JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.735

SPLITTING OFF H^f-SPACES AND THEIR DUALS

Yeon Soo Yoon*

Abstract. We obtain a splitting theorem which characterizes when a given space is a catesian product of an H^f-space, and also obtain a dual theorem for a co-H^g-space. Then we get Dula and Gottlieb's results as corollaries.

1. Introduction

In [2], Dula and Gottlieb obtained a splitting theorem as follows;

Theorem 1.1. ([2] Theorem 1.3) *Given spaces Z, X and Y the following statements are equivalent;*

(1) *X is an H-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$.*
(2) *There are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \in G(X, Z)$.*

In [17], we introduced concepts of H^f-spaces for maps which are generalizations of H-spaces and also introduced concepts of co-H^g-spaces for maps which are generalizations of co-H-spaces. In this paper, we would like to obtain some conditions which characterize when a given space is a cartesian product of an H^f-space and also to obtain some conditions for dual situations. In section 2, we show that if X is an H^f-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$, then there are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \circ f \in G(A, Z)$. Also, we show that if there are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \circ f \in G(A, Z)$, then X is an H^f-space and there exists a space Y such that $\pi_n(Z) \cong \pi_n(X \times Y)$ for all n. Moreover, we can obtain a necessary and sufficient condition for splitting an

Received September 30, 2014; Accepted October 20, 2014.
2010 Mathematics Subject Classification: Primary 55P45, 55P35.
Key words and phrases: f-cyclic maps, H^f-spaces, g-cocyclic maps, co-H^g-spaces.
The author was supported by Hannam University Research Fund, 2014.

H^f-space off a space under a condition as follows. If $f : A \to X$ has a right homotopy inverse, then X is an H^f-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$ if and only if there are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \circ f \in G(A, Z)$. However, taking $f = 1_X$, we can obtain the above Dula and Gottlieb's theorem as a corollary.

In section 3, we study some dual situations obtaining some conditions for splitting co-H^g-space off a space. We show that if X is a co-H^g-space and there exists a simply connected space Y such that Z is homotopy equivalent to $X \vee Y$, then there exist maps $i : Z \to X$ and $r : X \to Z$ such that $i \circ r \sim 1$ and $g \circ i \in DG(Z, A)$. Also, we show that if there exist maps $i : Z \to X$ and $r : X \to Z$ such that $i \circ r \sim 1$ and $g \circ i \in DG(Z, A)$, then X is a co-H^g-space and there exists a simply connected space Y such that $H_n(Z) \cong H_n(X \vee Y)$ for all n. Thus we can obtain a necessary and sufficient condition for splitting a co-H^g-space off a space which is a dual result of the result in section 2. Moreover, we can obtain a Dula and Gottlieb's result as a corollary.

Throughout this paper, space means a space of homotopy type of connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$.

2. Splitting H^f-spaces off a space

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [11] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow{(f \vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case, $f = 1_X : X \to X$, $g : B \to X$ is called cyclic [14]. It is clear that a space X is an H-space if and only if the identity map 1_X of X is cyclic. In [17], for a map $f : A \to X$, we defined a space X to be a *H^f-space* if the map $f : A \to X$ is cyclic. Clearly, any H-space is an H^f-space for any map $f : A \to X$, but the converse does not hold. Consider the natural pairing $\mu : S^3/S^1 \times S^3 \to S^3/S^1$. Then we know that the Hopf map $\eta : S^3 \to S^2$ is cyclic. Thus S^2 is an H^η-space for $\eta : S^3 \to S^2$, but S^2 is not an H-space. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$ the *Gottlieb set* denoted $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Gottlieb [4,5] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$.

In general, $G(B; X) \subset G(B; A, f, X) \subset [B, X]$ for any map $f : A \to X$ and any space B. However, there is an example [16] such that $G(B, X) \neq G(B; A, f, X) \neq [B, X]$.

