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A SEXTIC-ORDER SIMPLE-ROOT FINDER WITH RATIONAL WEIGHTING FUNCTIONS OF DERIVATIVE-TO-DERIVATIVE RATIOS

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Abstract. A three-step sextic simple-root finder is constructed with the use of weighting functions of derivative-to-derivative ratios. Their convergence and computational properties are investigated along with concrete numerical examples to verify the theoretical analysis.

1. Introduction

High-order multipoint root-finders have been developed for a given nonlinear equation $f(x) = 0$, some of which can be found in [7, 11, 13]. Simple-root finders using more than two derivatives are rarely found. Jarratt[8], in 1966, suggested a multipoint iterative methods of order 4 or 5 costing three derivatives and one function below in (1.1):

$$\begin{cases} y_n = x_n + \gamma \frac{f(x_n)}{f'(x_n)},\ \gamma \in \mathbb{R}, \\ z_n = x_n + \beta \frac{f(x_n)}{f'(x_n)} + \sigma \frac{f(x_n)}{f'(y_n)},\ \beta, \sigma \in \mathbb{R}, \\ x_{n+1} = x_n - \frac{f(x_n)}{a_1 f'(x_n) + a_2 f'(y_n) + a_3 f'(z_n)}, a_1, a_2, a_3 \in \mathbb{R}. \end{cases} \tag{1.1}$$

Definition 1.1. (Error equation, asymptotic error constant, order of convergence) Let $x_0, x_1, \cdots, x_n, \cdots$ be a sequence of numbers converging to α. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$. If constants $p \geq 1,\ c \neq 0$ exist in such a way that $e_{n+1} = c\, {e_n}^p + O(e_n^{p+1})$ called the *error equation*, then p and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty} \frac{e_{n+1}}{{e_n}^p}$. Some authors call c the asymptotic error constant.

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If we select $\alpha \neq -\frac{2}{3}, \theta \neq 0, \theta \neq \alpha$ with $\theta = \beta + \gamma$ in (1.1), then we find a two-parameter family of fourth-order methods if $a_1 = \frac{6\alpha\theta+3(\alpha+\theta)+2}{6\alpha\theta}, a_2 = \frac{3\theta+2}{6\alpha(\alpha-\theta)}, a_3 = \frac{3\alpha+2}{6\theta(\theta-\alpha)}, \beta = \theta - \frac{3\theta(\theta-\alpha)}{2\alpha(3\alpha+2)}, \gamma = \frac{3\theta(\theta-\alpha)}{2\alpha(3\alpha+2)}$ are chosen. For $\alpha = -\frac{2}{3}$, we have $\theta = \alpha = -\frac{2}{3}, a_1 = \frac{1}{4}, a_2 = \frac{3}{4} + \frac{3}{8\gamma}, a_3 = -\frac{3}{8\gamma}, \beta = -\frac{2}{3} - \gamma$ and hence a one-parameter family of fourth methods are found. In particular, for $\alpha = -1, \theta = -\frac{1}{2}, a_1 = \frac{1}{6}, a_2 = \frac{1}{6}, a_3 = \frac{2}{3}, \beta = -\frac{1}{8}, \gamma = -\frac{3}{8}$, we obtain a fifth-order method, whose asymptotic error constant is given by:

$$c_2^4 + \frac{c_2^2 c_3}{8} - \frac{c_3^2}{4} + \frac{c_2 c_4}{2} + \frac{c_5}{24}, \tag{1.2}$$

which Jarratt did not show.

Three-point sextic-order methods with two derivatives can be found in [6, 9, 15] as well as in the works of Chun[5] and Parhi et al.[10] respectively shown in (1.3) and (1.4) below.

