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DIRICHLET BOUNDARY VALUE PROBLEM FOR A CLASS OF THE ELLIPTIC SYSTEM

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Abstract. We get a theorem which shows the existence of at least three solutions for some elliptic system with Dirichlet boundary condition. We obtain this result by using the finite dimensional reduction method which reduces the infinite dimensional problem to the finite dimensional one. We also use the critical point theory on the reduced finite dimensioal subspace.

1. Introduction

Let Ω be a bounded subset of R^n with smooth boundary $\partial\Omega$, $n \ge 3$. Let $\lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots$ be the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ_k be the eigenfunction belonging to the eigenvalue λ_k, $k \ge 1$. Let $F : R^n \times R^n \to R$ be a C^2 function such that $F(x, \theta) = 0$, $\theta = (0, \cdots, 0)$. In this paper we are concerned with the multiple solutions for a class of the systems of the elliptic equations with Dirichlet boundary condition

$$
\begin{aligned}
-\Delta u_1 &= F_{u_1}(x, u_1, \cdots, u_n) \qquad \text{in } \Omega, \\
-\Delta u_2 &= F_{u_2}(x, u_1, \cdots, u_n) \qquad \text{in } \Omega, \\
&\vdots \qquad \vdots \qquad \vdots \\
-\Delta u_n &= F_{u_n}(x, u_1, \cdots, u_n) \qquad \text{in } \Omega, \\
u_i(x) &= 0, \qquad i = 1, \cdots, n, \qquad \text{on } \partial\Omega,
\end{aligned} \tag{1.1}
$$

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where $u_i(x) \in W_0^{1,2}(\Omega)$ and $F_{u_i}(x, u_1, \cdots, u_n) = \frac{\partial F(x,u_1,\cdots,u_n)}{\partial u_i}$, $i = 1, \cdots, n$. Let $U = (u_1, \cdots, u_n)$ and $\|\cdot\|_{R^n}$ denote the Euclidean norm in R^n. Let us denote $F_U(x, U) = \text{grad}_U F(x, U) = (F_{u_1}(x, u_1, \cdots, u_n), \cdots, F_{u_n}(x, u_1, \cdots, u_n))$.

We assume that F satisfies the following conditions:

$(F1)$ $F \in C^2(R^n \times R^n, R)$, $F(x, \theta) = 0$, $F_U(x, \theta) = \theta$, $x \in \Omega$, $\theta = (0, \cdots, 0)$,

$(F2)$ There exist constants α and β (α, β are not eigenvalues of the elliptic eigenvalue problem) such that $\alpha < \beta$ and

$$\alpha I \le d_U^2 F(x, U) \le \beta I \qquad \forall (x, U) \in R^n \times R^n$$

and there exists $k \in N^*$ such that $\alpha I < \lambda_k I < d_U^2 F(x, U) < \lambda_{k+1} I < \beta I$ for every U.

$(F3)$ There exist eigenvalues $\lambda_{h+1}, \cdots, \lambda_{h+m}$ such that

$$\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{h+m} < \beta < \lambda_{h+m+1},$$

where $h \ge 1$, $m \ge 1$.

$(F4)$ There exist γ and C such that $\lambda_{h+m} < \gamma < \beta$ and

$$F(x, U) \ge \frac{1}{2}\gamma \|U\|_{R^n}^2 - C, \qquad \forall (x, U) \in R^n \times R^n.$$

Some papers of Lee [4, 6, 7, 8] concerning the semilinear elliptic system and some papers of the other several authors [3, 5] have treated the system of this kind nonlinear elliptic equations. Some papers of Chang [1] and Choi and Jung [2] considered the existence and the multiplicity of the weak solutions for the nonlinear boundary value problems with asymptotically linear term. The authors obtained some results for those problems by approaching the variational method, the critical point theory and the topological method.

Let E be a cartesian product of the Sobolev spaces $W_0^{1,2}(\Omega, R)$, i.e., $E = W_0^{1,2}(\Omega, R) \times \cdots \times W_0^{1,2}(\Omega, R)$. We endow the Hilbert space E with the norm

$$\|U\|^2 = \sum_{i=1}^{n} \|u_i\|^2,$$

where $\|u_i\|^2 = \int_\Omega |\nabla u_i(x)|^2 dx$.

The system (1.1) can be rewritten by

$$-\Delta U = \text{grad}_U F(x, U), \qquad \text{in } \Omega, \tag{1.2}$$

$$U = \theta \qquad \text{on } \partial\Omega,$$

where $U = (u_1, \cdots, u_n)$ and $\theta = (0, \cdots, 0)$. In this paper we are looking for the weak solutions of system (1.1) in E, that is, $U = (u_1, \cdots, u_n) \in E$ such that

$$\int_\Omega [-\Delta U \cdot V]dx - \int_\Omega F_U(x, U) \cdot V = 0, \qquad \text{for all } \ V \in E.$$

Our main result is the following:

THEOREM 1.1. *Assume that F satisfies the conditions $(F1) - (F4)$. Then system (1.1) has at least three nontrivial weak solutions.*

The proof of Theorem 1.1 is organized as follows: We approach the variational method and use the finite dimensional reduction method which reduce the infinite dimensional problem to the finite dimensional one. We also use the critical point theory on the reduced finite dimensional subspace. In section 2, we approach the variational method and the reduction method. We show that the reduced functional satisfies the $(P.S.)_c$ condition for any real number $c \in R$. In section 3, we show that the graph of the reduced functional has at least three nontrivial critical points, and prove Theorem 1.1.