Proposition 2.1. [8] *X is an H-space if and only if $G(B, X) = [B, X]$ for any space B.*

Proposition 2.2. [17] *X is an H^f-space for a map $f : A \to X$ if and only if $G(B; A, f, X) = [B, X]$ for any space B.*

Proposition 2.3. [17] *X is an H-space if and only if for any space A and any map $f : A \to X$, X is an H^f-space for a map $f : A \to X$.*

We obtain a necessary condition for splitting H^f-space off a space as follows;

Theorem 2.4. *If X is an H^f-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$, then there are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \circ f \in G(A, Z)$.*

Proof. There are maps $h : Z \to X \times Y$, $k : X \times Y \to Z$ which are satisfying $k \circ h \sim 1_Z$ and $h \circ k \sim 1_{X \times Y}$. Let $r = p_1 \circ h : Z \to X$ and $i = k \circ i_1 : X \to Z$, where $p_1 : X \times Y \to X$ is the projection and $i_1 : X \to X \times Y$ is the inclusion. Then $r \circ i \sim p_1 \circ h \circ k \circ i_1 \sim 1_X$. Thus i has r as a left homotopy inverse. Since X is an H^f-space, there is a map $F : A \times X \to X$ such that $F \circ j = \nabla(f \vee 1)$, where $j : A \vee X \to A \times X$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. Consider the map $I = k \circ (F \times 1) \circ (1 \times h) : A \times Z \to Z$. Then $I \circ j' \sim \nabla(i \circ f \vee 1)$, where $j' : A \vee Z \to A \times Z$ is the inclusion and $\nabla : Z \vee Z \to Z$ is the folding map. Thus $i \circ f \in G(A, Z)$.

□

We do not know whether the converse of the above result holds or not. But we can obtain a little bit weaker result than the converse of the above theorem.

LEMMA 2.5. [7] *Let* $0 \to A \stackrel{\lambda}{\to} B \stackrel{\mu}{\to} C \to 0$ *be an exact sequence of groups in which* λA *is in the center of* B*. Then the following three assertions are equivalent;*

(1) *There is a homomorphism* $\bar{\mu} : C \to B$ *with* $\mu\bar{\mu} = 1$.
(2) *There is a homomorphism* $\bar{\lambda} : B \to A$ *with* $\bar{\lambda}\lambda = 1$.
(3) *There is a homomorphism* $\bar{\lambda} : B \to A$ *such that* $(\bar{\lambda}, \mu) : B \to A \times C$ *is an isomorphism.*

THEOREM 2.6. *If there are maps* $i : X \to Z$ *and* $r : Z \to X$ *such that* r *is a left homotopy inverse for* i *and* $i \circ f \in G(A, Z)$*, then* X *is an* H^f*-space and there exists a space* Y *such that* $\pi_n(Z) \cong \pi_n(X \times Y)$ *for all* n.

Proof. Let $\iota_r : Y \to Z$ be the inclusion of the homotopy fiber of $r : Z \to X$. By Milnor's result in [10], ΩZ is homotopy equivalent to a CW complex. Thus in the fiber sequence $\Omega Z \to Y \to Z$ the base and fiber are homotopy equivalent to a CW complex, hence by Stasheff's result in [13], Y is homotopy equivalent to a CW complex. Since $i \circ f : A \to Z$ is cyclic, there is a map $H : A \times Z \to Z$ such that $H \circ j \sim \nabla(i \circ f \vee 1)$. Then the composition $r \circ H \circ (1 \times i) : A \times X \to X$ establishes the fact that $f : A \to K$ is cyclic. Thus we know that X is an H^f-space. Moreover, we know, from the fact $r \circ i \sim 1 : X \to X$, that the fibration $Y \stackrel{\iota_r}{\to} Z \stackrel{r}{\to} X$ has a cross section $i : X \to Z$. Then the section i induces a splitting of sequence for the fibration so that we get a split short exact sequences $0 \to \pi_*(Y) \stackrel{\iota_*}{\to} \pi_*(Z) \underset{i_*}{\overset{r_*}{\rightleftarrows}} \pi_*(X) \to 0$. Since $Im\ \iota_* = Ker\ r_* \subset Z(\pi_*(Z))$ and $r_* \circ i_* = 1$, we know, from Lemma 2.5, that there is a homomorphism $\lambda : \pi_*(Z) \to \pi_*(Y)$ such that $(\lambda, r_*) : \pi_*(Z) \to \pi_*(Y) \times \pi_*(X)$ is an isomorphism. Thus we have $\pi_n(Z) \cong \pi_n(X \times Y)$ for all n. □