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - J_f(x_n) \cdot \frac{f(x_n)}{f'(x_n)}, J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{a(z_n-x_n)(z_n-y_n)+\frac{3}{2}J_f(x_n)f'(y_n)+(1-\frac{3}{2}J_f(x_n))f'(x_n)}, a \in \mathbb{R}. \end{cases} \tag{1.3}$$

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} = z_n - \frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)} \cdot \frac{f(z_n)}{f'(x_n)}. \end{cases} \tag{1.4}$$

The main objective of this is to develop a more general class of three-point sixth-order Jarratt-like methods requiring three derivatives and one function. To this end, by modifying and extending (1.1), we consider a general family of three-point methods in the following form:

$$\begin{cases} y_n = x_n - \gamma \cdot \frac{f(x_n)}{f'(x_n)}, \ \gamma \in \mathbb{R}, \\ z_n = x_n - (\beta + \sigma s) \cdot \frac{f(x_n)}{f'(x_n)}, \ s = \frac{f'(y_n)}{f'(x_n)}, \ \beta, \sigma \in \mathbb{R}, \\ x_{n+1} = x_n - \frac{(1+b_1(s-1)+b_2(s-1)^2+b_3(t-1)+b_4(t-1)^2}{(1+a_1(s-1)+a_2(s-1)^2+a_3(t-1)+a_4(t-1)^2} \cdot \frac{f(x_n)}{f'(x_n)}, \ t = \frac{f'(z_n)}{f'(x_n)}, \end{cases} \tag{1.5}$$

where $\gamma, \beta, \sigma \in \mathbb{R}$, $a_i, b_i (1 \leq i \leq 4)$ are parameters to be determined for sextic-order convergence. In Section 2, a successful development is described for a new family of sixth-order methods. Section 3 presents numerical experiments along with concluding remarks.

2. Convergence analysis

The desired conditions on parameters $\gamma, \beta, \sigma \in \mathbb{R}$, $a_i, b_i (1 \le i \le 4)$ will be obtained along with the proof of Theorem 2.1 describing the methodology and convergence analysis on iterative scheme (1.5).

THEOREM 2.1. *Assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic in a region containing α. Let $\Delta = f'(\alpha)$ and $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$ for $j = 2, 3, \cdots$. Let $\gamma, \beta, \sigma, a_i, b_i (1 \le i \le 4) \in \mathbb{R}$ be some parameters in (1.5). Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $\gamma = \frac{6-\sqrt{6}}{10}, \beta = \frac{6+\sqrt{6}}{10}, \sigma = -\frac{3+8\sqrt{6}}{25}$ and $a_1 = -\frac{(41+21\sqrt{6})}{12}$, $a_2 = -\frac{5(38+13\sqrt{6})}{48}, a_3 = \frac{31+9\sqrt{6}}{12}, a_4 = \frac{118-23\sqrt{6}}{48}$, $b_1 = -\frac{(139+64\sqrt{6})}{36}$, $b_2 = -\frac{5(8+3\sqrt{6})}{36}$, $b_3 = \frac{7(11+4\sqrt{6})}{36}, b_4 = \frac{54-19\sqrt{6}}{36}$. Then iterative scheme (1.5) defines a sextic-order simple-root finders satisfying the error equation below: for $n = 0, 1, 2, \cdots$,*

$$e_{n+1} = \big(\phi_1 c_2^5 + \phi_2 c_2^3 c_3 + \phi_3 c_2^2 c_4 + \phi_4 c_3 c_4 + \phi_5 c_2 c_3^2 + \phi_6 c_2 c_5 + \phi_7 c_6\big)\ e_n^6 + O(e_n^7), \tag{2.1}$$

where $\phi_1 = \frac{41-19\sqrt{6}}{5}$, $\phi_2 = \frac{-761+454\sqrt{6}}{100}$, $\phi_3 = \frac{53-12\sqrt{6}}{25}$, $\phi_4 = -\frac{2+3\sqrt{6}}{50}$, $\phi_5 = -\frac{224+171\sqrt{6}}{400}$, $\phi_6 = \frac{-4+\sqrt{6}}{20}$ and $\phi_7 = \frac{1}{100}$.

Proof. Taylor series expansion of $f(x_n)$ about α up to 6th-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = \Delta\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)\}. \tag{2.2}$$

It follows that

$$f'(x_n) = \Delta\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)\}. \tag{2.3}$$

