JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.763

# A CERTAIN EXAMPLE FOR A DE GIORGI CONJECTURE

Sungwon Cho*

Abstract. In this paper, we illustrate a counter example for the converse of a certain conjecture proposed by De Giorgi. De Giorgi suggested a series of conjectures, in which a certain integral condition for singularity or degeneracy of an elliptic operator is satisfied, the solutions are continuous. We construct some singular elliptic operators and solutions such that the integral condition does not hold, but the solutions are continuous.

## 1. Introduction

We consider a second order, linear, elliptic partial differential equation with divergence structure:

$$D_i(a_{ij}(x)D_ju(x)) = 0, \quad i,j = 1,2,\cdots,n, n \geq 2, x \in \Omega \subset \mathbb{R}^n. \tag{1.1}$$

Here $a_{ij}(x)$ satisfies the following ellipticity condition:

$$\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_j \leq \Lambda(x)|\xi|^2 \tag{1.2}$$

for some measurable, finite, positive function $\lambda(x), \Lambda(x)$, for all $\xi = (\xi_1, \xi_2, .., \xi_n) \in \mathbb{R}^n$, and almost every $x \in \Omega$.

The derivative $D_i, D_j$ is understood in a weak sense. Namely, the equation (1.1) means

$$-\int_\Omega a_{ij}(x)D_ju(x)D_i\phi(x)dx = 0 \tag{1.3}$$

Received October 23, 2014; Accepted November 10, 2014.

2010 Mathematics Subject Classification: Primary 35J70, 35J75; Secondary 35B44.

Key words and phrases: singular elliptic partial differential equation, Degenerate elliptic partial differential equation, Continuity of a weak solution.

This work is supported by Gwangju National University of Education Research Grant 2014.

for all $\phi(x) \in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is a set of all smooth functions with compact support in a given domain $\Omega$. Thus, we seek a solution $u$ in some Sobolev space, which guarantee that the integral of (1.3) is finite. More detailed definition of a weak solution, one may refer to Definition 2.1.

In the case of $a_{ij} = \delta_{ij}$ the elliptic operator reduces to the well known Laplace operator. The equation (1.1) is called uniformly elliptic if $\frac{\Lambda(x)}{\lambda(x)}$ is essentially bounded, and strictly elliptic if $\lambda(x) \geq \lambda_0$ for some positive constant $\lambda_0$.

On the contrary, if $\lambda(x)^{-1}$ is unbounded, it is called degenerate, and $\Lambda(x)$ is unbounded, it is called singular.

When the equation is uniformly and strictly elliptic, namely, $\lambda^{-1}$ and $\Lambda$ is essentially bounded, there are many established theories and results, for example, maximum principle, a regularity of solution, the existence of a solution, representation of solutions, etc. For example, see [10, 3].

Among them, we are interested in a regularity of solution, especially a continuity and discontinuity of a solution. Here, we briefly explain some classical works regarding a regularity property. In the planar case, Morrey proved that a weak solution is eventually Hölder continuous [4, 5]. For a higher dimensions ($\mathbb{R}^n$, $n \geq 3$), in the late 1950's, De Giorgi [1] and Nash [8] obtained Hölder continuity for elliptic and parabolic case, independently. A bit later, Moser proved the Harnack inequality, which leads to Hölder continuity [6, 7].

On the other hand, for the degenerate or singular case, to find optimal conditions for $\lambda(x)$ and $\Lambda(x)$ to guarantee the continuity of a solution is completely unsettled.

De Giorgi gave a talk in Italy regarding the continuity of solutions, and proposed some conjectures [2], some of which are enlisted below.

The first one is about the singular case in higher dimensions.

Conjecture 1.1. *Let* $n \geq 3$. *Suppose that* $a_{ij}$ *satisfies* (1.2) *with* $\lambda(x) = 1$ *and* $\Lambda(x)$ *satisfying*

$$\int_\Omega \exp(\Lambda(x))dx < \infty. \tag{1.4}$$

*Then all weak solutions of* (1.1) *are continuous in* $\Omega$.

The second one is concerned about the degenerate case in higher dimensions.

Conjecture 1.2. *Let* $n \geq 3$*. Suppose that* $a_{ij}$ *satisfies* (1.2) *with* $\Lambda(x) = 1$ *and* $\lambda(x)$ *satisfying*

$$\int_\Omega \exp(\lambda(x)^{-1})dx < \infty.$$

*Then all weak solutions of* (1.1) *are continuous in* $\Omega$*.*

The third one concerns the singular and degenerate case in higher dimensions.

