ldots$ has a period $d \geq 2$. Then, the stationary probabilities $\tilde{\pi}_j$ exist and the following equations*

$$\lim_{n \to \infty} \frac{P_{ij}^{(n)} + P_{ij}^{(n-1)} + \cdots + P_{ij}^{(n-d+1)}}{d} = \tilde{\pi}_j \tag{2.4}$$

hold for all states i, j in S.

Proof: The existence of the stationary probabilities is caused by the positive recurrence. The value of $P_{ij}^{(n)}$ as a function of n is 0 except times of d space after the instance at which $P_{ij}^{(n)}$ has the value larger than 0 firstly. Let n_{ij} denote the first n such that $P_{ij}^{(n)} > 0$ and let us define $f_{ij}^{(k)},\ k = 1, 2, \ldots$ as the probability that the chain enters state j for the first time at the k-th instance after the chain departs from state i. Then,

$$P_{ij}^{(n_{ij}+ld)} = \sum_{m=0}^{l} f_{ij}^{(n_{ij}+md)} P_{jj}^{((l-m)d)}.$$

For $\tilde{X}_n$, it holds that $\lim_{n\to\infty} P_{jj}^{(nd)} = d\tilde{\pi}_j$ (cf. Theorem 4.3.1 in Ross (1996)) and we obtain that

$$\lim_{l\to\infty} P_{ij}^{(n_{ij}+ld)} = \sum_{m=0}^{\infty} f_{ij}^{(n_{ij}+md)} \lim_{l\to\infty} P_{jj}^{((l-m)d)} = d\tilde{\pi}_j. \tag{2.5}$$

Observe that $d-1$ number of values of d consecutive $P_{ij}^{(n)}$ are 0 and remaining one satisfies the property (2.5). Thus, the result (2.4) follows the periodicity of the chain and (2.5). □

It can be seen in the previous theorem that the average of consecutive d transition probabilities into state j converges to $\tilde{\pi}_j$, regardless of the starting state i, which is the long-run proportion of the Markov chain in state j. The following theorem is an easy consequence of (2.3) of the iteration in Step 2 and (2.4) of Theorem 1.

Theorem 2. *For $\tilde{X}_n$ in Theorem 1 and $\tilde{\pi}_j^{(n)}$ in Step 2, the following equation*

$$\lim_{n\to\infty} \frac{\tilde{\pi}_j^{(n)} + \tilde{\pi}_j^{(n-1)} + \cdots + \tilde{\pi}_j^{(n-d+1)}}{d} = \tilde{\pi}_j \tag{2.6}$$

holds, regardless of the initial distribution $\tilde{\pi}_j^{(0)}$ in Step 1.

In the case of periodic embedded Markov chains, Step 2 for the numerical iterative solution should be adjusted. That is, a procedure to calculate the average of d number of consecutive $\tilde{\pi}_j^{(n)}$ is added.

Step 2*. For $n \geq 1$, iterate $\tilde{\pi}_j^{(n)} = \sum_{i\in S} \tilde{\pi}_i^{(n-1)} P_{ij},\ j \in S$ and define $\tilde{\pi}_j^{(n)*} = (\tilde{\pi}_j^{(n)} + \tilde{\pi}_j^{(n-1)} + \cdots + \tilde{\pi}_j^{(n-d+1)})/d$.

In the adjusted procedure, the convergence of the numerical iteration is guaranteed by Theorem 2. The stopping rule in Step 3 is applied to $\tilde{\pi}_j^{(n)*}$ and the stationary probabilities of the continuous time Markov chain are determined in Step 4 in the same way.

3. An Illustrative Example

In this section we investigate the procedure to obtain iterative numerical solutions of Markov chains through an appropriate example. In particular, periodic embedded Markov chains are considered, which frequently occur in queueing models such as birth-and-death processes and can be found in Na (2010).

First, we consider the following transition rate matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

for a continuous time Markov chain $X(t)$ with the state space $S = \{1, 2, 3, 4, 5\}$. The rates $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda_3 = 8$, $\lambda_4 = 5$, $\lambda_5 = 1$ are determined from the transition rate matrix. The limiting probabilities $(\pi_1, \ldots, \pi_5) = (8/39, 4/39, 3/39, 6/39, 18/39)$ are obtained from the balance equations of $X(t)$; however, we use the numerical iteration of this paper.