For brevity of notation, e_n will be denoted by e throughout the proof. With the aid of symbolic computation of Mathematica[16], we have:

$$y_n = x_n - \gamma \frac{f(x_n)}{f'(x_n)} = \alpha + (1-\gamma)e + c_2\gamma e^2 + 2\gamma(-c_2^2 + c_3)e^3 + Y_4 e^4 + Y_5 e^5 + Y_6 e_n^6 + O(e^7), \tag{2.4}$$

where $Y_4 = \gamma(4c_2^3 - 7c_2c_3 + 3c_4), Y_5 = -2\gamma(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$ and $Y_6 = \gamma(16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)$. In view of the fact that $f'(y_n) = f'(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$f'(y_n) = \Delta[1 + (1-\gamma)c_2 e + \{3(1-\gamma)^2 c_3 + 2\gamma c_2^2\}e^2$$

$$+D_3e^3 + \Sigma_{i=4}^{6}\ D_i\ e^i + O(e^7)], \tag{2.5}$$

where $D_3 = 2\{2(1-\gamma)^3c_4 - \gamma c_2(2c_2^2 + (3\gamma-5)c_3)\}, D_i = D_i(\gamma, c_2, c_3, \cdots, c_6)$ for $4 \le i \le 6$.

$$s = \frac{f'(y_n)}{f'(x_n)} = 1 - 2\gamma c_2 e + 3\gamma(2c_2^2 + (\gamma-2)c_3)e^2 + \Sigma_{i=3}^{6}\ E_i\ e^i + O(e^7), \tag{2.6a}$$

where $E_i = E_i(\gamma, c_2, c_3, \cdots, c_6)$ is a multivariate polynomial in $c_2, c_3, \cdots, c_6$ for $3 \le i \le 6$.

Hence with $\beta = 4 - \frac{5\sqrt{6}}{3}$ and $\sigma = -\frac{3+8\sqrt{6}}{25}$ we have:

$$z_n = x_n - (\beta + \sigma s) \cdot \frac{f(x_n)}{f'(x_n)} = (1-\beta)e + (\beta + 2\gamma\sigma)c_2e^2$$

$$+\Sigma_{i=3}^{6}\ F_i\ e^i + O(e^7), \tag{2.6b}$$

where $F_i = F_i(\beta, \gamma, \sigma, c_2, c_3, \cdots, c_6)$ for $3 \le i \le 6$.

In view of the fact that $f'(z_n) = f'(x_n)|_{e_n \to (z_n - \alpha)}$, we get:

$$f'(z_n) = \Delta[1 + 2(1-\beta)c_2e + [3c_3(1-\beta)^2 + 2c_2^2(\beta + 2\gamma\sigma)]e^2$$

$$+\Sigma_{i=3}^{6}\ G_i\ e^i + O(e^7)], \tag{2.7}$$

where $G_i = G_i(\beta, \gamma, \sigma, c_2, c_3, \cdots, c_6)$ for $3 \le i \le 6$.

By direct substitution of $z_n, f(x_n), f'(x_n), f'(y_n), f'(z_n)$ with $s = \frac{f'(y_n)}{f'(x_n)}, t = \frac{f'(z_n)}{f'(x_n)}$ in (1.5), we find

$$x_{n+1} = x_n - \frac{(1 + b_1(s-1) + b_2(s-1)^2 + b_3(t-1) + b_4(t-1)^2}{(1 + a_1(s-1) + a_2(s-1)^2 + a_3(t-1) + a_4(t-1)^2} \cdot \frac{f(x_n)}{f'(x_n)}$$

$$= \alpha + (1 - 2(a_3 - b_3)\beta - 2(a_1 - b_1)\gamma)e^2 + \sigma_{i=3}^{6}\ \Gamma_i\ e^i + O(e^7), \tag{2.8}$$

where $\Gamma_i = \Gamma_i(a_j, b_k, \beta, \gamma, \sigma, c_2, c_3, \cdots, c_6)$ for $3 \le i \le 6$ and $1 \le j, k \le 4$.

By solving $1 - 2(a_3 - b_3)\beta - 2(a_1 - b_1)\gamma = 0$ from (2.8) for a_1, we immediately obtain

$$a_1 = b_1 - \frac{1 - 2\beta(a_3 - b_3)}{2\gamma}. \tag{2.9}$$

By substituting a_1 into $\Gamma_3 = 0$, we find two relations independently of c_2 and c_3 below:

$$-2 + 6(a_3 - b_3)\beta(\beta - \gamma) + 3\gamma = 0,$$

$$1 + 4(a_4 - b_4)\beta^2 - 2(b_1\gamma + b_3\beta) + 4(a_2 - b_2)\gamma^2 + 4(a_3 - b_3)\gamma\sigma = 0.$$

