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Optimum multi-objective modified step-stress accelerated life test plan for the Burr type-XII distribution

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Abstract. This paper deals with formulation of optimum multi-objective modified step-stress accelerated life test (ALT) plan for Burr type-XII distribution under type-I censoring. Since it is impractical to estimate only one objective parameter after conducting costly ALT tests; also, it is not desirable to assume instantaneous changes in stress levels because of limited capacity of test equipments and the presence of undesirable failure modes, therefore, an optimum multi-objective modified step-stress ALT plan has been designed. The optimal test plan consists in determining the optimum low stress level and optimal time at which stress starts linearly increasing from low stress by minimizing the weighted sum of the asymptotic variances of the maximum likelihood estimator of quantile lifetimes at design constant stress. The method developed has been illustrated using an example. Sensitivity analysis has been carried out. Comparative study has also been done to highlight the merits of the proposed model.

Key Words: *Accelerated lfe test, Burr type-XII distribution, modified step-stress, multi-objective optimization*

NOTATIONS

β	Stress rate, $\beta > 0$
s_0	Design stress
s_1	Low stress
s_1^*	Optimal low stress
s_2	High stress
t_1	Time at which stress changes from s_0 to s_1
t_2	Time at which stress starts increasing at the rate β from s_1

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t_2^*	Optimal time at which stress starts increasing at the rate β from s_1
t_3	Time at which stress changes from s_1 to s_2
η	Censoring time, $\eta > 0$
n	Total number of test units
n_c	Number of items censored
burr(c, k, α)	Burr type-XII distribution with parameters c, k, α
c, k	Shape parameters of Burr type-XII distribution, $c > 0,\ k > 0$
α	Scale parameter of Burr type-XII distribution, $\alpha > 0$
γ_0, γ_1	Parameters of the inverse power law, $\gamma_1 > 0,\ -\infty < \gamma_0 < \infty$
s(t)	Stress at time t
ε(t)	Cumulative exposure at time t
$G_1(t)$	$\dfrac{e^{-\gamma_0} s_0^{-\gamma_1}((\beta t + s_0)^{(1+\gamma_1)} - s_0^{(1+\gamma_1)})}{\beta(1+\gamma_1)}$
$G_2(t)$	$\dfrac{e^{-\gamma_0} s_0^{-\gamma_1}(t - t_1)}{s_1^{-\gamma_1}}$
$G_3(t)$	$\dfrac{e^{-\gamma_0} s_0^{-\gamma_1}((\beta t - \beta t_2 + s_1)^{(1+\gamma_1)} - s_1^{(1+\gamma_1)})}{\beta(1+\gamma_1)}$
$G_4(t)$	$\dfrac{e^{-\gamma_0} s_0^{-\gamma_1}(t - t_3)}{s_2^{-\gamma_1}}$
Q_1	$kce^{-\gamma_0} s_0^{-\gamma_1}$
$R_1(t)$	$\dfrac{s_0{}^{(1+\gamma_1)} \ln s_0 - (\beta t + s_0)^{(1+\gamma_1)} \ln(\beta t + s_0)}{(\beta t + s_0)^{(1+\gamma_1)} - s_0{}^{(1+\gamma_1)}}$
$R_2(t)$	$\dfrac{s_1{}^{(1+\gamma_1)} \ln s_1 - (\beta t - \beta t_2 + s_1)^{(1+\gamma_1)} \ln(\beta t - \beta t_2 + s_1)}{(\beta t - \beta t_2 + s_1)^{(1+\gamma_1)} - s_1{}^{(1+\gamma_1)}}$
$A(t)$	$\dfrac{1}{(1+\gamma_1)^2} - \dfrac{(s_0{}^{(1+\gamma_1)}(\beta t + s_0)^{(1+\gamma_1)}(\ln s_0 - \ln(\beta t + s_0))^2)}{((\beta t + s_0)^{(1+\gamma_1)} - s_0{}^{(1+\gamma_1)})^2}$
$B(t)$	$G_1(t_1)A(t_1) + \left(\ln\left(\dfrac{s_0}{s_1}\right)\right)^2 G_2(t) + G_1(t_1)\left(\ln s_0 + \dfrac{1}{1+\gamma_1} + R_1(t_1)\right)^2$
$C(t)$	$\dfrac{1}{(1+\gamma_1)^2} - \dfrac{(s_1{}^{(1+\gamma_1)}(\beta t - \beta t_2 + s_1)^{(1+\gamma_1)}(\ln s_1 - \ln(\beta t - \beta t_2 + s_1))^2)}{((\beta t - \beta t_2 + s_1)^{(1+\gamma_1)} - s_1{}^{(1+\gamma_1)})^2}$
$D(t)$	$B(t_2) + G_3(t)C(t) + G_3(t)\left(\ln s_0 + \dfrac{1}{1+\gamma_1} + R_2(t)\right)^2$

$E(t)$	$D(t_3)+G_4(t)\left(\ln\left(\frac{s_0}{s_2}\right)\right)^2$
A_j	$\left\{t \middle\| t_j < t \le t_{j+1}\right\}$, $j = 0, 1, 2, 3, 4$, where $t_0 = 0, t_4 = \eta, t_5 = \infty$
$I_j = I_j(t)$	Indicator function: $I_j(t) = \begin{cases} 1, \text{ if } t \in A_j, \ j = 0,1,2,3,4 \\ 0, \text{ otherwise} \end{cases}$
^	Implies a ML estimate
q	Quantile
Asvar	Asymptotic variance
Ascov	Asymptotic covariance
Asvar^*	Asymptotic variance in case of multi-objective plan
Asvar_q	Asymptotic variance in case of single-objective plan at quantile q

1. INTRODUCTION

In today's technological environment, it is often difficult to get the failure time data of the experimental units by testing them at normal stress. So, ALT is used to induce early failures by subjecting the experimental units to stress conditions that are more severe than those encountered in normal use condition. The failure times observed under overstress conditions are analyzed in terms of a model and then extrapolated to quantify reliability characteristics of the product under use condition, for example, estimate of probability of failure under use condition, mean or quantile life under use condition, projected return and warranty costs. An optimal ALT plan helps in determining the number of test units at each stress level, test duration, and other experimental variables and is designed to improve the accuracy of the reliability estimates.

The stress can be applied in various ways, namely, constant, step, progressive, cyclic, random (see Nelson (1990)).

The traditional constant-stress and step-stress testing assume instantaneous changes in the stress levels. However, from a practical point of view, it is desirable to increase the stress at some finite rate, because a sudden jump in stress level may cause a stress (thermal) shock or undesirable failure modes which may not appear under the normal use condition. In addition, it may be impossible for some test units to jump instantaneously from a lower stress level to a higher level. This has necessitated the use of **modified constant-stress testing** and **modified step-stress testing** in each of which stress from one level to another higher level is increased at a finite rate. In both the cases two or more stress levels higher than the normal stress level are employed. Park and Yum (1998) are the first to propose the optimum ALT plan under modified stress loading methods.

