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ON HENSTROCK INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

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Abstract. In this paper we introduce the interval-valued Henstock integral on time scales and investigate some properties of these integrals.

1. Introduction and preliminaries

The Henstock integral for real functions was first defined by Henstock [2] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Wiener and Feynman integrals. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [5] in 2006. In 2000, Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral of interval-valued functions [6].

In this paper we introduce the concept of the Henstock delta integral of interval-valued function on time scales and investigate some properties of the integral.

A time scale T is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in T$ we define the forward jump operator $\sigma(t) = \inf\{s \in T : s > t\}$ where $\inf \phi = \sup\{T\}$, while the backward jump operator $\rho(t) = \sup\{s \in T : s < t\}$ where $\sup \phi = \inf\{T\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in T$ is defined by $\mu(t) = \sigma(t) - t$, whlie the backward graininess function $\nu(t)$ of $t \in T$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in T$ we denote the closed interval $[a, b]_T = \{t \in T : a \leq t \leq b\}$.

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$\delta = (\delta_L, \delta_R)$ is a Δ-gauge on $[a,b]_T$ if $\delta_L(t) > 0$ on $(a,b]_T$, $\delta_R(t) > 0$ on $(a,b]_T$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \mu(t)$ for each $t \in [a,b]_T$.

A collection $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of tagged intervals is δ-fine Henstock partition of $[a,b]_T$ if $U_{i=1}^{n}[t_{i-1}, t_i] = [a,b]_T$, $[t_{i-1}, t_i]_T \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_T$ for each $i = 1, 2, \cdots, n$.

Definition 1.1 ([5]). *A function $f : [a,b] \longrightarrow \mathbb{R}$ is Henstock delta integrable (or H_Δ-integrable) on $[a,b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a Δ-gauge δ on $[a,b]$ such that*

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - A \right| < \epsilon$$

for every δ-fine Henstock partition $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of $[a,b]$. The number A is called the Henstock delta integral of f on $[a,b]$ and we write $A = (H_\Delta) \int_a^b f$.

Definition 1.2. *Let $I_\mathbb{R} = \{I = [I^-, I^+]$ is the closed bounded interval on the real $\mathbb{R}\}$, where $I^- = \min\{x : x \in I\}$, $I^+ = \max\{x : x \in I\}$. For $A, B, C \in I_\mathbb{R}$, we define $A \leq B$ iff $A^- \leq B^-$ and $A^+ \leq B^+$, $A + B = C$ iff $A^- + B^- = C^-$ and $A^+ + B^+ = C^+$, and $AB = \{ab : a \in A, b \in B\}$, where $(AB)^- = \min(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$ and $(AB)^+ = \max(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$. Define $d(A,B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between A and B.*

Definition 1.3 ([6]). *An interval-valued function $F : [a,b] \longrightarrow I_\mathbb{R}$ is Henstock integrable to $I_0 \in I_\mathbb{R}$ on $[a,b]$ if for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of $[a,b]$ is a δ-fine Henstock partition of $[a,b]$. We write $(IH) \int_a^b F(x)dx = I_0$ and $F \in IH[a,b]$.

2. The interval-valued Henstock delta integral on time scales

In this section, we will define the Henstock integral of interval-valued function on time scales and investigate some properties of the integral.

Definition 2.1. *An interval-valued function $F : [a,b]_T \longrightarrow I_\mathbb{R}$ is Henstock delta integrable to $I_0 \in I_\mathbb{R}$ on $[a,b]_T$ if for every $\epsilon > 0$ there* *exists a* Δ*-gauge* δ *on* $[a,b]_T$ *such that*

$$d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) \,:\, 1 \le i \le n\}$ *is a* δ*-fine Henstock partition of* $[a,b]_T$*. We write* $(IH_\Delta)\int_a^b F(x)dx = I_0$ *and* $F \in IH_\Delta[a,b]_T$*.*

Remark 2.2. *It is clear, if* $F(x) = F^-(x) = F^+(x)$ *for all* $x \in [a,b]$*, then Definition 2.1 implies the real-valued Henstock integral on* $[a,b]$*.*

Remark 2.3. *If* $F \in IH_\Delta[a,b]_T$*, then the integral is unique.*

Theorem 2.4. *An interval-valued function* $F : [a,b]_T \longrightarrow I_{\mathbb{R}}$ *is Henstock delta integrable on* $[a,b]_T$ *if and only if* $F^-, F^+ \in H_\Delta[a,b]_T$ *and*

$$(IH_\Delta)\int_a^b F(x)dx = \left[(H_\Delta)\int_a^b F^-(x)dx, (H_\Delta)\int_a^b F^+(x)dx\right],$$

where $F(x) = [F^-(x), F^+(x)]$*.*

Proof. Let $F \in IH_\Delta[a,b]_T$. Then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for each $\epsilon > 0$ there exists a Δ-gauge δ on $[a,b]_T$ such that

$$d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) \,:\, 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$.