As a result, we find

$$a_3 = \frac{1 + 4(a_4 - b_4)\beta^2 - 2b_1\gamma + 4(a_2 - b_2)\gamma^2}{2\beta} + \frac{(2 - 3\gamma)(\beta + 2\gamma\sigma)}{6\beta^2(\beta - \gamma)},$$

$$b_3 = \frac{1 + 4(a_4 - b_4)\beta^2 - 2b_1\gamma + 4(a_2 - b_2)\gamma^2}{2\beta} + \frac{(2 - 3\gamma)\gamma\sigma}{3\beta^2(\beta - \gamma)}. \quad (2.10)$$

By substituting a_1, a_3, b_3 into $\Gamma_4 = 0$, we find three relations independently of c_2, c_3 and c_4 with $\beta(\beta - \gamma) \neq 0$ below:

$$3 - 4\gamma + \beta(6\gamma - 4) = 0,$$

$$\frac{2\gamma(3\gamma - 2)[-2 + (4a_2 + 3b_1 - 4b_2)\gamma^2 - 12(a_2 - b_2)\gamma^3]}{(6\gamma^2 - 8\gamma + 3)} - \frac{2\gamma(3\gamma - 2)^2(-1 - 12\gamma + 18\gamma^2)\sigma}{3(4\gamma - 3)(6\gamma^2 - 8\gamma + 3)}$$
$$+ \frac{(3 - 4\gamma)^2[(a_4 - b_4)(6\gamma^2 - 8\gamma + 3) - 3(2 - 3\gamma)^2(1 - 2b_1\gamma + 4(a_2 - b_2)\gamma^2)]}{4(3\gamma - 2)^3(6\gamma^2 - 8\gamma + 3)}$$
$$+ \frac{(4\gamma - 3)[4 + 3\gamma + (-8a_2 - 12b_1 + 8b_2)\gamma^2 + 36(a_2 - b_2)\gamma^3]}{2(6\gamma^2 - 8\gamma + 3)} = 0,$$

$$-\frac{(a_4 - 2b_4)(3 - 4\gamma)^2 + (2 - 3\gamma)^2[-1 + 4(a_2 - 2b_2)\gamma^2]}{(2 - 3\gamma)^2} + \frac{32\gamma^2(-2 + 3\gamma)^4\sigma^2}{3(-3 + 4\gamma)^2(6\gamma^2 - 8\gamma + 3)}$$
$$+ \frac{4\gamma\sigma(2 - 3\gamma)[1 - 6\gamma(b_1 - 2a_2\gamma + 2b_2\gamma)(6\gamma^2 - 8\gamma + 3)]}{3(3 - 4\gamma)(6\gamma^2 - 8\gamma + 3)} + \frac{4\gamma\sigma(-a_4 + b_4)(3 - 4\gamma)}{(2 - 3\gamma)} = 0. \quad (2.11)$$

Solving (2.11) for β, a_4, b_4 yields

$$\beta = \frac{3 - 4\gamma}{4 - 6\gamma},$$

$$a_4 = \frac{-6 + \beta[3 + 12\gamma(-b_1 + 3a_2\gamma - 4b_2\gamma)] + 4\gamma^2[3b_1 + 2(a_2 - 6a_2\gamma + 6b_2\gamma)]}{4\beta^2(-2 + 3\beta)}$$
$$- \frac{\gamma\sigma[-4 + 12\beta + (12 + 3(-3 + 4b_1)\beta)\gamma - 6(3 + 2b_1)\gamma^2]}{6\beta^3(-2 + 3\beta)(\beta - \gamma)} - \frac{\gamma^3\sigma[4(a_2 - b_2)(3\gamma - 2) - 3b_1]}{\beta^3(-2 + 3\beta)}$$
$$+ \frac{\gamma^2\sigma^2(-2 + 3\gamma)(-4 + 6\beta + 3\gamma)}{3\beta^4(-2 + 3\beta)(\beta - \gamma)},$$

$$b_4 = \frac{-1 + 3b_1\gamma(-\beta + \gamma) + 2\gamma^2[6a_2(\beta - \gamma) + b_2(2 - 9\beta + 6\gamma)]}{2\beta^2(-2 + 3\beta)}$$

$$+\frac{\gamma^2\sigma^2(-2+3\gamma)(-4+6\beta+3\gamma)}{3\beta^4(-2+3\beta)(\beta-\gamma)}$$
$$+\frac{\gamma\sigma[-2\beta(1+2b_1\gamma)+(3+4b_1)\gamma^2]}{2\beta^3(-2+3\beta)(\beta-\gamma)}$$
$$+\frac{\gamma^3\sigma[4(a_2-b_2)(2-3\gamma)+3b_1]}{\beta^3(-2+\beta)}. \tag{2.12}$$