In most of the existing ALT plans, the stress level changes instantaneously which is impractical. So, it is desirable to increase the stress at some finite rate, since a sudden jump in stress level may cause a thermal shock or undesirable failure modes which may

not occur at the normal use condition. In addition, it may not be desirable for some test unit to jump instantaneously from a lower stress level to higher stress level.
Further, most of the previous work on planning ALT has focused on a sole estimating objective, such as some specified 100pth quantile lifetime, the reliability of a product over some specified period of time, and accelerating factor. It is impractical to estimate only one objective parameter after conducting such costly tests. Optimum constant stress ALT test plans with multiple estimating objectives have been designed by Fei and Xu (2009).
Since in practicing ALT one is interested in lower tail of the life distribution therefore, single objective optimal plan would separately minimize asymptotic variance of maximum likelihood (ML) estimate of log of quantiles 1%, 10% and 50%. It is impractical to obtain optimal plan using single objective after conducting such costly tests. Therefore, multi-objective plan has been considered in this paper which can easily be reduced to single objective plan by taking relevant weight equal to one and rest of them equal to zero.
Many life distribution models have been used in the literature to analyze an ALT data such as exponential distribution, normal distribution, Weibull distribution, log normal distribution, extreme value distribution, logistic distribution, log logistic distribution, truncated logistic distribution. The need to analyze an ALT data with so many life distribution models is necessitated since the use of correct life distribution model especially in the presence of a limited source of data – as typically occurs with modern devices having high reliability helps in preventing the choice of unnecessary and expensive planned replacements.
In this paper, we have obtained optimum time censored multi-objective modified step-stress ALT test plan for Burr type-XII life distribution. The Burr type-XII life distribution has a non monotone hazard function, which can accommodate many shapes of hazard function. It has algebraic tails, which are effective for modeling failures that occur with less frequency than in corresponding models based on exponential tails. The Weibull and exponential life distributions are special limiting cases of this distribution (see Appendix). The log-logistic distribution is a particular case of this distribution. This distribution has been found appropriate for accelerated life testing experiments (see Soliman (2005)). Keats et al. (1998) have shown that the use of Burr type-XII model is appropriate for the data on times to breakdown of an insulating fluid between electrodes at a voltage of 34 KV (minutes) given in Nelson (1982, pg. 105). Cook and Johnson (1986) have used the Burr model to obtain better fits to a uranium survey data set.
The paper is organized as follows:
ML estimate of model parameters and their asymptotic covariance matrix are obtained. The optimum stress change time and optimum lower level stress are found. The data analysis is presented to validate the model (design) formulated. The method developed has been illustrated using an example. Confidence intervals involving design parameters have also been obtained and sensitivity analysis carried out. Finally the proposed model has been compared with Park and Yum (1998) model using the simulated failure time data set. Using the same data set the proposed multi-objective plan has been compared with single-objective plan with respect to changes in the values of the parameters, equal weights and unequal weights.

2. ASSUMPTIONS AND TEST PROCEDURE

Basic Assumptions

a) The lifetimes of test units are independent and identically distributed.
b) Failed items are not replaced during the test.
c) Two stress levels higher than design stress (s_0), viz., the low stress level, s_1, and the high stress level, s_2, are employed, with s_0 and s_2 assumed to be pre-specified.
d) The censoring time η is pre-specified.
e) For the effect of changing stress levels, a cumulative exposure model holds (see Nelson (1980)).
f) Between any two stress levels, the stress is increased at the rate β.
g) The shape parameter "k" does not depend on the stress level. k is assumed to be known for the sake of mathematical convenience.
h) At any constant stress s, the lifetime of a unit follows a Burr type-XII model with scale parameter $\alpha(s)$ and shape parameters c and k, and the inverse power law holds for $\alpha(s)$:

$$\alpha(s) = e^{\gamma_0}\left(\frac{s_0}{s}\right)^{\gamma_1}. \tag{2.1}$$

Test Procedure

The modified step-stress ALT proceeds as follows (Figure 2.1):
a) n test units are put on test at time t_0.
b) The stress is increased at the rate β until it reaches s_1, which occurs at time $t_1 = \dfrac{s_1 - s_0}{\beta}$.
c) From t_1 to t_2, the stress is maintained at the level s_1.
d) At time t_2, the stress is increased again at the rate β until it reaches s_2, which occurs at time $t_3 = t_2 + \dfrac{s_2 - s_1}{\beta}$.
e) After t_3, the stress is maintained at the level s_2 until the censoring time η is reached.

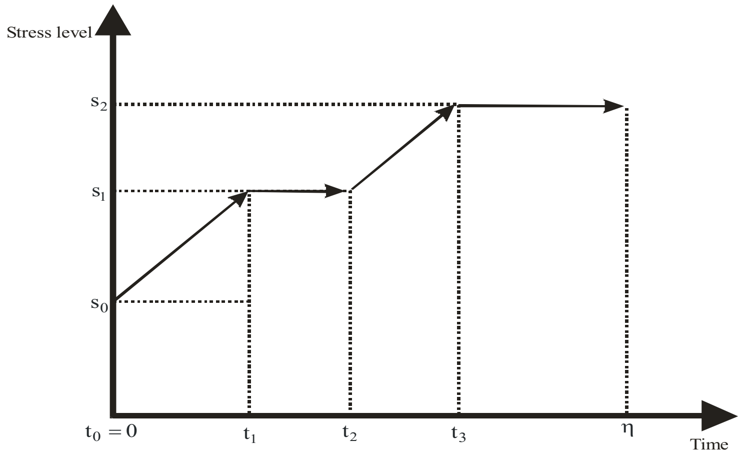

Figure 2.1. Modified step-stress ALT

3. MODEL FORMULATION AND PARAMETER ESTIMATION

Burr Type-XII Lifetime Distribution

The Burr type-XII distribution has a non-monotone hazard function, which can accommodate many shapes of hazard function.

The probability density function (pdf), and cumulative distribution function (cdf), respectively, of Burr type-XII distribution are

$$g(t;c,k,\alpha)=\left(\frac{kc}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{c-1}\left(1+\left(\frac{t}{\alpha}\right)^{c}\right)^{-(k+1)}, t\geq 0, c>0, k>0, \alpha>0,$$

$$G(t;c,k,\alpha)=1-\left(1+\left(\frac{t}{\alpha}\right)^{c}\right)^{-k}, t\geq 0, c>0, k>0, \alpha>0, \tag{3.1}$$

where c and k are shape parameters, and α is scale parameter. The Burr type-XII distribution is unimodal, and its mode is $T_{mode}=\alpha\left(\frac{c-1}{ck+1}\right)^{1/c}$ if $c>1$; and the pdf is L-shaped if $c\leq 1$.

The reliability function and hazard function are given, respectively by

$$R(t)=\left(1+\left(\frac{t}{\alpha}\right)^{c}\right)^{-k}, t\geq 0, c>0, k>0, \alpha>0,$$

$$h(t)=\left(\frac{kc}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{c-1}\left(1+\left(\frac{t}{\alpha}\right)^{c}\right)^{-1}, t\geq 0, c>0, k>0, \alpha>0.$$

The Burr type-XII distribution tends to Weibull distribution as $k \to \infty$, such that $\alpha = k^{1/c}$ (see Appendix A.1). Also, the Burr type-XII distribution tends to exponential distribution as $k \to \infty$, such that $\alpha = k^{1/c}$ and $c = 1$ (see Appendix A.2), and log-logistic distribution is a particular case of this distribution, as for k = 1, the distribution reduces to log-logistic distribution.

Soliman (2005), Abd-Elfattah et al. (2008), Lewis (1981) and Tadikamalla (1980) discuss the statistical and probabilistic properties of the Burr type-XII distribution and its relationship to other distributions used in reliability analysis.