The transition probability matrix of the embedded Markov chain $\tilde{X}_n$ is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

from Q. The period of $\tilde{X}_n$ is $d = 2$ and the balance equations of the stationary probabilities $\tilde{\pi}_j$ are

$$\begin{aligned}
\tilde{\pi}_1 &= \tilde{\pi}_2 \cdot \frac{2}{5}, \\
\tilde{\pi}_2 &= \tilde{\pi}_1 \cdot 1 + \tilde{\pi}_3 \cdot \frac{1}{2}, \\
\tilde{\pi}_3 &= \tilde{\pi}_2 \cdot \frac{3}{5} + \tilde{\pi}_4 \cdot \frac{2}{5}, \\
\tilde{\pi}_4 &= \tilde{\pi}_3 \cdot \frac{1}{2} + \tilde{\pi}_5 \cdot 1, \\
\tilde{\pi}_5 &= \tilde{\pi}_4 \cdot \frac{3}{5}.
\end{aligned}$$

Based on these equations, the iteration steps for our example can be summarized as:

Step 1. For initial values, set $(\tilde{\pi}_1^{(0)}, \ldots, \tilde{\pi}_5^{(0)}) = (0.2, \ldots, 0.2)$ or $(1, 0, \ldots, 0)$.

Step 2*. For $n \geq 1$, iterate

$$\begin{aligned}
\tilde{\pi}_1^{(n)} &= \tilde{\pi}_2^{(n-1)} \cdot \frac{2}{5}, \\
\tilde{\pi}_2^{(n)} &= \tilde{\pi}_1^{(n-1)} \cdot 1 + \tilde{\pi}_3^{(n-1)} \cdot \frac{1}{2}, \\
\tilde{\pi}_3^{(n)} &= \tilde{\pi}_2^{(n-1)} \cdot \frac{3}{5} + \tilde{\pi}_4^{(n-1)} \cdot \frac{2}{5}, \\
\tilde{\pi}_4^{(n)} &= \tilde{\pi}_3^{(n-1)} \cdot \frac{1}{2} + \tilde{\pi}_5^{(n-1)} \cdot 1, \\
\tilde{\pi}_5^{(n)} &= \tilde{\pi}_4^{(n-1)} \cdot \frac{3}{5}
\end{aligned}$$

Table 1: Iterative probabilities with equal initial distribution.

n	$\tilde{\pi}_j^{(n)*}$ $(\tilde{\pi}_j^{(n)})$	stpidx$^{(n)}$
0	· (0.2 0.2 0.2 0.2 0.2)	·
1	0.1400 0.2500 0.2000 0.2500 0.1600 (0.0800 0.3000 0.2000 0.3000 0.1200)	·
2	0.1000 0.2400 0.2500 0.2600 0.1500 (0.1200 0.1800 0.3000 0.2200 0.1800)	0.05000
3	0.0960 0.2250 0.2480 0.2750 0.1560 (0.0720 0.2700 0.1960 0.33000.1320)	0.01500
4	0.0900 0.2200 0.2450 0.2800 0.1650 (0.1080 0.1700 0.2940 0.2300 0.1980)	0.00900
5	0.0880 0.2125 0.2440 0.2875 0.1680 (0.0680 0.2550 0.1940 0.3450 0.1380)	0.00750
	⋮	
16	0.0802 0.2003 0.2401 0.2997 0.1798 (0.0962 0.1602 0.2881 0.2398 0.2157)	0.00014
17	0.0801 0.2002 0.2401 0.2998 0.1798 (0.0641 0.2402 0.1920 0.3598 0.1439)	0.00012
18	0.0801 0.2002 0.2400 0.2998 0.1799 (0.0961 0.1601 0.2880 0.2399 0.2159)	0.00007
π_j^*	0.2053 0.1027 0.0769 0.1538 0.4613	

and calculate $\tilde{\pi}_j^{(n)*} = (\tilde{\pi}_j^{(n)} + \tilde{\pi}_j^{(n-1)})/2,\ j = 1, \ldots, 5.$

Step 3. Define the numerical solution $\tilde{\pi}_j^*$ as the first $\tilde{\pi}_j^{(n)*}$ satisfying the stopping rule $|\tilde{\pi}_j^{(n)*} - \tilde{\pi}_j^{(n-1)*}| < \epsilon,\ j = 1, \ldots, 5.$

Step 4. Approximate the limiting probabilities π_j of $X(t)$ by

$$\pi_j^* = \frac{\tilde{\pi}_j^* \lambda_j^{-1}}{\sum_{i=1}^5 \tilde{\pi}_i^* \lambda_i^{-1}}, \quad j = 1, \ldots, 5.$$

