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IRREDUCIBLE REIDEMEISTER ORBIT SETS

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Abstract. The Reidemeister orbit set plays a crucial role in the Nielsen type theory of periodic orbits, much as the Reidemeister set does in Nielsen fixed point theory. Extending our work on Reidemeister orbit sets, we obtain algebraic results such as addition formulae for irreducible Reidemeister orbit sets. Similar formulae for Nielsen type irreducible essential orbit numbers are also proved for fibre preserving maps.

1. Introduction

Nielsen fixed point theory has been extended to a Nielsen type theory of periodic orbits [6, Section III.3]. In fixed point theory, the computation of the Nielsen number often relies on our knowledge of the Reidemeister set, that is the set of Reidemeister conjugacy classes in the fundamental group. Extending Ferrario's work [2] on Reidemeister sets, we in [7] made an algebraic study of the Reidemeister orbit set in relation to an invariant normal subgroup. We obtained addition formulae for Reidemeister orbit numbers, and applied them to the Nielsen type essential orbit number of fiber preserving maps. Our aim in this paper is similar to study the irreducible Reidemeister orbit set, to obtain some addition formulae for irreducible Reidemeister orbit numbers, and as application, to find addition formulae for Nielsen type irreducible essential orbit numbers of fiber preserving maps.

Given a group endomorphism $f : G \to G$, the Reidemeister set $\mathcal{R}(f)$ of f is the set of orbits of the left action of G on G via $\gamma \overset{g}{\mapsto} g\gamma f(g^{-1})$. For a given integer $n > 0$, f acts on the Reidemeister set $\mathcal{R}(f^n)$ of the n-th iterate f^n. An orbit of this action is called a Reidemeister orbit, the set of all such orbits is the Reidemeister orbit set $\mathcal{RO}^{(n)}(f)$. We define a

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subset $\mathcal{IRO}^{(n)}(f)$ of it to be the set of irreducible Reidemeister orbits. If $H \subset G$ is an f-invariant normal subgroup and $\bar{G} = G/H$, then the short exact sequence $1 \to H \overset{i}{\to} G \to \bar{G} \to 1$ induces an exact sequence

$$\mathcal{RO}^{(n)}(f_H) \overset{i_*}{\to} \mathcal{RO}^{(n)}(f) \to \mathcal{RO}^{(n)}(\bar{f}) \to 1$$

of Reidemeister orbit sets. Note that i_* does not preserve irreducibility. Under certain conditions, we can have an addition formula of the form

$$\sharp\mathcal{IRO}^{(n)}(f) = \sum_{j \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{IRO}^{(m_j)}(\theta'_j),$$

where $m_j = n/\ell_j$, ℓ_j being the length of the orbit j, and $\theta'_j : H \to H$ is a twisted version of $f_H^{\ell_j}$. In Example 2.10 we give a correction of [7, Example 1.16].

Turning to the topological context, we consider a fibre preserving map $f : E \to E$ of a Hurewicz fibration $p : E \to B$ of compact ANR's. It induces a map $\bar{f} : B \to B$. Let K be the kernel of the homomorphism $j_* : \pi_1(F_b) \to \pi_1(E)$ induced by the inclusion of a fiber. Denote by $IEO^{(n)}(f)$ the number of irreducible essential orbit classes of f, and by $IEO_K^{(m)}$ the number of mod K irreducible essential orbit classes on a fibre. The following formula is a correction of [7, Corollary 2.5]. Under suitable conditions, we have an addition formula of the form

$$IEO^{(n)}(f) = \sum_{b \in \xi} IEO_K^{(m)}(h_b),$$

where the summation runs over a set ξ of essential orbit representatives for $\bar{f}$, ℓ is the length of the essential $\bar{f}$-orbit class containing b, $m = n/\ell$, and $h_b : F_b \to F_b$ is a variant of $f^\ell|F_b$.

The paper consists of two sections. In the first section we show our results on the irreducible Reidemeister orbit sets using the method of our algebraic results in [7] on the Reidemeister orbit sets. In the second section we apply them to fibre preserving maps.

For the basics of Nielsen fixed point theory, the reader is referred to [1,6].

2. Irreducible Reidemeister n-orbit sets

Let $f : G \to G$ be a group endomorphism. The Reidemeister set $\mathcal{R}(f)$ of f is the set of equivalence classes for the following Reidemeister equivalence relation in G: $\gamma, \gamma' \in G$ are equivalent if and only if $\gamma' =$

$g\gamma f(g^{-1})$ for some $g \in G$. The Reidemeister class of $\gamma \in G$ is denoted by $[\gamma]_f$.

If $H \subset G$ is an f-invariant normal subgroup, then the short exact sequence

$$1 \to H \xrightarrow{i} G \xrightarrow{q} \bar{G} \to 1,$$

where $\bar{G} = G/H$, and $i : H \to G$ and $q : G \to \bar{G}$ are the inclusion and quotient homomorphisms, induces an exact sequence (in the category of pointed sets)

$$(\mathcal{R}(f_H), [1]_{f_H}) \xrightarrow{i_*} (\mathcal{R}(f), [1]_f) \xrightarrow{q_*} (\mathcal{R}(\bar{f}), [1]_{\bar{f}}) \to 1$$

of Reidemeister sets, where $\mathcal{R}(f_H)$ is the Reidemeister set of the restriction map $f_H : H \to H$, and $\mathcal{R}(\bar{f})$ is the Reidemeister set of the induced map $\bar{f} : \bar{G} \to \bar{G}$.