From the Whitehead's theorem, we have the following corollary.

COROLLARY 2.7. *If there is a map* $\rho : Z \to Y$ *such that* $\rho_* = \lambda : \pi_*(Z) \to \pi_*(Y)$ *in the proof of the above theorem, then* X *is an* H^f*-space and there exists a space* Y *such that* Z *is homotopy equivalent to* $X \times Y$ *if and only if there are maps* $i : X \to Z$ *and* $r : Z \to X$ *such that* r *is a left homotopy inverse for* i *and* $i \circ f \in G(A, Z)$.

In the proof of Theorem 2.6, if we can show that Z is homotopy equivalent to $X \times Y$, then we can obtain a necessary and sufficient condition for splitting H^f-space off a space as follows. Anyway we need a little bit more condition for this.

COROLLARY 2.8. *Suppose $f : A \to X$ has a right homotopy inverse $g : X \to A$. Then X is an H^f-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$ if and only if there are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \circ f \in G(A, Z)$.*

Proof. A necessary part follows from Theorem 2.4. For a sufficient part, from the proof of Theorem 2.6, we only need to show that Z is homotopy equivalent to $X \times Y$. Let $\iota_r : Y \to Z$ be the inclusion of the homotopy fiber of $r : Z \to X$. Let $H : A \times Z \to Z$ be a map satisfying $H \circ j \sim \nabla(i \circ f \vee 1)$. Then let $h = H \circ (1 \times \iota_r) \circ (g \times 1) : X \times Y \to Z$. Consider the following diagram;

$$\begin{array}{ccccc} Y & \xrightarrow{i_2} & X \times Y & \xrightarrow{p_1} & X \\ \| & & \downarrow{\scriptstyle g} & & \| \\ Y & \xrightarrow{\iota_r} & Z & \xrightarrow{r} & X \end{array}$$

By the definition of h, the left square commutes, while the right square commutes after π_* is applied. Thus h induces an isomorphism of homotopy groups, and as all spaces are homotopy equivalent to CW complexes, it follows that $h : X \times Y \to Z$ is a homotopy equivalence. $\square$

Taking $f = 1_X$, we have, from the fact H^1-space is just an H-space, that the following corollary for splitting an H-space off a space.

COROLLARY 2.9. [2] *The following conditions are equivalent;*

(1) *X is an H-space and there exists a space Y such that Z is homotopy equivalent to $X \times Y$.*
(2) *There are maps $i : X \to Z$ and $r : Z \to X$ such that r is a left homotopy inverse for i and $i \in G(X, Z)$.*

3. Splitting co-H^g-spaces off a space

Let $g : X \to A$ be a map. A based map $f : X \to B$ is called *g-coclic* [11] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ \Delta\downarrow & & j\downarrow \\ X \times X & \xrightarrow{(g\times f)} & A \times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a g-cocyclic map f.

In the case $g = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [14]. Clearly any cocyclic map is a g-cocyclic map and also $f : X \to B$ is g-cocyclic iff $g : X \to A$ is f-cocyclic. The *dual Gottlieb set* $DG(X, g, A; B)$ *for a map* $g : X \to A$ is the set of all homotopy classes of g-cocyclic maps from X to B. In the case $g = 1_X : X \to X$, we called such a set $DG(X, 1, X; B)$ the *dual Gottlieb set* denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map.