Life Distribution under Modified Step-Stress

Based on the inverse power law (see Assumption (h), (1)), we calculate the cumulative exposure function $\varepsilon(t)$ at time t under stress level s as follows:

For $0 < t \le t_1$, we have

$$\varepsilon(t) = \int_0^t \frac{1}{\alpha(s(y))} dy = \int_0^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{s(y)}\right)^{\gamma_1}} dy = \int_0^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{y\beta + s_0}\right)^{\gamma_1}} dy = \frac{e^{-\gamma_0} s_0^{-\gamma_1}((\beta t + s_0)^{1+\gamma_1} - s_0^{1+\gamma_1})}{\beta(1+\gamma_1)}$$

$$= G_1(t) . \tag{3.2}$$

For $t_1 < t \le t_2$, we have

$$\varepsilon(t) = \int_0^t \frac{1}{\alpha(s(y))} dy = \varepsilon(t_1) + \int_{t_1}^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{s_1}\right)^{\gamma_1}} dy = \varepsilon(t_1) + e^{-\gamma_0}\left(\frac{s_0}{s_1}\right)^{-\gamma_1}(t - t_1) = \varepsilon(t_1) + G_2(t) . \tag{3.3}$$

For $t_2 < t \le t_3$, we have

$$\varepsilon(t) = \int_0^t \frac{1}{\alpha(s(y))} dy = \varepsilon(t_2) + \int_{t_2}^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{s(y)}\right)^{\gamma_1}} dy = \varepsilon(t_2) + \int_{t_2}^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{\beta y - \beta t_2 + s_1}\right)^{\gamma_1}} dy$$

$$= \varepsilon(t_2) + \frac{e^{-\gamma_0} s_0^{-\gamma_1}((\beta t - \beta t_2 + s_1)^{1+\gamma_1} - s_1^{1+\gamma_1})}{\beta(1+\gamma_1)} = \varepsilon(t_2) + G_3(t) . \tag{3.4}$$

For $t_3 < t \le \eta$, we have

$$\varepsilon(t) = \int_0^t \frac{1}{\alpha(s(y))} dy = \varepsilon(t_3) + \int_{t_3}^t \frac{1}{e^{\gamma_0}\left(\frac{s_0}{s_2}\right)^{\gamma_1}} dy = \varepsilon(t_3) + e^{-\gamma_0}\left(\frac{s_0}{s_2}\right)^{-\gamma_1}(t - t_3) = \varepsilon(t_3) + G_4(t) . \tag{3.5}$$

Then, cdf of the lifetime T of a unit tested under modified step-stress is

$$F(t) = G(\varepsilon(t)), \tag{3.6}$$

where G (.) is the assumed cdf (see (3.1)) with the scale parameter α set equal to one (shown in Appendix A.3), $\varepsilon(t)$ is the cumulative exposure (damage) model. Hence, using (3.1) and (3.6), the cdf is

$$F(t) = 1 - \left\{1 + \varepsilon(t)^c\right\}^{-k} .$$

Therefore,

$$F(t) = \begin{cases} F_1(t), 0 < t \le t_1 \\ F_2(t), t_1 < t \le t_2 \\ F_3(t), t_2 < t \le t_3 \\ F_4(t), t_3 < t \le \eta \end{cases}, \tag{3.7}$$

where

$$F_1(t) = 1 - (1 + (G_1(t))^c)^{-k},$$
$$F_2(t) = 1 - (1 + (G_1(t_1) + G_2(t))^c)^{-k},$$
$$F_3(t) = 1 - (1 + (G_1(t_1) + G_2(t_2) + G_3(t))^c)^{-k},$$
$$F_4(t) = 1 - (1 + (G_1(t_1) + G_2(t_2) + G_3(t_3) + G_4(t))^c)^{-k}.$$

The pdf is given by

$$f(t) = \begin{cases} f_1(t), 0 < t \le t_1 \\ f_2(t), t_1 < t \le t_2 \\ f_3(t), t_2 < t \le t_3 \\ f_4(t), t_3 < t \le \eta \end{cases},$$

where

$$f_1(t) = Q_1(\beta t + s_0)^{\gamma_1} (G_1(t))^{c-1} (1 + (G_1(t))^c)^{-k-1},$$
$$f_2(t) = Q_1 s_1^{\gamma_1} (G_1(t_1) + G_2(t))^{c-1} (1 + (G_1(t_1) + G_2(t))^c)^{-k-1},$$
$$f_3(t) = Q_1(\beta t - \beta t_2 + s_1)^{\gamma_1} (G_1(t_1) + G_2(t_2) + G_3(t))^{c-1} (1 + (G_1(t_1) + G_2(t_2) + G_3(t))^c)^{-k-1},$$
$$f_4(t) = Q_1 s_2^{\gamma_1} (G_1(t_1) + G_2(t_2) + G_3(t_3) + G_4(t))^{c-1} (1 + (G_1(t_1) + G_2(t_2) + G_3(t_3) + G_4(t))^c)^{-k-1}.$$

Likelihood Function

The log-likelihood of a single observation at time t is

$$L(\gamma_0, \gamma_1, c) = L = \sum_{j=0}^{3} I_j \ln f_{j+1}(t) + I_4 \ln(1 - F_4(\eta)) . \tag{3.8}$$

Let the log-likelihood of unit j be L_j. The log-likelihood, L_0, for n s-independent observations is,

$$L_0 = L_1 + ... + L_n. \tag{3.9}$$

The first and second partial derivatives of (3.8) with respect to the model parameters for a single observation are given in the Appendix A.4.

Fisher Information Matrix

The Fisher information is obtained by taking expectations of the negative of the second partial derivatives of the log (likelihood) function with respect to γ_0, γ_1, and c. The Fisher information matrix for an observation is

$$F(\gamma_0,\gamma_1,c) = \begin{bmatrix} E\left[-\frac{\partial^2 L}{\partial \gamma_0^2}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial \gamma_1}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial c}\right] \\ E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial \gamma_1}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1 \partial c}\right] \\ E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial c}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1 \partial c}\right] & E\left[-\frac{\partial^2 L}{\partial c^2}\right] \end{bmatrix}.$$

Since for some set of parameters { γ_0, γ_1, s_0, s_1, s_2, η, β, k, c, t_1, t_2, t_3 }; $|F|$ or variance function may be negative, therefore, we choose only that parametric set for which $|F| > 0$ and variance function, is positive. A similar observation has been made by Balakrishnan et al. (2004).

Since n units are tested, the Fisher information matrix for the plan with a sample of n s-independent units is

$$F = n\,F(\gamma_0,\gamma_1,c).$$

Asymptotic Variance of ML Estimate of Log Quantile q at s_0

For any plan, the asymptotic variance-covariance matrix of the model parameters is given by the inverse of the corresponding Fisher information matrix, that is,

$$F^{-1} = \begin{bmatrix} \text{Asvar}(\hat{\gamma}_0) & \text{Ascov}(\hat{\gamma}_0,\hat{\gamma}_1) & \text{Ascov}(\hat{\gamma}_0,\hat{c}) \\ \text{Ascov}(\hat{\gamma}_0,\hat{\gamma}_1) & \text{Asvar}(\hat{\gamma}_1) & \text{Ascov}(\hat{\gamma}_1,\hat{c}) \\ \text{Ascov}(\hat{\gamma}_0,\hat{c}) & \text{Ascov}(\hat{\gamma}_1,\hat{c}) & \text{Asvar}(\hat{c}) \end{bmatrix},$$

where F is the Fisher information matrix.

The asymptotic variance of the log of quantile q at design constant stress s_0 is

$$\text{Asvar}\left[\log \hat{\theta}_q\right] = \text{Asvar}\left[\hat{h}(\hat{\gamma}_0,\hat{\gamma}_1,\hat{c})\right],$$

where

$$\text{As var}\left[\hat{h}(\hat{\gamma}_0,\hat{\gamma}_1,\hat{c})\right] = \underset{\sim}{\hat{H}}' F^{-1} \underset{\sim}{\hat{H}},$$

vector $\underset{\sim}{\hat{H}}'$ is a transpose of vector $\underset{\sim}{\hat{H}}$,

$$\hat{h} = \hat{\gamma}_0 + \frac{\ln\left(-1+\left(1-q\right)^{-1/k}\right)}{\hat{c}},$$

and

$$\underset{\sim}{\hat{H}} = \left[\frac{\partial \hat{h}}{\partial \hat{\gamma}_0} \quad \frac{\partial \hat{h}}{\partial \hat{\gamma}_1} \quad \frac{\partial \hat{h}}{\partial \hat{c}}\right]'.$$

Thus,

$$\text{Asvar}\left[\log \hat{\theta}_q\right] = \text{Asvar}\left[\hat{h}(\hat{\gamma}_0, \hat{\gamma}_1, \hat{c})\right] \tag{3.10}$$

4. FORMULATION OF A MULTI-OBJECTIVE OPTIMIZATION PROBLEM

A variety of approaches can be used to solve the multi-objective optimization problem. One popular approach is to combine these objectives into one single composite objective so that the traditional mathematical programming methods can be applied (see Li and Liao (2008)). The later approach is used in this paper for the proposed plan.