Let $n > 0$ be a given integer. Then f acts on the Reidemeister set $\mathcal{R}(f^n)$ by $[\gamma]_{f^n} \overset{f}{\mapsto} [f(\gamma)]_{f^n}$. The f-orbit of a Reidemeister class $[\gamma]_{f^n}$ is called a Reidemeister n-orbit, denoted by $[\gamma]_f^{(n)}$. The set of all such Reidemeister f-orbits is called the Reidemeister n-orbit set of f, denoted by $\mathcal{RO}^{(n)}(f)$. The length of the orbit $[\gamma]_f^{(n)}$ is the smallest integer $\ell = \ell([\gamma]_f^{(n)}) > 0$ such that $[\gamma]_{f^n} = [f^\ell(\gamma)]_{f^n}$. Furthermore, there is an exact sequence (in the category of pointed sets)

$$(\mathcal{RO}^{(n)}(f_H), [1]_{f_H}^{(n)}) \xrightarrow{i_*} (\mathcal{RO}^{(n)}(f), [1]_f^{(n)}) \xrightarrow{q_*} (\mathcal{RO}^{(n)}(\bar{f}), [1]_{\bar{f}}^{(n)}) \to 1$$

of Reidemeister orbit sets, where $\mathcal{RO}^{(n)}(f_H)$ is the Reidemeister orbit set of the restriction map $f_H : H \to H$, and $\mathcal{RO}^{(n)}(\bar{f})$ is the Reidemeister orbit set of the induced map $\bar{f} : \bar{G} \to \bar{G}$. For $\ell \mid n$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(f^\ell) & \xrightarrow{\ell,n} & \mathcal{R}(f^n) \\ \downarrow & & \downarrow \\ \mathcal{RO}^{(\ell)}(f) & \xrightarrow{\ell,n} & \mathcal{RO}^{(n)}(f), \end{array}$$

where the vertical maps are projections, and the horizontal maps are induced by the level-change function $\iota_{\ell,n} : G \to G$ defined (as in [5, 1.9]) by

$$\iota_{\ell,n}(\beta) := \beta f^\ell(\beta) f^{2\ell}(\beta) \cdots f^{n-\ell}(\beta).$$

We say that an f-orbit $[\alpha]_f^{(n)} \in \mathcal{RO}^{(n)}(f)$ is reducible to level h, if there exists a $[\beta]_f^{(h)} \in \mathcal{RO}^{(h)}(f)$ such that $\iota_{h,n}([\beta]_f^{(h)}) = [\alpha]_f^{(n)}$. The lowest level $d = d([\alpha]_f^{(n)})$ to which $[\alpha]_f^{(n)}$ reduces is its depth. A Reidemeister orbit $[\alpha]_f^{(n)} \in \mathcal{RO}^{(n)}(f)$ is said to have the full depth property if its depth equals its length, i.e., $d = \ell$ (see [7]).

From the above definitions, we have

LEMMA 2.1. *Let $f : G \to G$ and $g : G' \to G'$ be group endomorphisms. Let $h : G \to G'$ be a homomorphism. If the diagram*

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ G' & \xrightarrow{g} & G' \end{array}$$

commutes, then $\ell([h(\alpha)]_g^{(n)}) \mid \ell([\alpha]_f^{(n)})$ and $d([h(\alpha)]_g^{(n)}) \leq d([\alpha]_f^{(n)})$, where ℓ, d are length and depth of corresponding ones.

COROLLARY 2.2. *If $[h(\alpha)]_g^{(n)}$ is irreducible, then $[\alpha]_f^{(n)}$ is irreducible.*

COROLLARY 2.3. *If $[h(\alpha)]_g^{(n)}$ has the full depth property, then*

$$d([h(\alpha)]_g^{(n)}) \mid d([\alpha]_f^{(n)}).$$

DEFINITION 2.4. Let $f : G \to G$ be a group endomorphism. The f-orbit of an irreducible Reidemeister class will be called an *irreducible Reidemeister n-orbit*. The set of all such irreducible Reidemeister f-orbits will be called the *irreducible Reidemeister n-orbit set* of f, denoted by $\mathcal{IRO}^{(n)}(f)$.

Note that if $n = m\ell$, then the orbit $[\alpha]_{f^\ell}^{(m)} = \{[\alpha]_{f^n}, [f^\ell(\alpha)]_{f^n}, \ldots\}$ is a subset of the orbit $[\alpha]_f^{(n)} = \{[\alpha]_{f^n}, [f(\alpha)]_{f^n}, \ldots\}$. Thus this inclusion induces a map

$$\sigma : (\mathcal{RO}^{(m)}(f^\ell), [\alpha]_{f^\ell}^{(m)}) \to (\mathcal{RO}^{(n)}(f), [\alpha]_f^{(n)}).$$

LEMMA 2.5. *Let $n > 0$ and $\alpha \in G$. Suppose the orbit $[\bar{\alpha}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$ has length ℓ, and let $m := n/\ell$. We have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccc} (\mathcal{RO}^{(m)}(f^\ell), [\alpha]_{f^\ell}^{(m)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\bar{f}^\ell), [\bar{\alpha}]_{\bar{f}^\ell}^{(m)}) & \longrightarrow & 1 \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}} & & \\ (\mathcal{RO}^{(n)}(f), [\alpha]_f^{(n)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(n)}(\bar{f}), [\bar{\alpha}]_{\bar{f}}^{(n)}) & \longrightarrow & 1, \end{array}$$

where the vertical maps σ and $\bar{\sigma}$ are induced by inclusions, and they are surjective. σ restricts to a bijection

$$\sigma : q_*^{-1}([\bar{\alpha}]_{\bar{f}^\ell}^{(m)}) \to q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)}).$$

Furthermore, if $[\bar{\alpha}]_{\bar{f}}^{(n)}$ has the full depth property, then σ restricts to a bijection

$$\sigma : \mathcal{IRO}^{(m)}(f^\ell) \cap q_*^{-1}([\bar{\alpha}]_{\bar{f}^\ell}^{(m)}) \to \mathcal{IRO}^{(n)}(f) \cap q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)}).$$

Proof. For the first part, see [7, 1.7]. Now we show that if $[\beta]_{f^\ell}^{(m)} \in \mathcal{IRO}^{(m)}(f^\ell) \cap q_*^{-1}([\bar{\alpha}]_{\bar{f}^\ell}^{(m)})$, then $\sigma([\beta]_{f^\ell}^{(m)}) = [\beta]_f^{(n)}$ is irreducible. Assume that the depth $d = d([\beta]_f^{(n)}) < n$. Since $[\bar{\alpha}]_{\bar{f}}^{(n)} = q_*([\beta]_f^{(n)})$ has the full depth property, the length ℓ of it is its depth. Thus by Corollary 2.3 $\ell \mid d$. Let $d = \ell r$. Then we have $r < m$ and $r \mid m$. By definition of depth, there exists $\gamma \in G$ such that

$$\beta = \gamma f^d(\gamma) \cdots f^{n-d}(\gamma) = \gamma (f^\ell)^r(\gamma) \cdots (f^\ell)^{m-r}(\gamma).$$

This means that $[\beta]_{f^\ell}^{(m)}$ is reducible to level r. This is a contradiction.