In general, $DG(X; B) \subset DG(X, g, A; B) \subset [X, B]$ for any map $g : X \to B$ and any space B. However, there is an example in [15] such that $DG(X, B) \neq DG(X, g, A; B) \neq [X, B]$.

Proposition 3.1. [9] *X is a co-H-space if and only if $DG(X, B) = [X, B]$ for any space B.*

Definition 3.2. *A space X is called a co-H^g-space for a map $g : X \to A$ [17] if there is a map, a co-H^g-structure, $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times g)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map.*

Proposition 3.3. [17] *X is a co-H^g-space for a map $g : X \to A$ if and only if $DG(X, g, A; B) = [X, B]$ for any space B.*

Clearly we have, from Proposition 3.1 and Proposition 3.3, the following corollary.

Corollary 3.4. *X is a co-H-space if and only if for any space A and any map $g : X \to A$, X is a co-H^g-space for a map $g : X \to A$.*

From now on, every space is assumed to be homotopy equivalent to a connected simply connected CW complex.

Theorem 3.5. *If X is a co-H^g-space and there exists a simply connected space Y such that Z is homotopy equivalent to $X \vee Y$, then there exist maps $i : Z \to X$ and $r : X \to Z$ such that $i \circ r \sim 1$ and $g \circ i \in DG(Z, A)$.*

Proof. Since Z is homotopy equivalent to $X \vee Y$, there exist maps $h : Z \to X \vee Y$, $k : X \vee Y \to Z$ such that $k \circ h \sim 1_Z$ and $h \circ k \sim 1_{X \vee Y}$. Let $i = p_1 \circ h : X \overset{h}{\to} X \vee Y \overset{p_1}{\to} X$ and $r = k \circ i_1 : X \overset{i_1}{\to} X \vee Y \overset{k}{\to} Z$, where $p_1 : X \vee Y \to X$ is the projection and $i_1 : X \to X \vee Y$ is the inclusion. Then $i \circ r = p_1 \circ h \circ k \circ i_1 \sim p_1 \circ i_1 = 1_X$. Since X is a co-H^g-space, there is a map $\theta : X \to A \vee X$ such that $j \circ \theta = (g \times 1) \circ \Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Now we show that $g \circ i \in DG(Z, A)$. Consider the composite map $\rho : Z \overset{h}{\to} X \vee Y \overset{(\theta \vee 1)}{\to} A \vee X \vee Y \overset{(1 \vee k)}{\to} A \vee Z$. Consider the commutative diagram

$$\begin{array}{ccccccc} Z & \xrightarrow{h} & X \vee Y & \xrightarrow{\theta \vee 1} & A \vee X \vee Y & \xrightarrow{1 \vee k} & A \vee Z \\ \Delta \downarrow & & \Delta \downarrow & & \downarrow & & j \downarrow \\ Z \times Z & \xrightarrow{h \times h} & (X \vee Y) \times (X \vee Y) & \xrightarrow{g \circ p_1 \times 1} & A \times (X \vee Y) & \xrightarrow{1 \times k} & A \times Z. \end{array}$$

Then $j \circ \rho \sim (g \circ i \times 1) \circ \Delta$. Thus $g \circ i \in DG(Z, A)$. □

We do not know whether the converse of the above result holds or not. But we can obtain a little bit weaker result than the converse of the above theorem.