A modified step-stress test is specified by stress rate β. The s_1 and t_2 are determined by minimizing the weighted sum of asymptotic variances of ML estimators of quantile lifetimes at design constant stress. The asymptotic variance of the estimator is a function of s_0, s_1, s_2, η, β, γ_0, γ_1, c, k, t_1, t_2 and t_3.

Thus, the optimal design problem can be formulated as a nonlinear optimization problem:

$$\begin{aligned} &\text{Minimize } z = \sum_{j=1}^{b} w_j v_j \\ &\text{subject to } s_0 < s_1 < s_2 \;,\; t_1 < t_2 < t_3 \;, \end{aligned} \tag{4.1}$$

where $\sum_{j=1}^{b} w_j = 1$, b is the number of quantile lifetimes, and using (3.10)

$$\begin{aligned} v_j &= \text{As var}(\log \hat{\theta}_{q_j}) \\ &= \left[1 \quad 0 \quad -\frac{\ln((1-q_j)^{-1/k}-1)}{c^2}\right] F^{-1} \left[1 \quad 0 \quad -\frac{\ln((1-q_j)^{-1/k}-1)}{c^2}\right]', \; j = 1, 2..., b\,. \end{aligned}$$

Since, the optimum modified step-stress test depends on s_0, η, β, γ_0, γ_1, c, k, and s_2, one must obtain their values from experience, similar data, or a preliminary test. The optimum values of s_1 and t_2 can be found by using *NMinimize* option of *Mathematica 6*.

In practicing ALT one is interested in the lower tails of the life distribution, i.e., early failures so 1%, 10% and 50% quantiles need to be considered.
The weights are chosen according to needs of the plan designed.

5. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

In this section, a hypothetical modified step-stress ALT experiment is considered to illustrate the methods described in this paper with the following data set.
$n=35,\ \beta=9.8,\ \gamma_0=4.7,\ \gamma_1=5.4,\ s_0=30,\ s_2=60,\ \eta=10,\ k=3,\ c=1,\ b=3,$
$w_1=w_2=w_3=1/3$.

Optimal Plan

Optimal s_1 and t_2 are obtained by minimizing the weighted sum of the Asvar of the ML estimate of the log of quantiles 1%, 10% and 50% of the distribution at design constant stress (see (4.1)) using the NMinimize option of Mathematica 6.0. They are obtained as $s_1^*=38.2825$ and $t_2^*=7.24574$.
Thus, the optimum t_1^* and t_3^* are

$$t_1^*=\frac{s_1^*-s_0}{\beta}=0.845156 \text{ and } t_3^*=t_2^*+\frac{s_2-s_1^*}{\beta}=9.46181.$$

Table 5.1. Statistically optimal modified step-stress ALT plans with
$n=35,\ \gamma_0=4.7,\ \gamma_1=5.4,\ s_0=30,\ s_2=60,\ \eta=10,\ k=3,\ c=1, b=3, w_1=w_2=w_3=1/3$

β	s_1^*	t_2^*	As var^*
5	37.3838	5.73818	0.540229
7	37.9534	6.70461	0.448281
9	38.2147	7.13900	0.415200
50	38.5591	7.71454	0.350359
100	38.5141	7.69727	0.344256
200	38.4807	7.68049	0.341119
500	38.4572	7.66649	0.339143
1000	38.4491	7.66063	0.338448
10000	38.4418	7.65440	0.337796
100000	38.4411	7.65371	0.337728
1000000	38.4411	7.65364	0.337722
10000000	38.4411	7.65363	0.337721
100000000	38.4411	7.65363	0.337721
1000000000	38.4411	7.65363	0.337721
∞	38.4411	7.65363	0.337721

Table 5.1 shows that As var^* is not substantially affected by β when $\beta \geq 50$, for the modified step-stress ALT plan.

Simulated Data

Type-I censored sample is simulated under modified step-stress test using (3.7) and is displayed in Table 5.2.

Table 5.2. Simulated data set under type-I censoring
($n = 35,\ \beta = 9.8,\ \gamma_0 = 4.7,\ \gamma_1 = 5.4,\ s_0 = 30,\ s_2 = 60, \eta = 10,\ k = 3,\ c = 1,\ b = 3,$ and $n_c = 7$)

Intervals	Failure Times
$0 < t \leq t_1$	0.110754
$t_1 < t \leq t_2$	1.09155, 3.0254, 7.18592, 5.86256, 5.84415, 2.21903, 4.40102, 1.95597, 2.44792, 6.01589
$t_2 < t \leq t_3$	9.25951, 8.58422, 9.45862, 8.25858, 8.37894, 8.45721, 8.58104, 9.3228, 8.69834, 9.30489, 8.0258, 8.82896, 8.29714
$t_3 < t \leq \eta$	9.70105, 9.82526, 9.48501, 9.54313

ML Estimates of the Design Parameters

The ML estimates of the design parameters obtained using simulated data in Table 5.2 are: $\hat{\gamma}_0 = 6.42675,\ \hat{\gamma}_1 = 8.2459,\ \hat{c} = 0.817719$.
These are obtained by using the NMaximize option of Mathematica 6.0.

Confidence Intervals

The ML estimates $\hat{\gamma}_0, \hat{\gamma}_1, \text{and}\, \hat{c}$ are approximately normally distributed in large samples, therefore ($\hat{\gamma}_0, \hat{\gamma}_1, \hat{c}$) ~ N ((γ_0, γ_1, c), F^{-1}). The two-sided $100(1-\alpha_1)\%$ approximate confidence interval for the parameter γ_0 is given by $\hat{\gamma}_0 \pm z_{\alpha_1/2}\sqrt{\hat{\text{var}}(\hat{\gamma}_0)}$, where $z_{\alpha_1/2}$ is the $(1-\alpha_1/2)^{\text{th}}$ quantile of a standard normal distribution , and $\sqrt{\hat{\text{var}}(\hat{\gamma}_0)}$ is obtained by taking square root of the first diagonal element of F^{-1}. Similarly two-sided $100(1-\alpha_1)\%$ approximate confidence interval for the parameter γ_1 and c can be obtained. The main disadvantage of approximate $100(1-\alpha_1)\%$ confidence interval is that it may yield negative lower bound though the parameter takes only positive values. In such a case the negative value is replaced by zero. Alternatively, Escobar and Meeker (1998) have suggested the use of a log transformation to obtain approximate confidence intervals for the parameters that take positive values. Thus, the approximate two sided $100(1-\alpha_1)\%$ confidence intervals for γ_1 and c are

$$\left(\hat{\gamma}_1 e^{[-z_{\alpha_1/2}\sqrt{\hat{var}(\hat{\gamma}_1)}\,/\,\hat{\gamma}_1]},\ \hat{\gamma}_1 e^{[z_{\alpha_1/2}\sqrt{\hat{var}(\hat{\gamma}_1)}\,/\,\hat{\gamma}_1]}\right),$$

and

$$\left(\hat{c}\, e^{[-z_{\alpha_1/2}\sqrt{\hat{var}(\hat{c})}\,/\,\hat{c}]},\ \hat{c}\, e^{[z_{\alpha_1/2}\sqrt{\hat{var}(\hat{c})}\,/\,\hat{c}]}\right),$$

respectively.