On the other hand, in the proof of [7,1.7] if $[\beta]_f^{(n)} \in q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)})$, then

$$\sigma^{-1}([\beta]_f^{(n)}) \cap q_*^{-1}([\bar{\alpha}]_{f^\ell}^{(m)}) = \{[\beta]_{f^\ell}^{(m)}\}.$$

It is sufficient to show that if $[\beta]_f^{(n)} \in \mathcal{IRO}^{(n)}(f) \cap q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)})$, then $[\beta]_{f^\ell}^{(m)}$ is irreducible. Assume that the depth $d = d([\beta]_{f^\ell}^{(m)}) < m$. Then there exists $\gamma \in G$ such that

$$\beta = \gamma (f^\ell)^d(\gamma) \cdots (f^\ell)^{m-d}(\gamma) = \gamma f^{\ell d}(\gamma) \cdots f^{n-\ell d}(\gamma).$$

This means that $[\beta]_f^{(n)}$ is reducible to level ℓd. □

Notation. For $\alpha \in G$, let $\tau_\alpha : G \to G$ denote the conjugation defined by $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$.

From [7, 1.10] there is a canonical bijection of the Reidemeister orbit sets of $\tau_\alpha f$ and f, denoted by $\alpha_* : \mathcal{RO}^{(n)}(\tau_\alpha f) \to \mathcal{RO}^{(n)}(f)$, given by

$$\alpha_*([g]_{\tau_\alpha f}^{(n)}) = [g\alpha f(\alpha) \cdots f^{n-1}(\alpha)]_f^{(n)}.$$

Lemma 2.6. *The canonical bijection α_* preserves irreducibility.*

Proof. For $\ell \mid n$, we will show the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{RO}^{(\ell)}(\tau_\alpha f) & \xrightarrow{\alpha_*} & \mathcal{RO}^{(\ell)}(f) \\ {\scriptstyle \iota_{\ell,n}}\downarrow & & \downarrow{\scriptstyle \iota_{\ell,n}} \\ \mathcal{RO}^{(n)}(\tau_\alpha f) & \xrightarrow{\alpha_*} & \mathcal{RO}^{(n)}(f) \end{array}$$

where the vertical maps are level-change functions. Then α_* preseves irreducibility. Clearly we have

$$(\tau_\alpha f)^n(g) = (\alpha f(\alpha)\cdots f^{n-1}(\alpha))f^n(g)(\alpha f(\alpha)\cdots f^{n-1}(\alpha))^{-1}$$

for all $g \in G$. Then

$$\begin{aligned} \alpha_*\iota_{\ell,n}([g]^{(\ell)}_{\tau_\alpha f}) &= \alpha_*([g(\tau_\alpha f)^\ell(g)\cdots(\tau_\alpha f)^{n-\ell}(g)]^{(n)}_{\tau_\alpha f}) \\ &= [g(\tau_\alpha f)^\ell(g)\cdots(\tau_\alpha f)^{n-\ell}(g)\alpha\cdots f^{n-1}(\alpha)]^{(n)}_f \\ &= [g(\alpha f(\alpha)\cdots f^{\ell-1}(\alpha))f^\ell(g)(\alpha f(\alpha)\cdots f^{\ell-1}(\alpha))^{-1}\cdots \\ &\quad (\alpha f(\alpha)\cdots f^{n-\ell-1}(\alpha))f^{n-\ell}(g)(\alpha f(\alpha)\cdots f^{n-\ell-1}(\alpha))^{-1} \\ &\quad \alpha\cdots f^{n-1}(\alpha)]^{(n)}_f \\ &= [(g\alpha f(\alpha)\cdots f^{\ell-1}(\alpha))f^\ell(g\alpha f(\alpha)\cdots f^{\ell-1}(\alpha))\cdots \\ &\quad \cdots f^{n-\ell}(g\alpha f(\alpha)\cdots f^{\ell-1}(\alpha))]^{(n)}_f \\ &= \iota_{\ell,n}([g\alpha f(\alpha)\cdots f^{\ell-1}(\alpha)]^{(\ell)}_f \\ &= \iota_{\ell,n}\alpha_*([g]^{(\ell)}_{\tau_\alpha f}). \end{aligned}$$

This is exactly what we need. □

Applying Lemma 2.5 and Lemma 2.6 to [7, Corollary 1.11], we have

Corollary 2.7. *Suppose $n > 0$ and $\alpha \in G$ are given. Suppose the orbit $[\bar\alpha]^{(n)}_{\bar f} \in \mathcal{RO}^{(n)}(\bar f)$ has depth d, and let $m := n/d$ and $\bar\iota_{d,n}(\bar\beta) = \bar\alpha$ for some $\beta \in G$. Then we have a commutative diagram in the category of pointed sets:*

$$\begin{array}{ccc} (\mathcal{RO}^{(m)}(\tau_\beta f^d), [1]^{(m)}_{\tau_\beta f^d}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\tau_{\bar\beta}\bar f^d), [1]^{(m)}_{\tau_{\bar\beta}\bar f^d}) \\ {\scriptstyle \beta_*}\downarrow & & \downarrow{\scriptstyle \bar\beta_*} \\ (\mathcal{RO}^{(m)}(f^d), [\beta_d^{(m)}]^{(m)}_{f^d}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\bar f^d), [\bar\beta_d^{(m)}]^{(m)}_{\bar f^d}) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \bar\sigma} \\ (\mathcal{RO}^{(n)}(f), [\alpha]^{(n)}_f) & \xrightarrow{q_*} & (\mathcal{RO}^{(n)}(\bar f), [\bar\alpha]^{(n)}_{\bar f}), \end{array}$$