THEOREM 3.6. *If there exist maps $i : Z \to X$ and $r : X \to Z$ such that $i \circ r \sim 1$ and $g \circ i \in DG(Z, A)$, then X is a co-H^g-space and there exists a simply connected space Y such that $H_n(Z) \cong H_n(X \vee Y)$ for all n.*

Proof. Let $\iota : Z \to Y$ be the projecton of the homotopy cofiber of $r : X \to Z$. That is, we have a cofibration $X \overset{r}{\to} Z \overset{\iota}{\to} Y (= C_r)$. It is clear that $Y = C_r$ is homotopy equivalent to CW complex, where C_r is the mapping cone of $r : X \to Z$. Since $g \circ i \in DG(Z, A)$, there is a map $\rho : Z \to A \vee Z$ such that $j \circ \rho \sim (g \circ i \times 1) \circ \Delta$. Let $\theta : X \overset{r}{\to} Z \overset{\rho}{\to} A \vee Z \overset{(1 \vee i)}{\to} A \vee X$. Then $j \circ \theta \sim (g \times 1) \circ \Delta$ and X is a co-H^g-space. Since $i \circ r \sim 1$, the cofibration $X \overset{r}{\to} Z \overset{\iota}{\to} Y$ gives rise to a split short exact sequence $0 \to H_n(X) \underset{i_*}{\overset{r_*}{\rightleftarrows}} H_n(Z) \overset{\iota_*}{\to} H_n(Y) \to 0$. Since $Im\, r_* = Ker\, \iota_* \subset Z(H_*(Z))$ and $i_* \circ r_* = 1$, we know, from Lemma 2.5, that there is an isomorphism $(i_*, \iota_*) : H_*(Z) \to H_*(X) \times H_*(Y)$. Thus we have, from the fact $H_*(X) \times H_*(Y) \cong H_*(X \vee Y)$, that $H_n(Z) \cong H_n(X \vee Y)$ for all n. □

In the proof of Theorem 3.6, if we can show that Z is homotopy equivalent to $X \vee Y$, then we can obtain a necessary and sufficient condition for splitting so-H^g-space off a space as follows. Anyway we need a little bit more condition for this.

COROLLARY 3.7. *Suppose $g : X \to A$ has a left homotopy inverse $f : A \to X$. Then X is a co-H^g-space and there exists a space Y such that Z is homotopy equivalent to $X \vee Y$ if and only if there are maps $i : Z \to X$ and $r : X \to Z$ such that r is a right homotopy inverse for i and $g \circ i \in DG(Z, A)$.*

Proof. A necessary part follows from Theorem 3.5. For a sufficient part, from the proof of Theorem 3.6, we only need to show that Z is homotopy equivalent to $X \vee Y$. Let $\iota : Z \to Y$ be the projecton of the homotopy cofiber of $r : X \to Z$. Let $\rho : Z \to A \vee Z$ be a map satisfying $j \circ \rho \sim (g \circ i \times 1) \circ \Delta$. Then let $h = (1 \vee \iota) \circ (f \vee 1) \circ \rho : Z \to X \vee Y$. Then the following diagram of cofiber sequence is homotopy commute;

$$\begin{array}{ccccc} X & \xrightarrow{r} & Z & \xrightarrow{\iota} & Y \\ \| & & h\downarrow & & \| \\ X & \xrightarrow{\iota_r} & X \vee Y & \xrightarrow{p_2} & Y \end{array}$$

By the definition of h, the left square commutes, while the right square commutes after H_* is applied. Thus h induces an isomorphism of homology groups, and as all spaces are homotopy equivalent to CW complexes and $\pi_1(Z) = \pi_1(X \vee Y) = 0$, it follows that $h : Z \to X \vee Y$ is a homotopy equivalence. □

Taking $g = 1_X$, we have, from the fact co-H^1-space is just a co-H-space, that the following corollary for splitting a co-H-space off a space.

COROLLARY 3.8. [2] *The following conditions are equivalent;*

(1) *X is a co-H-space and there exists a space Y such that Z is homotopy equivalent to $X \vee Y$.*
(2) *There are maps $i : Z \to X$ and $r : X \to Z$ such that r is a right homotopy inverse for i and $i \in DG(Z, X)$.*

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Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: yoon@hannam.ac.kr