By substituting $a_1, a_3, b_3, \beta, a_4, b_4$ into $\Gamma_5 = 0$, we find five relations independently of c_2, c_3, c_4 and c_5 below:

$$3-12\gamma+10\gamma^2=0,\ 3+\gamma^2(6-24\sigma)+18\gamma^3\sigma+\gamma(-9+8\sigma)=0,$$
$$125(137609+57501\sqrt{6})+648(28357+11923\sqrt{6})a_2$$
$$-2250(4202+1753\sqrt{6})b_1-648(28357+11923\sqrt{6})b_2=0,$$
$$325(229308+91387\sqrt{6})-72(945441+409574\sqrt{6})a_2$$
$$+90(647706+261659\sqrt{6})b_1+72(655416+304849\sqrt{6})b_2=0,$$
$$-25(36683358+14824337\sqrt{6})+72(2673882+1215523\sqrt{6})a_2$$
$$-540(627727+253603\sqrt{6})b_1+288(672867+227813\sqrt{6})b_2=0. \tag{2.13}$$

Solving (2.13) for $\gamma, \sigma, b_1, a_2, b_2$ and simplifying, we obtain:

$$\sigma=-\frac{(3+8\sqrt{6})}{25}, \gamma=\frac{6-\sqrt{6}}{10},$$
$$a_2=-\frac{5(38+13\sqrt{6})}{48}, b_1=-\frac{(139+64\sqrt{6})}{36}, b_2=-\frac{5(8+3\sqrt{6})}{36}. \tag{2.14}$$

After substituting (2.14) into (2.9), (2.10),(2.12), we further simplify:

$$a_1=-\frac{(41+21\sqrt{6})}{12}, a_3=\frac{31+9\sqrt{6}}{12}, b_3=\frac{7(11+4\sqrt{6})}{36},$$
$$\beta=\frac{6+\sqrt{6}}{10}, a_4=\frac{118-23\sqrt{6}}{48}, b_4=\frac{54-19\sqrt{6}}{36}. \tag{2.15}$$

We finally substitute overall relations found so far into Γ_6 to obtain $\Gamma_6 = \left(\phi_1 c_2^5 + \phi_2 c_2^3 c_3 + \phi_3 c_2^2 c_4 + \phi_4 c3c4 + \phi_5 c_2 c_3^2 + \phi_6 c_2 c_5 + \phi_7 c_6\right)$, where ϕ_i are described in (2.1). This completes the proof. □

REMARK 2.2. From the first two equations of (2.13), the other pair of values for σ, γ is possible with $\sigma = -\frac{(3-8\sqrt{6})}{25}, \gamma = \frac{6+\sqrt{6}}{10}$. In this case, $\beta = \frac{6-\sqrt{6}}{10}$ and all coefficients a_i, b_j can be obtained with $\sqrt{6}$ replaced by $-\sqrt{6}$.

3. Numerical experiments and concluding remarks

Many numerical analysis arising in engineering problems often requires a large number of working-precision digits. Computing asymptotic error constants $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ with several significant digits of accuracy would encounter extreme calculations due to the indeterminate form of a small-number division near the root α.

In the current numerical experiments, high-precision computing with programming language *Mathematica* (Version 7) has been performed with 100 working-precision digits, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. In addition, the error bound $\epsilon = \frac{1}{2} \times 10^{-80}$ was used. The values of initial guess x_0 were selected close to α to guarantee the convergence of the iterative methods. Only 15 significant digits of approximated roots x_n are displayed in Tables 1-3 due to the limited paper space, although 80 significant digits are available. Numerical experiments have been carried out on a personal computer equipped with an AMD 3.1 Ghz dual-core processor and 64-bit Windows 7 operating system.