where the vertical maps β_* and $\bar{\beta}_*$ are induced by the element $\beta \in G$. The notation $\beta_d^{(m)}$ stands for $\beta f^d(\beta)\cdots f^{n-d}(\beta)$, and similarly for $\bar{\beta}_d^{(m)}$. Furthermore, if $[\bar{\alpha}]_{\bar{f}}^{(n)}$ has the full depth property, then we have a bijection

$$\sigma \circ \beta_* : \mathcal{IRO}^{(m)}(\tau_\beta f^d) \cap q_*^{-1}([1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m)}) \longrightarrow \mathcal{IRO}^{(n)}(f) \cap q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)}),$$

where $\beta_* : \mathcal{RO}^{(m)}(\tau_\beta f^d) \to \mathcal{RO}^{(m)}(f^d)$ is a canonical bijection given by $\beta_*([g]_{\tau_\beta f^d}^{(m)}) = [g\beta f^d(\beta)\cdots f^{n-d}(\beta)]_{f^d}^{(m)}$.

Proof. By [7, Corollary 1.11] the diagram commutes and we have a bijection

$$\sigma \circ \beta_* : q_*^{-1}([1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m)}) \longrightarrow q_*^{-1}([\bar{\alpha}]_{\bar{f}}^{(n)}).$$

Since β_* and σ preserves irreducibility, we have the assersion. □

LEMMA 2.8. *Suppose $n > 0$ and $\alpha \in G$ are given. Suppose the orbit $[\bar{\alpha}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$ has depth d, and let $m := n/d$ and $\bar{\iota}_{d,n}(\bar{\beta}) = \bar{\alpha}$ for some $\beta \in G$. If $\mathrm{Fix}(\tau_{\bar{\alpha}}\bar{f}^n) = \{1\}$ and $\bar{\iota}_{m',m}$ is injective for every $m' \mid m$, then the restriction*

$$i_* : \mathcal{IRO}^{(m)}(\tau_\beta f_H^d) \longrightarrow \mathcal{IRO}^{(m)}(\tau_\beta f^d) \cap q_*^{-1}([1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m)})$$

is a bijection.

Proof. When $\mathrm{Fix}((\tau_{\bar{\beta}}\bar{f}^d)^m) = \mathrm{Fix}(\tau_{\bar{\alpha}}\bar{f}^n) = \{1\}$, by [7, 1.6] the map

$$i_* : \mathcal{RO}^{(m)}(\tau_\beta f_H^d) \to q_*^{-1}([1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m)})$$

is a bijection. First we will show that if $[\gamma]_{\tau_\beta f_H^d}^{(m)} \in \mathcal{IRO}^{(m)}(\tau_\beta f_H^d)$, then $i_*([\gamma]_{\tau_\beta f_H^d}^{(m)}) = [\gamma]_{\tau_\beta f^d}^{(m)} \in \mathcal{IRO}^{(m)}(\tau_\beta f^d)$. Assume the depth $m' = d([\gamma]_{\tau_\beta f^d}^{(m)}) < m$ and $\iota_{m',m}(\delta) = \gamma$ for some $\delta \in \mathcal{RO}^{(m')}(\tau_\beta f^d)$. Then we have a commutative diagram of exact sequences of pointed sets:

$$\begin{array}{ccccc} & & (\mathcal{RO}^{(m')}(\tau_\beta f^d), [\delta]_{\tau_\beta f^d}^{(m')}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m')}(\tau_{\bar{\beta}}\bar{f}^d), [\bar{\delta}]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m')}) \\ & & \iota_{m',m}\downarrow & & \downarrow \bar{\iota}_{m',m} \\ (\mathcal{RO}^{(m)}(\tau_\beta f_H^d), [\gamma]_{\tau_\beta f_H^d}^{(m)}) & \xrightarrow{i_*} & (\mathcal{RO}^{(m)}(\tau_\beta f^d), [\gamma]_{\tau_\beta f^d}^{(m)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\tau_{\bar{\beta}}\bar{f}^d), [1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m)}) \end{array}$$

Since $\bar{\iota}_{m',m}$ is injective, we have $[\bar{\delta}]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m')} = [1]_{\tau_{\bar{\beta}}\bar{f}^d}^{(m')}$, and so there exists $[\delta']_{\tau_\beta f_H^d}^{(m')} \in \mathcal{RO}^{(m')}(\tau_\beta f_H^d)$ such that $i_*([\delta']_{\tau_\beta f_H^d}^{(m')}) = [\delta]_{\tau_\beta f^d}^{(m')}$. Since i_* is injective in the lower sequence, we have $\iota_{m',m}([\delta']^{(m')}_{\tau_\beta f^d_H} = [\gamma]^{(m)}_{\tau_\beta f^d_H}$. This is a contradiction.

On the other hand, Corollary 2.2 tells us

$$i_*^{-1}(\mathcal{IRO}^{(m)}(\tau_\beta f^d)\cap q_*^{-1}([1]^{(m)}_{\tau_{\bar\beta}\bar f^d}))\subset \mathcal{IRO}^{(m)}(\tau_\beta f^d_H).$$

□

THEOREM 2.9. *Suppose $n>0$ is given. For an orbit $[\bar\alpha]^{(n)}_{\bar f}\in\mathcal{RO}^{(n)}(\bar f)$, let d_α be the depth of $[\bar\alpha]^{(n)}_{\bar f}$, $m_\alpha := n/d_\alpha$, and $\bar\iota_{d_\alpha,n}(\bar\beta_\alpha)=\bar\alpha$ for some $\beta_\alpha\in G$. If $\mathrm{Fix}(\tau_{\bar\alpha}\bar f^n)=\{1\}$ for all $[\bar\alpha]^{(n)}_{\bar f}\in\mathcal{RO}^{(n)}(\bar f)$, and $\bar\iota_{m,m_\alpha}$ is injective for every m, m_α with $m\mid m_\alpha$, then*

$$\sharp\mathcal{IRO}^{(n)}(f)=\sum_{[\bar\alpha]^{(n)}_{\bar f}\in\mathcal{RO}^{(n)}(\bar f)}\sharp\mathcal{IRO}^{(m_\alpha)}(\tau_{\beta_\alpha}f_H^{d_\alpha}).$$