Let $P = \{([t_{i-1}, t_i]_T, \xi_i) \,:\, 1 \le i \le n\}$ be a δ-fine Henstock partition of $[a,b]_T$. Since

$$\begin{aligned} &d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\right) \\ &= \max\left(\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - I_0^-\right|, \left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - I_0^+\right|\right), \end{aligned}$$

$$\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - I_0^-\right| < \epsilon, \left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - I_0^+\right| < \epsilon.$$

Conversely, let $F^-, F^+ \in H_\Delta[a,b]_T$. then there exist $H_1, H_2 \in \mathbb{R}$ with the property that given Δ-gauge δ on $[a,b]_T$ such that

$$\left|\sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - H_1\right| < \epsilon, \left|\sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - H_2\right| < \epsilon.$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$. We define $I_0 = [H_1, H_2]$, then if $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$. We have

$$d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\right) < \epsilon.$$

Hence $F : [a,b]_T \longrightarrow I_{\mathbb{R}}$ is Henstock delta integrable on $[a,b]_T$. □

From Theorem 2.4 and the properties of Henstock delta integral ([6]), we can easily obtain the following theorems.

THEOREM 2.5. *Let $F, G \in IH_\Delta[a,b]_T$ and $\beta, \gamma \in \mathbb{R}$. Then*

(1) $\beta F + \gamma G \in IH_\Delta[a,b]_T$ *and* $(IH_\Delta)\int_a^b (\beta F + \gamma G)\, dx = \beta (IH_\Delta)\int_a^b F dx + \gamma (IH_\Delta)\int_a^b G dx$
(2) *If* $F(x) \le G(x)$ a.e. *in* $[a,b]_T$*, then* $(IH_\Delta)\int_a^b F dx \le (IH_\Delta)\int_a^b G dx$

THEOREM 2.6. *Let $F \in IH_\Delta[a,c]_T$ and $F \in IH_\Delta[c,b]_T$. Then $F \in IH_\Delta[a,b]_T$ and*

$$(IH_\Delta)\int_a^b F dx = (IH_\Delta)\int_a^c F dx + (IH_\Delta)\int_c^b F dx$$

THEOREM 2.7. *Let $F, G \in IH_\Delta[a,b]_T$ and $d(F,G)$ is Lebesgue delta integrable on $[a,b]_T$. Then*

$$d\left((IH_\Delta)\int_a^b F dx, (IH_\Delta)\int_a^b G dx\right) \le (L_\Delta)\int_a^b d(F,G) dx$$

Proof. By definition of distance, we have

$$
\begin{aligned}
&d\left((IH_\Delta)\int_a^b F dx, (IH_\Delta)\int_a^b G dx\right) \\
&= \max\left(\left|\left((IH_\Delta)\int_a^b F dx\right)^- - \left((IH_\Delta)\int_a^b G dx\right)^-\right|,\right. \\
&\qquad\qquad \left.\left|\left((IH_\Delta)\int_a^b F dx\right)^+ - \left((IH_\Delta)\int_a^b G dx\right)^+\right|\right) \\
&= \max\left(\left|(IH_\Delta)\int_a^b (F^- - G^-)dx\right|, \left|(IH_\Delta)\int_a^b (F^+ - G^+)dx\right|\right) \\
&= \max\left((L_\Delta)\int_a^b \left|F^- - G^-\right| dx, (L_\Delta)\int_a^b \left|F^+ - G^+\right| dx\right) \\
&\le (L_\Delta)\int_a^b d(F,G)dx.
\end{aligned}
$$

□

3. The Henstock delta integral of fuzzy number valued functions

DEFINITION 3.1 ([1]). *Let $\widetilde{A} \in F(\mathbb{R})$ be a fuzzy subset on $\mathbb{R}$. If for any $\lambda \in [0,1]$, $A_\lambda = [A_\lambda^-, A_\lambda^+]$ and $A_1 \neq \emptyset$, where $A_\lambda = \{x : \widetilde{A}(x) \ge \lambda\}$, then $\widetilde{A}$ is called a fuzzy number.*

Let $\widetilde{\mathbb{R}}$ denote the set of all fuzzy numbers.