The $100(1-\alpha_1)\%$ confidence intervals for the parameters are obtained using the inverse of the observed Fisher information matrix $\hat{F}^{-1}$ and is given by:

$$\hat{F}^{-1} = \begin{pmatrix} 2.0651 & 3.18914 & -0.336455 \\ 3.18914 & 5.22423 & -0.507458 \\ -0.336455 & -0.507458 & 0.0662519 \end{pmatrix}.$$

The observed value of F^{-1}, that is, $\hat{F}^{-1}$, is determined by substituting the estimated parameters $\hat{\gamma}_0, \hat{\gamma}_1$ and $\hat{c}$ for the true parameters in the asymptotic covariance matrix. The square root of a diagonal element of $\hat{F}^{-1}$ gives standard error of an estimator.
Thus, the 95% approximate confidence intervals for γ_0, γ_1, and c, respectively, are $3.61015 \le \gamma_0 \le 9.24336$, $4.78952 \le \gamma_1 \le 14.1966$, and $0.44123 \le c \le 1.51545$.

Sensitivity Analysis

To use an optimum test plan, one needs estimates of the design parameters γ_0, γ_1, and c. These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor estimate of the quantile at design constant stress. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting ALT plan.

The percentage deviations of the optimal settings are measured by $PD = \left(\frac{|Z^{**} - Z^{*}|}{Z^{*}}\right) \times 100$, where Z^{*} is the setting obtained with the given design parameters, and Z^{**} is the one obtained when the parameter is misspecified. Table 5.3 shows the optimal test plans for various deviations from the design parameter estimates. The results show that the optimal plan is robust to the small deviations from baseline parameter estimates.

Table 5.3. Sensitivity analysis with $s_1^* = 38.2825$ and $t_2^* = 7.24574$

Parameter	% change	s_1	t_2	PD%
$\hat{\gamma}_0$	+0.1%	38.2914	7.24647	0.2060810
$\hat{\gamma}_0$	–0.1%	38.2736	7.24502	0.2049600
$\hat{\gamma}_1$	+0.1%	38.2824	7.24593	0.0673720
$\hat{\gamma}_1$	–0.1%	38.2826	7.24555	0.0675216
$\hat{c}$	+0.1%	38.2838	7.24585	0.1402610
$\hat{c}$	–0.1%	38.2813	7.24563	0.1407650

Comparative Study

In this section, comparative study has been carried out to highlight the merits of the proposed plan. In Table 5.4, the proposed modified step-stress ALT model have been compared with the one designed by Park and Yum (1998) in terms of likelihood functions using the hypothetical failure time data set under modified step-stress ALT with type-I censoring given in Table 5.2. Table 5.5 depicts comparison of multi-objective plan and single-objective plan with respect to change in parameters and Table 5.6 shows the comparison of multi-objective plan with equal weights and unequal weights.

Table 5.4. Comparative study of modified step-stress ALT models

ALT model	Log-likelihood function
Proposed Model	- 72.6653
Park and Yum (1998) model	- 72.9099

Table 5.4 shows that the proposed model performs better than the other modified step-stress ALT models; exist in the literature for the given data set.

Table 5.5. Comparison of multi-objective plan and single-objective plan with respect to change in parameters ($n = 35,\ \beta = 9.8,\ s_0 = 30,\ s_2 = 60,\ \eta = 10,\ k = 3,$ $b = 3, w_1 = w_2 = w_3 = 1/3$)

γ_0	γ_1	c	As var*	As var$_{0.01}$	As var$_{0.1}$	As var$_{0.5}$
14.2	15.4	1.0	48.8589	28.8940	48.7402	59.9847
14.7	15.4	1.0	81.9250	61.4429	81.8324	99.3590
15.2	15.4	1.0	138.773	106.390	138.745	166.752
14.7	15.9	1.0	60.9606	45.3858	60.8740	74.0811
14.7	16.4	1.0	45.6476	26.5458	45.6476	55.6625
14.7	15.4	1.5	486.695	326.862	488.407	544.094
14.7	15.4	2.0	2685.71	1658.36	2685.71	2903.62

Table 5.5 shows that multi-objective plan yields smaller variance as compared to single-objective plan at 50% quantile.

Table 5.6. Comparison of multi-objective plan with equal weights and unequal weights ($n = 35,\ \beta = 9.8,\ \gamma_0 = 14.7,\ \gamma_1 = 15.4,\ s_0 = 30,\ s_2 = 60,\ \eta = 10,\ k = 3,\ c = 1, b = 3$).

w_1	w_2	w_3	As var*
0.33	0.33	0.33	81.9250
0.50	0.40	0.10	74.2890
0.50	0.10	0.40	80.1030
0.10	0.40	0.50	89.1038
0.40	0.10	0.50	83.7835
0.40	0.50	0.10	76.2293
0.10	0.50	0.40	87.3294

Table 5.6 shows that multi-objective plan with equal weights yields smaller variance as compared to multi-objective plan with unequal weights when more weight is given to asymptotic variance corresponding to quantile 50% and when more weight is given to asymptotic variance corresponding to quantile 10% and next higher weight is given to asymptotic variance corresponding to quantile 50%.

6. CONCLUDING REMARKS

In contrast to traditional step-stress loading where stress levels are changed instantaneously, in modified step-stress loading stress from one level to another higher level is increased at a finite rate thereby preventing occurrence of a stress (thermal) shock or undesirable failure modes which may not appear under the normal operating condition. In this paper, an optimum time censored multi-objective modified step-stress ALT test plan for Burr type-XII life distribution has been obtained. The Burr type-XII life distribution model can be widely and effectively used in reliability applications because it has many different forms of reliability function and hazard function. The optimal test plan consists in determining the optimum low stress level and optimal time at which stress starts linearly increasing from low stress by minimizing the weighted sum of the asymptotic variances of the maximum likelihood estimator of quantile lifetimes at design constant stress. The procedure developed has been explained using an example and sensitivity analysis carried out. The results of sensitivity analysis show that optimum plan is robust for small deviations in the true values of the model parameters. Comparative study has also been carried out.

APPENDIX

A.1.
The Burr type-XII distribution tends to the Weibull life distribution as $k \to \infty$*, such that* $\alpha = k^{1/c}$.
Consider, the cdf of Burr type-XII distribution as,

$$G(t;c,k,\alpha) = 1 - \left(1 + \left(\frac{t}{\alpha}\right)^{c}\right)^{-k}, t \geq 0, c > 0, k > 0, \alpha > 0.$$

Let $\alpha = k^{1/c}$.Therefore, we get,

$$G(t;c,k) = 1 - \left(1 + \left(\frac{t}{k^{1/c}}\right)^{c}\right)^{-k}, t \geq 0, c > 0, k > 0$$

$$= 1 - \left(1 + \left(\frac{t^{c}}{k}\right)\right)^{-k}, t \geq 0, c > 0, k > 0$$

$$= 1 - \exp\left[-k \ln\left(1 + \left(\frac{t^{c}}{k}\right)\right)\right], t \geq 0, c > 0, k > 0$$

$$= 1 - \exp\left[-k\left(\left(\frac{t^{c}}{k}\right) - \left(\frac{1}{2}\right)\left(\frac{t^{c}}{k}\right)^{2} + \cdots\right)\right], t \geq 0, c > 0, k > 0$$

$$= 1 - \exp\left[-\left(t^{c} - \frac{(t^{c})^{2}}{2k} + \cdots\right)\right], t \geq 0, c > 0, k > 0 .$$

As, $k \to \infty$, we get,
$G(t;c,k) = 1 - \exp(-t^{c}), t \geq 0, c > 0$ which is the cdf of Weibull distribution.