Proof. Clearly the irreducible Reidemeister n-orbit set $\mathcal{IRO}^{(n)}(f)$ is the disjoint union of $\mathcal{IRO}^{(n)}(f)\cap q_*^{-1}([\bar\alpha]^{(n)}_{\bar f})$ for all $[\bar\alpha]^{(n)}_{\bar f}\in\mathcal{RO}^{(n)}(\bar f)$. When $\mathrm{Fix}(\tau_{\bar\alpha}\bar f^n)=\{1\}$, [7, 1.12] tells us $[\bar\alpha]^{(n)}_{\bar f}$ has the full depth property. By Corollary 2.7 and Lemma 2.8

$$i_*:\mathcal{IRO}^{(m_\alpha)}(\tau_{\beta_\alpha}f_H^{d_\alpha})\to\mathcal{IRO}^{(n)}(f)\cap q_*^{-1}([\bar\alpha]^{(n)}_{\bar f})$$

is a bijection for every $[\bar\alpha]^{(n)}_{\bar f}\in\mathcal{RO}^{(n)}(\bar f)$. This completes the proof of the theorem. □

In the next eample we have a correction of [7, Example 1.16].

EXAMPLE 2.10. (The Klein bottle). Let G be the fundamental group of the Klein bottle, i.e. $G:=\langle\alpha,\beta\mid\beta\alpha=\alpha^{-1}\beta\rangle$. The subgroup $H:=\langle\alpha\rangle$ is a fully invariant normal subgroup of G and if $M:\mathbb{Z}\to\mathrm{Aut}(\mathbb{Z})$ is the homomorphism defined by $M_k=M(k)=(-1)^k$ for all $k\in\mathbb{Z}$ then G is the external semidirect product of H and $A:=\mathbb{Z}$ via M, it is the set of all pairs $(a,h)\in A\times H$, with the group operation $(a_1,h_1)*(a_2,h_2)=(a_1+a_2,M_{a_2}(h_1)+h_2)$. The subgroup $H\cong\{0\}\times H\subset G$ is normal in G and $G/H\cong A\cong A\times\{0\}\subset G$. Let $f:G\to G$ be an endomorphism. Then $f_H:H\to H$ and $\bar f:A\to A$ are defined by elements of $\mathrm{Mat}_{1,1}(\mathbb{Z})$, thus they are integers u and w. In other words, $f_H(h)=uh$ for $h\in H$ and $\bar f(a)=wa$ for $a\in A$ (see [2, Example 3 and 4]).

We will calculate $\mathcal{IRO}^{(2)}(f)$ when $u=2$ and $w=3$. Since the following sequence

$$0 \to \mathbb{Z} \xrightarrow{1-\bar{f}^2} \mathbb{Z} \to \mathbb{Z}/\mathrm{Im}(1-\bar{f}^2) \to 0$$

is exact, we can identify the 2-periodic point classes of $\bar{f}$, i.e. the elements of $\mathcal{R}(\bar{f}^2)$, with the elements of $\mathbb{Z}_8$. Thus the Reidemeister 2-orbit set of $\bar{f}$ is

$$\mathcal{RO}^{(2)}(\bar{f}) = \{[0]^{(2)}_{\bar{f}}, [1]^{(2)}_{\bar{f}}, [2]^{(2)}_{\bar{f}}, [4]^{(2)}_{\bar{f}}, [5]^{(2)}_{\bar{f}}\},$$

where $[0]^{(2)}_{\bar{f}} = \{[0]_{\bar{f}^2}\}$, $[1]^{(2)}_{\bar{f}} = \{[1]_{\bar{f}^2}, [3]_{\bar{f}^2}\}$, $[2]^{(2)}_{\bar{f}} = \{[2]_{\bar{f}^2}, [6]_{\bar{f}^2}\}$, $[4]^{(2)}_{\bar{f}} = \{[4]_{\bar{f}^2}\}$ and $[5]^{(2)}_{\bar{f}} = \{[5]_{\bar{f}^2}, [7]_{\bar{f}^2}\}$. For level 1, the set $\mathcal{RO}^{(1)}(\bar{f}) = \mathcal{R}(\bar{f}) \cong \mathbb{Z}_2$ is $\{[0]_{\bar{f}}, [1]_{\bar{f}}\}$. Since $\bar{\iota}_{1,2}$ is multiplication by $(1+3) = 4$, the $\bar{f}$-orbits $[0]^{(2)}_{\bar{f}}$ and $[4]^{(2)}_{\bar{f}}$ are reducible to level 1, i.e. $\bar{\iota}_{1,2}(0) = 0$ and $\bar{\iota}_{1,2}(1) = 4$, the others are irreducible. For all $a \in A$ and $h \in H$ the conjugation is $(a,0) * (0,h) * (-a,0) = (0, M_{-a}h)$, hence $\tau_a f_H(h) = (-1)^{-a}uh = (-1)^a uh$. Since $\mathrm{Fix}(\tau_j \bar{f}^2) = \{0\}$ for all $[j]^{(2)}_f \in \mathcal{RO}^{(2)}(\bar{f})$, by [7, Theorem 1.14] we have

$$\begin{aligned}
\sharp\mathcal{RO}^{(2)}(f) &= \sharp\mathcal{RO}^{(2)}(\tau_0 f_H) + \sharp\mathcal{RO}^{(2)}(\tau_1 f_H) + \sum_{j=1,2,5} \sharp\mathcal{R}(\tau_j f_H^2) \\
&= \sharp\{[0]^{(2)}_{f_H}, [1]^{(2)}_{f_H}\} + \sharp\{[0]^{(2)}_{\tau_1 f_H}, [1]^{(2)}_{\tau_1 f_H}, [2]^{(2)}_{\tau_1 f_H}\} + \sum_{j=1,2,5} \mid 1 - (-1)^j 2^2 \mid \\
&= 18.
\end{aligned}$$