DEFINITION 3.2 ([3]). *Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathbb{R}}$, we define $\widetilde{A} \le \widetilde{B}$ iff $A_\lambda \le B_\lambda$ for all $\lambda \in (0,1]$, $\widetilde{A} + \widetilde{B} = \widetilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$, $\widetilde{A} \cdot \widetilde{B} = \widetilde{D}$ iff $A_\lambda \cdot B_\lambda = D_\lambda$ for any $\lambda \in (0,1]$. For $D(\widetilde{A}, \widetilde{B}) = \sup_{\lambda \in [0,1]} d(A_\lambda, B_\lambda)$ is called the distance between $\widetilde{A}$, $\widetilde{B}$.*

LEMMA 3.3 ([1]). *If a mapping $H : [0,1] \longrightarrow I_{\mathbb{R}}, \lambda \mapsto H(\lambda) = [m_\lambda, n_\lambda]$, satisfies $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ when $\lambda_1 < \lambda_2$, then*

$$\widetilde{A} := \bigcup_{\lambda \in (0,1]} \lambda H(\lambda) \in \widetilde{\mathbb{R}}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n),$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$.

DEFINITION 3.4. *Let* $\widetilde{F} : [a,b]_T \longrightarrow \widetilde{\mathbb{R}}$. *If the interval-valued function* $F_\lambda(x) = \left[F_\lambda^-(x), F_\lambda^+(x)\right]$ *is Henstock delta integrable on* $[a,b]_T$ *for any* $\lambda \in (0,1]$, *then we say that* $\widetilde{F}(x)$ *is Henstock delta integrable on* $[a,b]_T$ *and the integral is defined by Henstock delta integral is defined by*

$$(FH_\Delta)\int_a^b \widetilde{F}(x)dx := \bigcup_{\lambda\in(0,1]} \lambda (IH_\Delta)\int_a^b F_\lambda(x)dx$$
$$= \bigcup_{\lambda\in(0,1]} \lambda \left[(H_\Delta)\int_a^b F_\lambda^- dx, (H_\Delta)\int_a^b F_\lambda^+ dx\right].$$

We will write $\widetilde{F} \in FH_\Delta[a,b]_T$.

THEOREM 3.5. $\widetilde{F} \in FH_\Delta[a,b]_T$, *then* $(FH_\Delta)\int_a^b \widetilde{F}(x)dx \in \widetilde{\mathbb{R}}$ *and*

$$\left[(FH_\Delta)\int_a^b \widetilde{F}(x)dx\right]_\lambda = \bigcap_{n=1}^{\infty} (IH_\Delta)\int_a^b F_{\lambda_n}(x)dx,$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$.

Proof. Let $H : (0,1] \longrightarrow I_{\mathbb{R}}$ be defined by

$$H(\lambda) = \left[(H_\Delta)\int_a^b F_\lambda^-(x)dx, (H_\Delta)\int_a^b F_\lambda^+(x)dx\right].$$

Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are increasing and decreasing on λ, respectively, therefore, when $0 < \lambda_1 \le \lambda_2 \le 1$ we have $F_{\lambda_1}^-(x) \le F_{\lambda_2}^-(x)$, $F_{\lambda_1}^+(x) \ge F_{\lambda_2}^+(x)$ on $[a,b]_T$. Thus from Theorem 2.5, we have

$$\left[(H_\Delta)\int_a^b F_{\lambda_1}^-(x)dx, (H_\Delta)\int_a^b F_{\lambda_1}^+(x)dx\right]$$
$$\supset \left[(H_\Delta)\int_a^b F_{\lambda_2}^-(x)dx, (H_\Delta)\int_a^b F_{\lambda_2}^+(x)dx\right].$$

Using Theorem 2.5 and Lemma 3.3 we obtain

$$(IH_\Delta)\int_a^b \widetilde{F}(x)dx := \bigcup_{\lambda\in(0,1]} \lambda \left[(H_\Delta)\int_a^b F_\lambda^-(x)dx, (H_\Delta)\int_a^b F_\lambda^+(x)dx\right] \in \widetilde{\mathbb{R}}$$

and for all $\lambda \in (0,1]$,

$$\left[(FH_\Delta)\int_a^b \widetilde{F}(x)dx\right]_\lambda = \bigcap_{n=1}^{\infty} (IH_\Delta)\int_a^b F_{\lambda_n}(x)dx,$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$. □

Using Theorem 3.5 and the properties of (IH) integral, we can obtain the properties of (FH_Δ) integral. For examples, we get the linearity, monotonicity and interval additivity properties of (FH_Δ) integral.

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