Conjecture 1.3. *Let* $n \geq 3$*. Suppose that* $a_{ij}$ *satisfies* (1.2) *with* $\Lambda(x) = \lambda(x)^{-1}$ *satisfying*

$$\int_\Omega \exp(\Lambda(x)^2)dx < \infty.$$

*Then all weak solutions of* (1.1) *are continuous in* $\Omega$*.*

The fourth one concerns the degenerate case in planar case, $n = 2$.

Conjecture 1.4. *Let* $n = 2$*. Suppose that* $a_{ij}$ *satisfies* (1.2) *with* $\Lambda(x) = 1$ *and with* $\lambda(x)$ *satisfying*

$$\int_\Omega \exp(\sqrt{\lambda(x)^{-1}})dx < \infty.$$

*Then all weak solutions of* (1.1) *are continuous in* $\Omega$*.*

Conjectures 1–3 still remains open. In this direction, the best known result is due to Trudinger [11]. Regarding Conjecture 4, Onninen and Zhong [9] proved that all weak solution are continuous under the assumption that

$$\int_\Omega \exp(\alpha\sqrt{\lambda(x)^{-1}})dx < \infty.$$

for some $\alpha > 1$.

In his talk, De Giorgi also conjectured that the previous conditions are optimal. For example, in Conjecture 1, one can not replace (1.4) by

$$\int_\Omega \exp(\alpha\sqrt{\Lambda(x)^{1-\delta}})dx < \infty$$

for some $\delta > 0$ and any $\alpha > 0$. He gave a hint how one construct a counter example. Following his idea, Zhong [12] constructed some discontinuous solutions which illustrate that Conjectures 1,2,4 are optimal. For more details, one may refer to [12].

Obviously, any constant function is a solution of (1.1). In this paper, we prove Theorem 3.1 in Section 3, showing that the converse of

Conjecture 1 does not hold. Namely, we construct some singular elliptic operators and non-constant continuous solutions, and the singularity function $\Lambda$ does satisfy

$$\int_\Omega \exp(\Lambda(x))dx = \infty. \tag{1.5}$$

For the definition of weak solutions and weak solution spaces are discussed in Section 2.

## 2. Preliminaries

We give, here, the definition of weak solutions. Following Trudinger [11], we define scalar products:

$$\mathcal{A}(u,v) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_i v(x) dx;$$

$$\mathcal{A}_1(u,v) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_i v(x) + \Lambda(x) u(x) v(x) dx$$

on the spaces $C_0^1(\Omega)$, $C^1(\Omega)$, respectively. The weighted Sobolev spaces, $H_0^1(\mathcal{A},\Omega)$ and $H^1(\mathcal{A},\Omega)$ are then defined as the completions of $C_0^1(\Omega)$, $C^1(\Omega)$ under $\mathcal{A}, \mathcal{A}_1$, respectively, and become Hilbert spaces.

Definition 2.1. We define a weak solution of (1.1) as a function $u$ in $H^1(\mathcal{A},\Omega)$ satisfying

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_i \phi(x) dx = 0$$

for all non-negative functions $\phi \in H_0^1(\mathcal{A},\Omega)$.

One may refer to [11] for more detailed results on the properties of the weighted Sobolev spaces $H_0^1(\mathcal{A},\Omega)$ and $H^1(\mathcal{A},\Omega)$, and the weak solutions.

Lemma 2.2. *Let $a_{ij}(x) = \delta_{ij}|x|^\alpha$, $u(x) = |x|^\beta$, then $u \in H^1(\mathcal{A}, B_1(0))$ if and only if $\alpha + 2\beta + n > 2$.*

*Proof.* It is enough to check that

$$\int_{B_1(0)} |x|^\alpha |\nabla u|^2 + |x|^\alpha u^2 dx < \infty.$$

For this, note that

$$\int_{B_1(0)} |x|^\alpha |\nabla u|^2 dx = \int_{B_1(0)} \beta^2 |x|^\alpha |x|^{2\beta-2} dx$$
$$= \omega_n \int_0^1 \beta^2 r^\alpha r^{2\beta-2} r^{n-1} dr,$$

where $\omega_n$ is a surface area of a unit ball in $\mathbb{R}^n$. Also,

$$\int_{B_1(0)} |x|^\alpha u^2 dx = \int_{B_1(0)} |x|^\alpha |x|^{2\beta} dx$$
$$= \omega_n \int_0^1 r^{\alpha+2\beta+n-1} dr.$$

Thus $u \in H^1(\mathcal{A}, B_1(0))$ if and only if $\alpha + 2\beta + n - 3 > -1$, equivalently, $\alpha + 2\beta + n > 2$. □