For level 1, the set $\mathcal{RO}^{(1)}(f_H) = \mathcal{R}(f_H) \cong \mathbb{Z}_1$ is $\{[0]_{f_H}\}$, and so $[1]^{(2)}_{f_H}$ is irreducible. The set $\mathcal{RO}^{(1)}(\tau_1 f_H)$ is $\{[0]_{\tau_1 f_H}, [1]_{\tau_1 f_H}, [2]_{\tau_1 f_H}\}$, so all Reidemeister 2-orbits of $\tau_1 f_H$ are reducible. Thus by Theorem 2.9 we have

$$\begin{aligned}
\sharp\mathcal{IRO}^{(2)}(f) &= \sharp\mathcal{IRO}^{(2)}(\tau_0 f_H) + \sharp\mathcal{IRO}^{(2)}(\tau_1 f_H) + \sum_{j=1,2,5} \sharp\mathcal{R}(\tau_j f_H^2) \\
&= 1 + 0 + \sum_{j=1,2,5} \mid 1 - (-1)^j 2^2 \mid \\
&= 14.
\end{aligned}$$

3. Nielsen type irreducible essential n-orbit numbers.

Let X be a compact connected ANR. Let $f : X, \to X$ be a map. We denote by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of f. The set of fixed point classes is denoted by $\mathcal{FP}(f)$.

Let $n > 0$ be a given integer. Then f acts on the set $\mathcal{FP}(f^n)$ of n-periodic point classes of f by $\mathbf{F}_{f^n} \mapsto f(\mathbf{F}_{f^n})$. The f-orbit of a class $\mathbf{F}_{f^n}$ is called an n-orbit class, denoted by $\mathbf{F}_f^{(n)}$. The set of n-orbit classes is denoted by $\mathcal{O}^{(n)}(f)$. Let $\mathcal{E}(f^n)$ be the set of essential periodic point classes of f. We denoted the set of essential n-orbit classes by $\mathcal{EO}^{(n)}(f)$. The essential n-orbit number $EO^{(n)}(f)$ is the the cardinality of it (see [7]).

Let x be the base point in X, and take a path w from x to $f(x)$ as the base path for f. The induced endomorphism $f_*^w : \pi_1(X, x) \to \pi_1(X, x)$ is defined by $f_*^w(\langle\gamma\rangle) = \langle wf(\gamma)w^{-1}\rangle$ for any loop γ at x. If w is the constant path, f_*^w will be denoted by f_*^x. It is well known that every fixed point class of f is assigned a Reidemeister class in $\mathcal{R}(f_*^x)$, called its coordinate. We get an injection $\rho : \mathcal{FP}(f) \hookrightarrow \mathcal{R}(f_*^x)$, where $\mathcal{R}(f_*^x)$ is the Reidemeister set in $\pi_1(X, x)$, defined by $\rho(\mathbf{A}_f) := [cf(c^{-1})w^{-1}]_{f_*^x}$ for any path c from x to a point x' in $\mathbf{A}_f$. Thus we also get an injection $\rho : \mathcal{O}^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(f_*^x)$, defined by

$$\rho(\mathbf{A}_f^{(n)}) := [cf^n(c^{-1})f^{n-1}(w^{-1}) \cdots f(w^{-1})w^{-1}]_{f_*^x}^{(n)}$$

for any path c from x to a point x' in $\mathbf{A}_f^{(n)}$.

We define the irreducible essential n-orbit number $IEO^{(n)}(f)$ to be the cardinality of the set $\mathcal{IEO}^{(n)}(f)$ of irreducible essential n-orbit classes. This number is a homotopy invariance (see [J, III.3.3, 3.4]). It is a Nielsen type number in the general sence of [J, III.4.8].

We will need the mod K version of the Nielsen theory. If f, X and x are as above, and if K is an f_*^x-invariant normal subgroup of $\pi_1(X, x)$, then we denote the induced homomorphism on $\pi_1(X, x)/K$ by $f_{*/K}^x$. We then have the set $\mathcal{RO}^{(n)}(f_{*/K}^x)$ of Reidemeister $f_{*/K}^x$-orbits, and the mod K essential n-orbit number $EO_K^{(n)}(f)$, that is the cardinality of the set $\mathcal{EO}_K^{(n)}(f)$ of mod K essential n-orbit classes. We also have an injection ρ_K from the set of mod K n-orbit classes to the set of Reidemeister $f_{*/K}^x$-orbits, i.e.,

$$\rho_K : \mathcal{O}_K^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(f_{*/K}^x).$$

The mod K irreducible essential n-orbit number $IEO_K^{(n)}(f)$ is defined by the cardinality of the set $\mathcal{IEO}_K^{(n)}(f)$ of mod K irreducible essential n-orbit classes.

In this paper we will assume that all of our fibrations $F \hookrightarrow E \to B$ (with projection $p : E \to B$) are Hurewicz fibrations with typical fibre,

E and B path-connected (see [8]). For any $b \in \mathrm{Fix}(\bar{f}^n)$, we will denote the restricted map on the fibre $F_b := p^{-1}(b)$ by f_b^n. For $x \in E$ let $j : F_{p(x)} \to E$ be the inclusion.

PROPOSITION 3.1. [7, 2.4] *Let $p : E \to B$ be a fibration of compact connected ANR's with path-connected fibres, and let $f : E \to E$ be a fibre preserving map. If $x \in E$ is in an essential n-orbit class $\mathbf{F}_f^{(n)}$ of f, and $p(x)$ is a fixed point of $\bar{f}^\ell$, where $\ell \mid n$, then the sequence (with $m := n/\ell$)*

$$(\mathcal{EO}_K^{(m)}(f_{p(x)}^\ell), {}_K\mathbf{F}_{f_{p(x)}^\ell}^{(m)}) \xrightarrow{j_{\mathcal{E}}} (\mathcal{EO}^{(m)}(f^\ell), \mathbf{F}_{f^\ell}^{(m)}) \xrightarrow{p_{\mathcal{E}}} (\mathcal{EO}^{(m)}(\bar{f}^\ell), \mathbf{F}_{\bar{f}^\ell}^{(m)})$$

is an exact sequence of based sets, where $j_{\mathcal{E}}$ and $p_{\mathcal{E}}$ are induced by the inclusion $j : F_{p(x)} \to E$ and the projection $p : E \to B$ respectively, and the base points are the essential orbit classes containing either x or $p(x)$.