Iterative method (1.5) was identified by **Y1** and has shown acceptable performance when applied to a test function:

$$F_1(x) = \sin(x+1) - x + 2, \alpha \approx 2.07076672714204.$$

Definition 3.1. (Asymptotic Convergence Order) Assume that the asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ is known. Then we can define the *asymptotic convergence order* $p_a = \lim_{n\to\infty} \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$, being abbreviated by ACO.

Method **Y1** in Table 1 clearly confirmed hexic-order convergence. Table 1 lists iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $\eta_n = |\frac{e_n}{e_{n-1}{}^6}|$ as well as the theoretical asymptotic error constant η and computational asymptotic convergence order $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$. The values of η_n agree up to 7 significant digits with η. As expected, the computed asymptotic order of convergence well approaches 6.

Additional functions below are tested to verify the convergence behavior of proposed scheme (1.5):

n	x_n	$\lvert F(x_n)\rvert$	$\lvert e_n\rvert$	$\frac{e_n}{e_{n-1}{}^6}$	η	p_n
0	1.9	0.339249	0.170767			
1	2.07076671448853	2.527×10^{-8}	1.265×10^{-8}	0.0005102599209	0.0002274623374	5.54288
2	2.07076672714204	1.864×10^{-51}	9.336×10^{-52}	0.0002274623568		6.00000
3	2.07076672714204	0.0×10^{-99}	0.0×10^{-99}			

$p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$, $^*\binom{0.95}{1.5} = 0.95 + 1.5i$

TABLE 1. Convergence for sample test functions $F_1(x)$ with method **Y1**

f	x_0	$\lvert x_n-\alpha\rvert$	**JA**	**CH**	**PG**	**Y1**
f_1	-0.31	$\lvert x_1-\alpha\rvert$	2.39e-8*	9.17e-10	3.67e-9	1.06e-9
		$\lvert x_2-\alpha\rvert$	3.95e-38	*3.69e-54*	6.54e-50	6.66e-54
		$\lvert x_3-\alpha\rvert$	0.0e-100	0.0e-100	0.0e-100	0.0e-100
f_2	1.2	$\lvert x_1-\alpha\rvert$	4.14e-5	3.76e-6	1.85e-5	3.02e-6
		$\lvert x_2-\alpha\rvert$	4.92e-21	3.78e-29	9.59e-27	*6.06e-32*
		$\lvert x_3-\alpha\rvert$	0.0e-99	0.0e-99	0.0e-99	0.0e-99
f_3	0.45	$\lvert x_1-\alpha\rvert$	1.05e-7	8.84e-9	1.28e-8	1.90e-8
	$+0.85i$	$\lvert x_2-\alpha\rvert$	1.99e-36	*9.56e-50*	1.36e-48	1.75e-47
		$\lvert x_3-\alpha\rvert$	0.0e-100	0.0e-99	0.0e-99	0.0e-100
f_4	0.1	$\lvert x_1-\alpha\rvert$	8.32e-7	5.00e-7	2.41e-6	3.87e-8
		$\lvert x_2-\alpha\rvert$	5.39e-32	1.01e-38	5.08e-34	*1.91e-45*
		$\lvert x_3-\alpha\rvert$	0.0e-130	0.0e-137	0.0e-199	0.0e-144
f_5	0.84	$\lvert x_1-\alpha\rvert$	2.55e-7	2.22e-9	8.79e-9	5.24e-9
		$\lvert x_2-\alpha\rvert$	2.87e-33	*98.93e-53*	1.04e-48	4.14e-50
		$\lvert x_3-\alpha\rvert$	0.0e-100	0.0e-100	0.0e-100	0.0e-100
f_6	1.1	$\lvert x_1-\alpha\rvert$	2.77e-6	1.78e-7	2.91e-7	3.91e-7
		$\lvert x_2-\alpha\rvert$	1.12e-27	*1.09e-40*	7.98e-40	9.86e-39
		$\lvert x_3-\alpha\rvert$	0.0e-99	0.0e-99	0.0e-99	0.0e-99
f_7	0.3	$\lvert x_1-\alpha\rvert$	5.33e-10	1.08e-11	7.57e-12	4.84e-12
		$\lvert x_2-\alpha\rvert$	3.31e-48	8.90e-68	6.59e-69	*2.21e-70*
		$\lvert x_3-\alpha\rvert$	0.0e-100	0.0e-100	0.0e-100	0.0e-100