## 3. Main result

THEOREM 3.1. *For $n \geq 2$, there exist some singular elliptic operators and non-constant continuous solutions such that*

$$\int_\Omega \exp(\Lambda(x)) dx = \infty. \tag{3.1}$$

*Proof.* Let $Lu = \sum_{i,j=1}^n D_i(\delta_{ij}|x|^\alpha D_j u)$ for some $\alpha < 0$, which will be fixed later. In this case, $a_{ij}(x)$ is $\delta_{ij}|x|^\alpha$, thus our operator $L$ is a singular elliptic operator with $\lambda = 1$, $\Lambda(x) = |x|^\alpha$. Note that

$$Lu = |x|^\alpha \Delta u + \sum_{i,j=1|}^n \delta_{ij} D_i |x|^\alpha D_j u.$$

We choose $u(x) = |x|^\beta$ for some $\beta > 0$, then

$$\Delta u = \beta(\beta-1)|x|^{\beta-2} + \frac{(n-1)\beta|x|^{\beta-1}}{|x|}$$
$$= \beta(\beta+n-2)|x|^{\beta-2}$$

and

$$\sum_{i,j=1|}^n \delta_{ij} D_i |x|^\alpha D_j u = \nabla |x|^\alpha \nabla |x|^\beta$$
$$= \alpha |x|^{\alpha-1} \beta |x|^{\beta-1}.$$

Thus in all,

$$\begin{aligned} Lu &= \beta(\beta+n-2)|x|^{\alpha+\beta-2} + \alpha\beta|x|^{\alpha+\beta-2} \\ &= \beta(\alpha+\beta+n-2)|x|^{\alpha+\beta-2}. \end{aligned}$$

Then $u(x) = |x|^\beta$ is a solution if and only if

$$\alpha + \beta + n - 2 = 0. \tag{3.2}$$

Also $u \in H^1$ if and only if

$$\alpha + 2\beta + n > 2 \tag{3.3}$$

by Lemma 2.2. We can choose $\alpha < 0$, $\beta > 0$ which satisfy (3.2) and (3.3). For example, $\alpha = \frac{3}{2} - n$, $\beta = \frac{1}{2}$. Note that from the fact that $\beta > 0$, a solution $u$ is continuous, and it is immediate to see that, for $\alpha < 0$,

$$\int_\Omega \exp(|x|^\alpha) \geq \int_\Omega \exp(c \ln |x|^{-n}) = \infty$$

for some positive constant $c$. □

A couple of remarks are in order.

Remark 3.2. One may try for the degenerate case in a similar way. But we need a condition $\alpha > 0$, which is impossible from $\alpha+\beta+n-2 = 0$ for $n \geq 2$.

Remark 3.3. If $n = 1$, then we can find $\alpha, \beta > 0$. But this 1-D case is not suggested by De Giorgi. In fact, we can find necessary and sufficient conditions in 1-D case. Note that, from $D(a(x)Du(x)) = 0$, $Du(x) = \frac{C}{a(x)}$, $u(x) = \int_0^x \frac{C}{a(t)} dt + u(0)$. Thus $u$ is continuous near 0 if and only if $\int_0^\epsilon \frac{1}{a(x)} dx$ is integrable for small $\epsilon > 0$.

## References

[1] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari,* Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. **3** (1957), no. 3, 25-43.

[2] E. De Giorgi, *Congetture sulla continuit delle soluzioni di equazioni lineari ellittiche autoaggiunte a coefficienti illimitati,* talk given in Lecce, Unpublished, January 4, 1995.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Second Edition, Springer-Verlag, Berlin-Hedelberg-New York-Tokyo, 1983.

[4] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126-166.

[5] C. B. Morrey, *Multiple integral problems in the calculus of variations and related topics*, Univ. California Publ. Math. (N.S.) **1** (1943), 1-130.

[6] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations,* Comm. Pure Appl. Math. **13** (1960), 457-468.

[7] J. Moser, *On Harnack's theorem for elliptic differential equations,* Comm. Pure Appl. Math. **14** (1961), 577-591.

[8] J. Nash, *Continuity of solutions of parabolic and elliptic equations,* Am. J. Math. **80** (1958), 931-954.

[9] J. Onninen, X. Zhong, *Continuity of solutions of linear, degenerate elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **6** (2007), no. 5, 103-116.

[10] H. C. Pak and S. Chang, *Nonlinear heat equations with transcendental nonlinearity in Besov spaces*, J. Chungcheong Matheamtical society **23** (2010), no. 4, 773-784.

[11] N. S. Trudinger, *On the regularity of generalized solutions of linear, nonuniformly elliptic equations*, Arch. Ration. Mech. Anal. **42** (1971), 51-62.

[12] X. Zhong, *Discontinuous solutions of linear, degenerate elliptic equations*, J. Math. Pures Appl. **90** (2008), 31-41.

*

Department of Mathematics Education
Gwangju National University of Education
Gwangju 500-703, Republic of Korea
*E-mail*: scho@gnue.ac.kr