We call a subset $\xi \subset \mathrm{Fix}(\bar{f}^n)$ a set of essential n-orbit representatives for $\bar{f}$ if ξ contains exactly one point from each essential n-orbit class $\mathbf{F}_{\bar{f}}^{(n)} \in \mathcal{EO}^{(n)}(\bar{f})$.

THEOREM 3.2. *Suppose $p : E \to B$ is a fibration of compact connected ANR's with path-connected fibres, and $f : E \to E$ is a fibre preserving map. Let $\xi = \{b_1, b_2, \ldots, b_k\}$ be a set of essential n-orbit representatives for $\bar{f}$. Let d_i be the depth of the n-orbit class of $\bar{f}$ containing b_i, and $m_i = n/d_i$ for all i. If $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$ for every $b_i \in \xi$, and $\bar{\iota}_{m,m_i}$ is injective for every m, m_i with $m \mid m_i$, then we have*

$$IEO^{(n)}(f) = \sum_{b_i \in \xi} IEO_K^{(m_i)}(g_{b_i}^{d_i}),$$

where g is the fibre preserving map from the Reducing Lemma, K is the kernel of the homomorphism $j_* : \pi_1(F_{b_i}) \to \pi_1(E)$ induced by the inclusion of the fibre. In the condition $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$, the endomorphism $(\bar{f}^n)_*^{b_i} : \pi_1(B, b_i) \to \pi_1(B, b_i)$ is meant to have the constant path at b_i as base path.

Proof. By Reducing Lemma [7, 2.2] there exists a homotopy $\bar{H}$ connecting $\bar{f}$ and $\bar{g}$ such that $b_i \in \mathrm{Fix}(\bar{g}^{d_i})$ for every $b_i \in \xi$. By the homotopy lifting property of the fibration p, the homotopy $\bar{H}$ in B lifts to a fibre preserving homotopy $H = \{h_t : E \to E\}_{t \in I}$ connecting $f = h_0$ to some $g = h_1$. By homotopy invariance we have $EO^{(n)}(f) = EO^{(n)}(g)$. So without loss of generality (by rewriting g as f) we may assume that $b_i \in \mathrm{Fix}(\bar{f}^{d_i})$ and g is the same as f.

For each $b_i \in \xi$, let $\mathbf{F}^{(n)}_{\bar{f},i}$ be the essential n-orbit class containing it. Clearly $\mathcal{IEO}^{(n)}(f) = \bigcup_i \mathcal{IEO}^{(n)}(f) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar{f},i})$. So we only need to show

$$|\mathcal{IEO}^{(n)}(f) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar{f},i})| = IEO^{(m_i)}_K(f^{d_i}_{b_i}).$$

In the following proof we shall drop the subscript i from our notation.

Suppose $b \in \xi$ is in the essential n-periodic point class $\mathbf{F}_{\bar{f}^n}$ which in turn is in the essential n-orbit class $\mathbf{F}^{(n)}_{\bar{f}}$. Since d is the depth of $\mathbf{F}^{(n)}_{\bar{f}}$, $\mathbf{F}_{\bar{f}^n}$ alone constitutes an essential m-orbit class $\mathbf{F}^{(m)}_{\bar{f}^d} \subset \mathbf{F}^{(n)}_{\bar{f}}$. Suppose $\mathcal{IEO}^{(m)}(f^d) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(m)}_{\bar{f}^d}) \neq \emptyset$. Choose $\mathbf{F}^{(m)}_{f^d} \in \mathcal{IEO}^{(m)}(f^d) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(m)}_{\bar{f}^d})$ and $x \in \mathbf{F}^{(m)}_{f^d}$ such that $p(x) = b$. Then we have an exact sequence

$$1 \to \pi_1(F_b, x)/K \xrightarrow{j_*} \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, b) \to 1.$$

Let ${}_K\mathbf{F}^{(m)}_{f^d_b} \in \mathcal{EO}^{(m)}_K(f^d_b)$ be the mod K essential orbit class containing x. Then by Proposition 3.1 we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccc}
(\mathcal{EO}^{(m)}_K(f^d_b), {}_K\mathbf{F}) & \xrightarrow{j_{\mathcal{E}}} & (\mathcal{EO}^{(m)}(f^d), \mathbf{F}^{(m)}_{f^d}) & \xrightarrow{p_{\mathcal{E}}} & (\mathcal{EO}^{(m)}(\bar{f}^d), \mathbf{F}^{(m)}_{\bar{f}^d}) \\
{\scriptstyle \rho_K}\big\downarrow & & {\scriptstyle \rho}\big\downarrow & & {\scriptstyle \bar{\rho}}\big\downarrow \\
(\mathcal{RO}^{(m)}((f^d_b)^x_{*/K}), \rho({}_K\mathbf{F})) & \xrightarrow{j^x_*} & (\mathcal{RO}^{(m)}((f^d)^x_*), \rho(\mathbf{F}^{(m)}_{f^d})) & \xrightarrow{p_*} & (\mathcal{RO}^{(m)}((\bar{f}^d)^b_*), [1]).
\end{array}$$

The notation ${}_K\mathbf{F}$ stands for ${}_K\mathbf{F}^{(m)}_{f^d_b}$ and $[1]$ stands for $[1]^{(m)}_{(\bar{f}^d)^b_*}$. Note that here we regard f^d as a self map of the pair (E, F_b). The base path is taken to be a path in F_b from x to $f^d(x)$, whose image in B is the constant path at b. Hence the coordinate of $\mathbf{F}^{(m)}_{\bar{f}^d}$ is the $[1]^{(m)}_{(\bar{f}^d)^b_*}$ in the lower right corner.