* 2.39e-8 denotes 2.39 $\times10^{-8}$

TABLE 2. Comparison of $|x_n-\alpha|$ for $f_1(x)-f_7(x)$ among listed methods

$$f_1(x) = x\cos\tfrac{3\pi}{2}x - \log(x^2 - \tfrac{1}{x} - \tfrac{19}{9}),\ \alpha = -1/3,\ x_0 = -0.31,$$
$$f_2(x) = \sqrt{2}x\cos x^2 - \log(e + 8x^2 - 4\pi) + 1,\ \alpha = \sqrt{\tfrac{\pi}{2}},\ x_0 = 1.2,$$
$$f_3(x) = \cos(x^2 - x + \tfrac{37}{36}) + 3x - \tfrac{5}{2} - i\sqrt{7},\ \alpha = \tfrac{1}{2} + i\tfrac{\sqrt{7}}{3},\ x_0 = 0.45 + 0.85i,\ i = \sqrt{-1},$$
$$f_4(x) = x^3 - 2 + (x+2)\log(e + x^2),\ \alpha = 0,\ x_0 = 0.1,$$
$$f_5(x) = x^5 + x^3 + e^{2x} - 7,\ \alpha \approx 0.878720933693359,\ x_0 = 0.84,$$
$$f_6(x) = 4\cos^2 x + \log(e^2 + 9x^2 - \pi^2) - 3,\ \alpha = \pi/3,\ x_0 = 1.1,$$
$$f_7(x) = 3x^2 + xe^{1-x^2} + \sin(x^3 + 2) - 2,\ \alpha \approx 0.323329877529435,\ x_0 = 0.3,$$

with $\log z$ ($z \in \mathbb{C}$) representing a principal analytic branch such that $-\pi \le \mathrm{Im}(\log z) < \pi$.

f	x_0	p_n	**JA**	**CH**	**PG**	**Y1**
f_1	-0.31	p_1	5.10138	6.02347	6.04048	5.90624
		p_2	5.00000	6.00000	6.00000	6.00000
f_2	1.2	p_1	4.70107	5.28779	5.57932	5.82983
		p_2	4.99994	5.99998	6.00003	6.00000
f_3	0.45	p_1	5.02463	5.99413	6.01727	5.93624
	$+0.85i$	p_2	5.00000	6.00000	6.00000	6.00000
f_4	0.1	p_1	5.20917	6.11028	6.03205	7.16720
		p_2	5.00000	6.00000	6.00000	6.00000
f_5	0.84	p_1	4.96969	6.03203	5.95361	6.07670
		p_2	5.00000	6.00000	6.00000	6.00000
f_6	1.1	p_1	5.00471	5.69594	5.21100	5.35830
		p_2	5.00000	6.00000	6.00000	6.00000
f_7	0.3	p_1	4.99851	5.93970	5.92075	5.85203
		p_2	5.00000	6.00000	6.00000	6.00000

TABLE 3. Comparison of computational ACO $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ for $f_1(x) - f_7(x)$

For the purpose of comparison, we first identify methods (1.1), (1.3), (1.4) by **JA**, **CH**, **PG**, respectively. Table 2 displays the values of $|x_n - \alpha|$ for methods **JA**, **CH**, **PG**, **Y1**. As a result of Table 2, proposed method shows favorable or equivalent performance as compared with existing methods **JA**, **CH** and **PG**. In Table 2, italicized numbers indicates the least errors $|x_n - \alpha|$. When the same order of convergence is known, one should note that the local convergence of $|x_n - \alpha|$ behaves differently depending on c_j, namely $f(x)$ and α. Table 3 well exhibits computational convergence orders of the listed methods, among which method **Y1** clearly shows the convergence order of 6.

During the current numerical experiments, **CH** has shown best accuracy for f_3, f_5, while **Y1** for f_2, f_4, f_7. On the other hand, two methods **CH, Y1** have shown similar performance for f_1 and three methods **CH, PG, Y1** for f_6. Computational accuracy is sensitively dependent on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. We should be aware that no iterative method always shows best accuracy for all the test functions. The corresponding efficiency index for the proposed method (1.5) is found to be $6^{1/4}$, which is same as those of listed sextic-order methods but better than that of fourth-order Jarratt's method [2]. A new development of a family of higher-order iterative methods will be pursued in the near future using the current approach employing three derivatives and one function.

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