A.2.
The Burr type-XII distribution tends to the exponential life distribution as $k \to \infty$*, such that* $\alpha = k^{1/c}$ and $c = 1$.
Consider, the cdf of Burr type-XII distribution as,

$$G(t;c,k,\alpha) = 1 - \left(1 + \left(\frac{t}{\alpha}\right)^{c}\right)^{-k}, t \geq 0, c > 0, k > 0, \alpha > 0 .$$

Let $\alpha = k^{1/c}$ and $c = 1$.Therefore, we get,

$$G(t;c,k) = 1-\left(1+\frac{t}{k}\right)^{-k}, t \geq 0, k > 0$$

$$= 1-\exp\left[-k\ln\left(1+\frac{t}{k}\right)\right], t \geq 0, k > 0$$

$$= 1-\exp\left[-k\left(\left(\frac{t}{k}\right)-\left(\frac{1}{2}\right)\left(\frac{t}{k}\right)^2+\cdots\right)\right], t \geq 0, k > 0$$

$$= 1-\exp\left[-\left((t)-\frac{(t)^2}{2k}+\cdots\right)\right], t \geq 0, k > 0 \text{ .}$$

As $k \to \infty$, we get,

$G(t;c,k) = 1-\exp(-t), t \geq 0$ which is the cdf of exponential distribution.

A.3.

The reason for setting $\alpha = 1$ *is explained as follows:*

Consider the cdf of Burr type-XII distribution as

$$G(t;c,k,\alpha) = 1-\left(1+\left(\frac{t}{\alpha}\right)^c\right)^{-k}, t \geq 0, c > 0, k > 0, \alpha > 0$$

$$\Rightarrow G(\varepsilon(t);c,k,\alpha) = 1-\left(1+\left(\frac{\varepsilon(t)}{\alpha}\right)^c\right)^{-k}, t \geq 0, c > 0, k > 0, \alpha > 0 \text{ .}$$

For $\alpha = 1$

$$G(\varepsilon(t)) \equiv G(\varepsilon(t);c,k,1)$$

$$= 1-(1+(\varepsilon(t))^c)^{-k}, t \geq 0, c > 0, k > 0$$

$$= \begin{cases} 1-\left(1+\left(G_1(t)\right)^c\right)^{-k} & ;0 < t \leq t_1 \text{ ; } c, k > 0 \quad \text{(Using (2.2))} \\ 1-\left(1+\left(\varepsilon(t_1)+G_2(t)\right)^c\right)^{-k} & ;t_1 < t \leq t_2\text{; } c, k > 0 \quad \text{(Using (2.3))} \\ 1-\left(1+\left(\varepsilon(t_2)+G_3(t)\right)^c\right)^{-k} & ;t_2 < t \leq t_3\text{; } c, k > 0 \quad \text{(Using (2.4))} \\ 1-\left(1+\left(\varepsilon(t_3)+G_4(t)\right)^c\right)^{-k} & ;t_3 < t \leq \eta \text{ ; } c, k > 0 \quad \text{(Using (2.5))} \end{cases}$$

$$= \begin{cases} F_1(t) \text{ ; } 0 < t \leq t_1 \\ F_2(t) \text{ ;} t_1 < t \leq t_2 \\ F_3(t) \text{ ;} t_2 < t \leq t_3 \\ F_4(t) \text{ ;} t_3 < t \leq \eta \end{cases}$$

$$= F(t) \text{ .}$$

It is necessary to set $\alpha = 1$, as then only F(t) will be equal to $G(\varepsilon(t))$ and linear cumulative exposure model will be hold.

A.4.

Calculations of derivatives of the log-likelihood and the elements of Fisher information matrix given in section 3 have been shown below:

The first partial derivatives are,

$$\frac{\partial L}{\partial \gamma_0} = -c\sum_{j=0}^{3} I_j + (k+1)c\left(\frac{I_0(G_1(t))^c}{(1+(G_1(t))^c)} + \frac{I_1(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)}\right.$$
$$\left. + \frac{I_2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)} + \frac{I_3(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)}\right)$$
$$+ \frac{I_4 kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)}, \tag{A1}$$

$$\frac{\partial L}{\partial \gamma_1} = I_0\left[-c\ln s_0 + \ln(\beta t + s_0) - (c-1)\left(\frac{1}{1+\gamma_1} + R_1(t)\right) + \left(\frac{(k+1)c(G_1(t))^c}{1+(G_1(t))^c}\right)\right.$$
$$\left.\left(\ln s_0 + \frac{1}{1+\gamma_1} + R_1(t)\right)\right] + I_1\left[-c\ln s_0 + \ln s_1 + \frac{(c-1)\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)} - (c-1)\left(\frac{1}{1+\gamma_1} + R_1(t_1)\right)\right.$$
$$\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right) + \left(\frac{(k+1)c(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}\right)\left(\left(\frac{1}{1+\gamma_1} + R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right) + \ln s_0\right.$$
$$\left.\left. - \frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right)\right] + I_2\left[-c\ln s_0 + \ln(\beta t - \beta t_2 + s_1) + \frac{(c-1)\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)} - \left(\frac{c-1}{1+\gamma_1}\right)\right.$$
$$\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right) - \frac{(c-1)(R_1(t_1)G_1(t_1)+R_2(t)G_3(t))}{G_1(t_1)+G_2(t_2)+G_3(t)}$$
$$+\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}\right)\left(\left(\frac{1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\right.$$
$$\left.\left. + \frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t) - \ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)} + \ln s_0\right)\right] + I_3\left[-c\ln s_0 + \ln s_2\right.$$

$$+\frac{(c-1)(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}-\left(\frac{c-1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)$$

$$-\frac{(c-1)(R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}+\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}\right)$$

$$\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\left(\frac{1}{1+\gamma_1}\right)\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}+\ln s_0\right)\Bigg]$$

$$+\left(\frac{I_4 kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right.$$

$$\left.\left(\frac{1}{1+\gamma_1}\right)+\ln s_0+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right], \qquad \text{(A2)}$$

$$\frac{\partial L}{\partial c}=\sum_{j=0}^{3}\frac{I_j}{c}+I_0\ln G_1(t)+I_1\ln(G_1(t_1)+G_2(t))+I_2\ln(G_1(t_1)+G_2(t_2)+G_3(t))$$

$$+I_3\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))-\left(\frac{I_1\ln(G_1(t_1)+G_2(t))(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}\right.$$

$$+\frac{I_0\ln G_1(t)(G_1(t))^c}{1+(G_1(t))^c}+\frac{I_2\ln(G_1(t_1)+G_2(t_2)+G_3(t))(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}$$

$$\left.+\frac{I_3\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}\right)(k+1)$$

$$-\frac{I_4 k\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}, \qquad \text{(A3)}$$

The likelihood equations are obtained by setting (A1) – (A3) to zero.
The parameter values that solve "these equations summed over all test units" are the ML estimates. As the system of likelihood equations has no closed form solution in $\gamma_0, \gamma_1,$ and c, therefore the maximum likelihood estimates $\hat{\gamma}_0, \hat{\gamma}_1,$ and $\hat{c}$ are obtained by maximizing (3.9) using NMaximize option of Mathematica 6.
The second partial derivatives are

$$\frac{\partial^2 L}{\partial \gamma_0^2} = -\left\{(k+1)c^2\left(\frac{I_0(G_1(t))^c}{(1+(G_1(t))^c)^2} + \frac{I_1(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right.\right.$$
$$\left. + \frac{I_2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2} + \frac{I_3(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$
$$\left. + \frac{I_4 kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}\right\}, \tag{A4}$$