We execute a program for the foregoing procedure and observe the convergence of the numerical solution, which is summarized in Table 1 and Table 2. Table 1 corresponds to the result based on the assumption of equal probabilities for the initial distribution, whereas Table 2 assumes the Markov chain starting from state 1. Tables contain the values of $\tilde{\pi}_j^{(n)},\ j = 1, \ldots, 5$ in parenthesis and the values of $\tilde{\pi}_j^{(n)*},\ j = 1, \ldots, 5$ in each row n, which are computed in Step 2*. The last column of each table represents the index values to check the stopping rule in Step 3, which are defined by

$$\text{stpidx}^{(n)} = \max_{1 \le j \le 5} \left| \tilde{\pi}_j^{(n)*} - \tilde{\pi}_j^{(n-1)*} \right|$$

and we use $\epsilon = 10^{-4}$. The last row of each table denotes the iterative numerical solution π_j^* computed in Step 4.

As seen in the results, the number of iterations for the stopping rule $\epsilon = 10^{-4}$ is 18 or 24, from which we can conclude that the power method in this paper converges rapidly. It can be reported for reference that a few more iterations are needed to achieve a smaller criterion $\epsilon = 10^{-5}$, for example. It is observed that the numerical values π_j^* in Table 1 and Table 2 are almost the same as the analytic solution π_j for the stationary probabilities of the transition rate matrix Q, which proves the usefulness

Table 2: Iterative probabilities with degenerate initial distribution.

n	$\tilde{\pi}_j^{(n)*}$ $(\tilde{\pi}_j^{(n)})$	stpidx$^{(n)}$
0	. (1 0 0 0 0)	.
1	0.5000 0.5000 0.0000 0.0000 0.0000 (0.0000 1.0000 0.0000 0.0000 0.0000)	.
2	0.2000 0.5000 0.3000 0.0000 0.0000 (0.4000 0.0000 0.6000 0.0000 0.0000)	0.30000
3	0.2000 0.3500 0.3000 0.1500 0.0000 (0.0000 0.7000 0.0000 0.3000 0.0000)	0.15000
4	0.1400 0.3500 0.2700 0.1500 0.0900 (0.2800 0.0000 0.5400 0.0000 0.1800)	0.09000
5	0.1400 0.2750 0.2700 0.2250 0.0900 (0.0000 0.5500 0.0000 0.4500 0.0000)	0.07500
	⋮	
21	0.0802 0.2003 0.2401 0.2997 0.1796 (0.0000 0.4006 0.0000 0.5994 0.0000)	0.00029
22	0.0801 0.2003 0.2401 0.2997 0.1798 (0.1602 0.0000 0.4801 0.0000 0.3596)	0.00018
23	0.0801 0.2001 0.2401 0.2999 0.1798 (0.0000 0.4003 0.0000 0.5997 0.0000)	0.00015
24	0.0801 0.2001 0.2400 0.2999 0.1799 (0.1601 0.0000 0.4801 0.0000 0.3598)	0.00009
π_j^*	0.2053 0.1026 0.0769 0.1538 0.4613	

of the stopping rule in Step 3. Note that the values of $\tilde{\pi}_j^{(n)}$ in Step 2* oscillate and do not converge, which is caused by the periodicity of the embedded Markov chain. On the contrary, the values of $\tilde{\pi}_j^{(n)*}$ consistently converge to the stationary probabilities; consequently, the proposed adjusted power method in this paper becomes necessary in case of periodic Markov chains.

4. Concluding Remarks

In this paper we studied an iterative numerical method for the limiting or stationary probabilities of Markov chains. The power method was investigated for the calculation of stationary probabilities for discrete time embedded Markov chains. The power method in this paper has two advantages of the direct use of balance equations and the fast speed of convergence. Furthermore, we proposed an adjusted power method applicable to periodic embedded Markov chains. The implementation and the approximation result were examined through an illustrative example. It was observed that marginal probabilities, which are computed iteratively, converge to the stationary probabilities rapidly and stably. In a future study, we intend to deal with the analysis of the real complex models where the analytical solution is difficult and the proposed method of this paper is applicable for the numerical solution. A comparison study of the iteration method in this paper with other numerical methods using several convergence measures based on a big sized transition matrix can be also considered.

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Received August 19, 2014; Revised November 10, 2014; Accepted November 11, 2014