By the above commutative diagram and Corollary 2.2, ${}_K\mathbf{F}^{(m)}_{f^d_b}$ is irreducible. When $\mathrm{Fix}((\bar{f}^d)^{bm}_*) = \mathrm{Fix}((\bar{f}^n)^b_*) = \{1\}$, Lemma 2.8 tells us

$$j^x_* : \mathcal{IRO}^{(m)}((f^d_b)^x_{*/K}) \to \mathcal{IRO}^{(m)}((f^d)^x_*) \cap p_*^{-1}([1]^{(m)}_{(\bar{f}^d)^b_*})$$

is bijective. Thus we have a bijection

$$j_{\mathcal{E}} : \mathcal{IEO}^{(m)}_K(f^d_b) \to \mathcal{IEO}^{(m)}(f^d) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(m)}_{\bar{f}^d}).$$

Since p_* and σ preserve essentiality, we have a commutative diagram in the category of pointed sets:

$$\begin{array}{ccc} (\mathcal{EO}^{(m)}(f^d), \mathbf{F}^{(m)}_{f^d}) & \xrightarrow{p_{\mathcal{E}}} & (\mathcal{EO}^{(m)}(\bar f^d), \mathbf{F}^{(m)}_{\bar f^d}) \\ \sigma \Big\downarrow & & \Big\downarrow \bar\sigma \\ (\mathcal{EO}^{(n)}(f), \mathbf{F}^{(n)}_{f}) & \xrightarrow{p_{\mathcal{E}}} & (\mathcal{EO}^{(n)}(\bar f), \mathbf{F}^{(n)}_{\bar f}). \end{array}$$

Let w be the base path for f from x to $f(x)$ in E, and $p(w) = \bar w$ is the base path for $\bar f$ from b to $\bar f(b)$ in B. Then $\bar w_n = \bar w \bar f(\bar w) \cdots \bar f^{n-1}(\bar w)$ is the base path for $\bar f^n$ from b to $\bar f^n(b)$ in B, and so $[\langle \bar w_n^{-1} \rangle]^{(n)}_{\bar f^{\bar w}_*}$ is the coordinate of $\mathbf{F}^{(n)}_{\bar f}$. When $\mathrm{Fix}(\tau_{\langle \bar w_n^{-1}\rangle}(\bar f^{\bar w}_*)^n) = \mathrm{Fix}((\bar f^n)^b_*) = \{1\}$, by [7, 1.12] the orbit $\bar\rho(\mathbf{F}^{(n)}_{\bar f})$ has the full depth property. Thus Lemma 2.5 tells us

$$\sigma : \mathcal{IEO}^{(m)}(f^d) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(m)}_{\bar f^d}) \to \mathcal{IEO}^{(n)}(f) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar f})$$

is bijective. We get the desired equality $|\mathcal{IEO}^{(n)}(f) \cap p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar f})| = IEO^{(m)}_K(f^d_b)$. □

Note that $IEO^{(n)}(f) = (1/n)NP_n(f)$ and $IEO^{(n)}_K(f) = (1/n)NP_{n,K}(f)$ (as defined in [6, III]). The following consequence is comparable to [3, 3.4].

Corollary 3.3. *Under the conditions of Theorem 3.2, we also have*

$$NP_n(f) = \sum_{b_i \in \xi} d_i \cdot NP_{m_i,K}(g^{d_i}_{b_i}).$$

The principal application is to fibrations over tori. It should be useful in calculations on nil and solvemanifolds.

Corollary 3.4. *Suppose $p : E \to B$ is a fibration over a torus (of any dimension). Then for any fibre preserving map $f : E \to E$, the summation formulae of Theorem 3.2 and Corollary 3.3 hold true:*

$$IEO^{(n)}(f) = \sum_{b_i \in \xi} IEO^{(m_i)}_K(f^{d_i}_{b_i}) \quad \text{and} \quad NP_n(f) = \sum_{b_i \in \xi} d_i \cdot NP_{m_i,K}(f^{d_i}_{b_i}).$$

Example 3.5. (The Klein bottle). Represent the Klein bottle K^2 as the quotient $\mathbb{R}^2/G$, where G is the group of automorphisms on $\mathbb{R}^2$ generated by $A : (x, y) \mapsto (x, y + 1)$ and $B : (x, y) \mapsto (x + 1, -y)$. By defining $p : \mathbb{R}^2/G \to S^1$ to be projection on the first factor we get the standard fibration $S^1 \hookrightarrow K^2 \xrightarrow{p} S^1$ of the Klein bottle. Given the map $(s, t) \mapsto (rs, qt)$ on $\mathbb{R}^2$ induces a well-defined fibre preserving map f of

K^2. For $r \neq 1$, since the degree of $\bar{f}$ is r, then $\bar{f}$ has $|r-1|$ fixed point $x_j \in \mathrm{Fix}(\bar{f})$, where $x_j = j/|r-1|$ with $j \in \{0, 1, \ldots, |r-1|-1\}$. The fibre maps f_{x_j} have degree $(-1)^j q$ (see [4, 4.6] and [3, 4.1]).

When $r = 3$ and $q = -1$, we will calculate $IEO^{(3)}(f)$. Note that $\mathrm{Fix}(\bar{f}^3) = \{x_j \mid j = 0, 1, \ldots, 25\}$, where $x_j = j/26$. Then $\xi = \{x_j \mid j = 0, 1, 2, 4, 5, 7, 8, 13, 14, 17\}$ is a set of essential 3-orbit representatives for $\bar{f}$. Since $\bar{\iota}_{1,3}$ is multiplication by $1 + 3^1 + 3^2 = 13$, by Corollary 3.4 we have

$$IEO^{(3)}(f) = \sum_{j=0,13} IEO^{(3)}(f_{x_j}) + \sum_{j \neq 0,13} N(f^3_{x_j}).$$

When $x_j \in \xi$, the self map $f^3_{x_j}$ has degree $(-1)^j(-1)^3$, and so $N(f^3_{x_j}) = 1 + (-1)^j$. Thus we have $IEO^{(3)}(f) = 8$. Also, we have $NP_3(f) = 3 \cdot IEO^{(3)}(f) = 24$.

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