$$\frac{\partial^2 L}{\partial \gamma_1\, \partial \gamma_0} = -(k+1)c^2\left\{\left(\frac{I_0(G_1(t))^c}{(1+(G_1(t))^c)^2}\right)\left[\ln s_0 + \frac{1}{1+\gamma_1} + R_1(t)\right] + \left(\frac{I_1(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right)\right.$$
$$\left[\ln s_0 + \left(\frac{1}{1+\gamma_1} + R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right) - \frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]$$
$$+\left(\frac{I_2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}\right)\left[\ln s_0 + \left(1 - \frac{G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\left(\frac{1}{1+\gamma_1}\right)\right.$$
$$\left. + \frac{R_1(t_1)G_1(t_1) + R_2(t)G_3(t) - \ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]$$
$$+\left(\frac{I_3(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\right.$$
$$\left.\left.\left(\frac{1}{1+\gamma_1}\right) + \ln s_0 + \frac{R_1(t_1)G_1(t_1) + R_2(t_3)G_3(t_3) - (\ln s_1 G_2(t_2) + \ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]\right\}$$
$$-\left(\frac{I_4 kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}\right)\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right.$$
$$\left.\left(\frac{1}{1+\gamma_1}\right) + \ln s_0 + \frac{R_1(t_1)G_1(t_1) + R_2(t_3)G_3(t_3) - (\ln s_1 G_2(t_2) + \ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right], \tag{A5}$$

$$\frac{\partial^2 L}{\partial c\, \partial \gamma_0} = \frac{1}{c}\frac{\partial L}{\partial \gamma_0} + (k+1)c\left(\frac{I_0 \ln G_1(t)(G_1(t))^c}{(1+(G_1(t))^c)^2} + \frac{I_1 \ln(G_1(t_1)+G_2(t))(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right.$$
$$+\frac{I_2 \ln(G_1(t_1)+G_2(t_2)+G_3(t))(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}$$
$$\left. + \frac{I_3 \ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$
$$+\frac{I_4 kc \ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}, \tag{A6}$$

$$\frac{\partial^2 L}{\partial \gamma_1^2} = -I_0\left\{\left[\ln s_0 + \frac{1}{1+\gamma_1} + R_1(t)\right]^2 \left(\frac{(k+1)c^2(G_1(t))^c}{(1+(G_1(t))^c)^2}\right)\left(A(t)\left(\frac{(k+1)c(G_1(t))^c}{1+(G_1(t))^c}\right) - (c-1)\right)\right\}$$

$$-I_1\left\{\left[\ln s_0 + \left(\frac{1}{1+\gamma_1} + R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right) - \frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]^2\right.$$

$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2} - \frac{(k+1)c(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c} + (c-1)\right)$$

$$\left.+\left(\frac{(k+1)c(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c} - (c-1)\right)\left(\frac{B(t)}{G_1(t_1)+G_2(t)}\right)\right\} - I_2\left\{\left[\ln s_0 + \left(\frac{1}{1+\gamma_1}\right)\right.\right.$$

$$\left.\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right) + \frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]^2$$

$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2} - \frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c} + (c-1)\right)$$

$$\left.+\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c} - (c-1)\right)\left(\frac{D(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\right\}$$

$$-I_3\left\{\left[\ln s_0 + \left(\frac{1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\right.\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]^2$$

$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right.$$

$$\left.-\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c} + (c-1)\right)$$

$$+\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c} - (c-1)\right)$$

$$\left.\left(\frac{E(t)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\right\} - I_4\left\{\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right.\right.$$

$$\left.\left(\frac{1}{1+\gamma_1}\right) + \ln s_0 + \frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right]^2$$

$$\left(\frac{kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}-\frac{kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)$$

$$+\left(\frac{kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)\left(\frac{E(\eta)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\Bigg\}, \qquad \text{(A7)}$$

$$\frac{\partial^2 L}{\partial\gamma_1\,\partial c}=I_0\left[\ln s_0+\frac{1}{1+\gamma_1}+R_1(t)\right]\left(\frac{(k+1)c\ln G_1(t)(G_1(t))^c}{(1+(G_1(t))^c)^2}+\frac{(k+1)(G_1(t))^c}{1+(G_1(t))^c}-1\right)$$

$$+I_1\left[\ln s_0+\left(\frac{1}{1+\gamma_1}+R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right)-\frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]\left(\frac{(k+1)(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}\right.$$

$$\left.-1+\frac{(k+1)c\ln(G_1(t_1)+G_2(t))(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right)+I_2\left[\left(\frac{1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}+\ln s_0\right]\left(\frac{(k+1)(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}-1\right.$$

$$\left.\frac{(k+1)c\ln(G_1(t_1)+G_2(t_2)+G_3(t))(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}\right)+I_3\left[\ln s_0+\left(\frac{1}{1+\gamma_1}\right)\right.$$

$$\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]$$

$$\left(\frac{(k+1)c\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}-1\right.$$

$$\left.+\frac{(k+1)(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}\right)+I_4\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right.$$

$$\left.\left(\frac{1}{1+\gamma_1}\right)+\ln s_0+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right]$$

$$\left(\frac{kc\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right.$$

$$\left.+\frac{k(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right), \qquad \text{(A8)}$$

$$\frac{\partial^2 L}{\partial c^2} = -\sum_{j=0}^{3}\frac{I_j}{c^2} - (k+1)\left(\frac{I_0(\ln G_1(t))^2(G_1(t))^c}{(1+(G_1(t))^c)^2} + \frac{I_1(\ln(G_1(t_1)+G_2(t)))^2(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right.$$
$$+\frac{I_2(\ln(G_1(t_1)+G_2(t_2)+G_3(t)))^2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}$$
$$\left.+\frac{I_3(\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)))^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$
$$-\frac{I_4k(\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)))^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}, \tag{A9}$$

The elements of the Fisher information matrix for an observation are the negative expectations of these second partial derivatives (A4 – A9), and are obtained with the aid of :

$$E[I_4(t)] = 1 - F_4(\eta),\ \ E\left[\frac{\partial L}{\partial \gamma_i}\right] = 0,\ \text{for } i = 0,1,\ \ E\left[\frac{\partial L}{\partial c}\right] = 0.$$

Thus, the expectations are:

$$E\left[-\frac{\partial^2 L}{\partial \gamma_0^2}\right] = (k+1)c^2\left(\int_0^{t_1}\frac{(G_1(t))^c f_1(t)}{(1+(G_1(t))^c)^2}dt + \int_{t_2}^{t_3}\frac{(G_1(t_1)+G_2(t_2)+G_3(t))^c f_3(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}dt\right.$$
$$\left.+\int_{t_1}^{t_2}\frac{(G_1(t_1)+G_2(t))^c f_2(t)}{(1+(G_1(t_1)+G_2(t))^c)^2}dt + \int_{t_3}^{\eta}\frac{(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c f_4(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$
$$+\frac{kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{2+k}},$$

$$E\left[-\frac{\partial^2 L}{\partial \gamma_1\,\partial \gamma_0}\right] = (k+1)c^2\left\{\int_0^{t_1}\left(\frac{f_1(t)(G_1(t))^c}{(1+(G_1(t))^c)^2}\right)\left[\ln s_0 + \frac{1}{1+\gamma_1} + R_1(t)\right]dt\right.$$
$$+\int_{t_1}^{t_2}\left(\frac{f_2(t)(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right)\left[\ln s_0 + \left(\frac{1}{1+\gamma_1} + R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right)\right.$$
$$\left.-\frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]dt + \int_{t_2}^{t_3}\left(\frac{f_3(t)(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}\right)\left[\ln s_0 + \left(\frac{1}{1+\gamma_1}\right)\right.$$
$$\left.\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right) + \frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]dt$$
$$+\int_{t_3}^{\eta}\left(\frac{f_4(t)(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$

$$\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]dt\Bigg\}$$

$$+\left(\frac{kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{2+k}}\right)$$

$$\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right],$$

$$E\left[-\frac{\partial^2 L}{\partial\gamma_1\,\partial\gamma_0}\right]=(k+1)c^2\left\{\int_0^{t_1}\left(\frac{f_1(t)(G_1(t))^c}{(1+(G_1(t))^c)^2}\right)\left[\ln s_0+\frac{1}{1+\gamma_1}+R_1(t)\right]dt\right.$$

$$+\int_{t_1}^{t_2}\left(\frac{f_2(t)(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right)\left[\ln s_0+\left(\frac{1}{1+\gamma_1}+R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right)\right.$$

$$\left.-\frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]dt+\int_{t_2}^{t_3}\left(\frac{f_3(t)(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}\right)\left[\ln s_0+\left(\frac{1}{1+\gamma_1}\right)\right.$$

$$\left.\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)+\frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]dt$$

$$+\int_{t_3}^{\eta}\left(\frac{f_4(t)(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right)$$

$$\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]dt\Bigg\}$$

$$+\left(\frac{kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{2+k}}\right)$$

$$\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right],$$

$$E\left[-\frac{\partial^2 L}{\partial c\,\partial\gamma_0}\right]=-(k+1)c\left(\int_0^{t_1}\frac{\ln G_1(t)(G_1(t))^c f_1(t)}{(1+(G_1(t))^c)^2}dt+\int_{t_1}^{t_2}\frac{\ln(G_1(t_1)+G_2(t))(G_1(t_1)+G_2(t))^c f_2(t)}{(1+(G_1(t_1)+G_2(t))^c)^2}dt\right.$$
$$+\int_{t_2}^{t_3}\frac{\ln(G_1(t_1)+G_2(t_2)+G_3(t))(G_1(t_1)+G_2(t_2)+G_3(t))^c f_3(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}dt$$
$$\left.+\int_{t_3}^{\eta}\frac{\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c f_4(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}dt\right)$$
$$-\frac{kc\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{k+2}},$$

$$E\left[-\frac{\partial^2 L}{\partial\gamma_1^2}\right]=\int_0^{t_1}\left\{\left[\ln s_0+\frac{1}{1+\gamma_1}+R_1(t)\right]^2\left(\frac{(k+1)c^2(G_1(t))^c}{(1+(G_1(t))^c)^2}\right)\left(A(t)\left(\frac{(k+1)c(G_1(t))^c}{1+(G_1(t))^c}\right.\right.\right.$$
$$\left.\left.\left.-(c-1)\right)\right\}f_1(t)dt+\int_{t_1}^{t_2}\left\{\left[\ln s_0+\left(\frac{1}{1+\gamma_1}+R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right)-\frac{\ln s_1 G_2(t)}{G_1(t_1)+G_2(t)}\right]^2\right.$$
$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}-\frac{(k+1)c(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}+(c-1)\right)+\left(\frac{B(t)}{G_1(t_1)+G_2(t)}\right)$$
$$\left.\left(\frac{(k+1)c(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}-(c-1)\right)\right\}f_2(t)dt+\int_{t_2}^{t_3}\left\{\left[\left(\frac{1}{1+\gamma_1}\right)\right.\right.$$
$$\left.\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)+\ln s_0+\frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1 G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]^2$$
$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}-\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}+(c-1)\right)$$
$$\left.+\left(\frac{D(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}-(c-1)\right)\right\}f_3(t)dt$$
$$+\int_{t_3}^{\eta}\left\{\left[\left(\frac{1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)+\ln s_0\right.\right.$$
$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]^2$$
$$\left(\frac{(k+1)c^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right.$$
$$\left.-\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}+(c-1)\right)$$

$$+\left(\frac{(k+1)c(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}-(c-1)\right)$$

$$\left.\left(\frac{E(t)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)\right\}f_4(t)dt+\left\{\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right.\right.$$

$$\left.\left(\frac{1}{1+\gamma_1}\right)+\ln s_0+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1G_2(t_2)+\ln s_2G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right]^2$$

$$\left(\frac{kc^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}\right.$$

$$\left.-\frac{kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)+\left(\frac{kc(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)$$

$$\left.\left(\frac{E(\eta)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\right\}(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{-k},$$

$$E\left[-\frac{\partial^2L}{\partial\gamma_1\partial c}\right]=-\int_0^{t_1}\left[\ln s_0+\frac{1}{1+\gamma_1}+R_1(t)\right]\left(\frac{(k+1)c\ln G_1(t)(G_1(t))^c}{(1+(G_1(t))^c)^2}+\frac{(k+1)(G_1(t))^c}{1+(G_1(t))^c}-1\right)f_1(t)dt$$

$$-\int_{t_1}^{t_2}\left[\ln s_0+\left(\frac{1}{1+\gamma_1}+R_1(t_1)\right)\left(\frac{G_1(t_1)}{G_1(t_1)+G_2(t)}\right)-\frac{\ln s_1G_2(t)}{G_1(t_1)+G_2(t)}\right]$$

$$\left(\frac{(k+1)(G_1(t_1)+G_2(t))^c}{1+(G_1(t_1)+G_2(t))^c}-1+\frac{(k+1)c\ln(G_1(t_1)+G_2(t))(G_1(t_1)+G_2(t))^c}{(1+(G_1(t_1)+G_2(t))^c)^2}\right)f_2(t)dt$$

$$-\int_{t_2}^{t_3}\left[\left(\frac{G_1(t_1)+G_3(t)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t)G_3(t)-\ln s_1G_2(t_2)}{G_1(t_1)+G_2(t_2)+G_3(t)}\right]\left(\frac{(k+1)(G_1(t_1)+G_2(t_2)+G_3(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t))^c}\right.$$

$$\left.-1+\frac{(k+1)c\ln(G_1(t_1)+G_2(t_2)+G_3(t))(G_1(t_1)+G_2(t_2)+G_3(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}\right)f_3(t)dt$$

$$-\int_{t_3}^{\eta}\left[\left(\frac{1}{1+\gamma_1}\right)\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right)+\ln s_0\right.$$

$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1G_2(t_2)+\ln s_2G_4(t))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)}\right]$$

$$\left(\frac{(k+1)c\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}\right.$$

$$\left.-1+\frac{(k+1)(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c}\right)f_4(t)dt$$

$$-\left[\left(\frac{G_1(t_1)+G_3(t_3)}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right)\left(\frac{1}{1+\gamma_1}\right)+\ln s_0\right.$$
$$\left.+\frac{R_1(t_1)G_1(t_1)+R_2(t_3)G_3(t_3)-(\ln s_1 G_2(t_2)+\ln s_2 G_4(\eta))}{G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)}\right]$$
$$\left(\frac{kc\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^2}\right.$$
$$\left.+\frac{k(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}\right)(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{-k},$$

$$E\left[-\frac{\partial^2 L}{\partial c^2}\right]=\int_0^{t_1}\frac{f_1(t)}{c^2}dt+\int_{t_1}^{t_2}\frac{f_2(t)}{c^2}dt+\int_{t_2}^{t_3}\frac{f_3(t)}{c^2}dt+\int_{t_3}^{\eta}\frac{f_4(t)}{c^2}dt+(k+1)$$
$$\left(\int_0^{t_1}\frac{(\ln G_1(t))^2(G_1(t))^c f_1(t)}{(1+(G_1(t))^c)^2}dt+\int_{t_1}^{t_2}\frac{(\ln(G_1(t_1)+G_2(t)))^2(G_1(t_1)+G_2(t))^c f_2(t)}{(1+(G_1(t_1)+G_2(t))^c)^2}dt\right.$$
$$+\int_{t_2}^{t_3}\frac{(\ln(G_1(t_1)+G_2(t_2)+G_3(t)))^2(G_1(t_1)+G_2(t_2)+G_3(t))^c f_3(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t))^c)^2}dt$$
$$\left.+\int_{t_3}^{\eta}\frac{(\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t)))^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c f_4(t)}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(t))^c)^2}dt\right)$$
$$+\frac{k(\ln(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta)))^2(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c}{(1+(G_1(t_1)+G_2(t_2)+G_3(t_3)+G_4(\eta))^c)^{2+k}}.$$

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