2. Reduction approach

We assume that $F \in C^2(R^n \times R^n, R)$, $F(x, \theta) = 0$, $F_U(x, \theta) = \theta$, $\theta = (0, \cdots, 0)$ and there exist constants α and β (α, β are not eigenvalues of the elliptic eigenvalue problem) such that $\alpha < \beta$ and

$$\alpha I \leq d_U^2 F(x, U) \leq \beta I \qquad \forall (x, U) \in R^n \times R^n$$

and there exists $k \in N^*$ such that $\alpha I < \lambda_k I < d_U^2 F(x, U) < \lambda_{k+1} I < \beta I$ for every U, where $U = (u_1, \cdots, u_n)$ and there exist eigenvalues λ_{h+1}, $\cdots$, λ_{h+m} such that

$$\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{h+m} < \beta < \lambda_{h+m+1},$$

where $h \geq 1$, $m \geq 1$.

LEMMA 2.1. *Let $F_{u_i}(x, U) \in L^2(\Omega)$. Then all the solutions of*

$$-\Delta U = \mathrm{grad}_U F(x, U)$$

belong to E.

Proof. Let $F_{u_i}(x,U) \in L^2(\Omega)$, $U=(u_1,\cdots,u_n)$. We note that $\{\lambda_n : |\lambda_n| < |c|\}$ is finite. Then $F_{u_i}(x,u_1,\cdots,u_n) \in L^2(\Omega)$, $i=1,\cdots,n$, can be expressed by

$$F_{u_i}(x,u_1,\cdots,u_n) = \sum_{k=1}^{\infty} h_k\phi_k, \qquad \sum_{k=1}^{\infty} h_k^2 < \infty, \qquad \text{for each } i=1,\cdots,n.$$

Then

$$(-\Delta)^{-1}F_{u_i}(x,u_1,\cdots,u_n) = \sum \frac{1}{\lambda_k} h_k\phi_k.$$

Hence we have the inequality

$$\|(-\Delta)^{-1}F_{u_i}(x,u_1,\cdots,u_n)\|^2 = \sum \lambda_n^2 \frac{1}{\lambda_k^2} h_k^2 \le \sum h_k^2,$$

which means that

$$\|(-\Delta)^{-1}F_{u_i}(x,u_1,\cdots,u_n)\| \le \|\mathrm{grad}_{u_i} F(x,u_1,\cdots,u_n)\|_{L^2(\Omega)}.$$

□

By the following Lemma 2.2, the weak solutions of system (1.1) coincide with the critical points of the associated functional I

$$I \in C^{1,1}(E,R), \tag{2.1}$$
$$I(U) = \int_\Omega [\frac{1}{2}|\nabla U|^2 - F(x,U)]dx,$$

where $U=(u_1,\cdots,u_n)$ and $\int_\Omega \|\nabla U\|_{R^n}^2 dx = \sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dx$, $n \ge 1$.

LEMMA 2.2. *Assume that H satisfies the conditions $(F1)$-$(F4)$. Then the functional $I(U)$ is continuous, Fréchet differentiable with Fréchet derivative*

$$DI(U)\cdot V = \int_\Omega [(-\Delta U)\cdot V - F_U(x,U)\cdot V]dx.$$

Moreover $DI \in C$. That is $I \in C^1$.

Proof. First we shall prove that $I(U)$ is continuous. For $U, V \in E$,

$$\begin{aligned} |I(U+V) - I(U)| &= \Big|\frac{1}{2}\int_\Omega (-\Delta U - \Delta V)\cdot(U+V)dx - \int_\Omega F(x,U+V)dx \\ &\quad -\frac{1}{2}\int_\Omega (-\Delta U)\cdot U dx + \int_\Omega F(x,U)dx\Big| \\ &= \Big|\frac{1}{2}\int_\Omega [(-\Delta U\cdot V - \Delta V\cdot U - \Delta V\cdot V)dx \\ &\quad - \int_\Omega (F(x,U+V) - F(x,U))dx\Big|. \end{aligned}$$

We have

$$\begin{aligned}(2.2)\qquad &\Big|\int_\Omega [F(x,U+V)-F(x,U)]dx\Big| \\ &\le \Big|\int_\Omega [F_U(x,U)\cdot V + O(\|V\|_{R^n})]dx\Big| = O(\|V\|_{R^n}).\end{aligned}$$

Thus we have

$$|I(U+V)-I(U)| = O(\|V\|_{R^n}). \tag{2.3}$$

$$|I(U+V)-I(U)-DI(U)\cdot V| = O(\|V\|^2_{R^n}). \tag{2.4}$$

Next we shall prove that $I(U)$ is *Fréchet* differentiable. For $U, V \in E$,

$$\begin{aligned}&|I(U+V)-I(U)-DI(U)\cdot V| \\ &= |\frac{1}{2}\int_\Omega (-\Delta U - \Delta V)\cdot(U+V)dx - \int_\Omega F(x,U+V)dx \\ &\quad - \frac{1}{2}\int_\Omega (-\Delta U)\cdot U dx + \int_\Omega F(x,U)dx - \int_\Omega (-\Delta U - F_U(x,U))\cdot V dx| \\ &= |\frac{1}{2}\int_\Omega [-\Delta U\cdot V - \Delta V\cdot U - \Delta V\cdot V]dx \\ &\quad - \int_\Omega [F(x,u+v)-F(x,U)]dx - \int_\Omega [(-\Delta U - F_U(x,U))\cdot V]dx|.\end{aligned}$$

By (2.2),

$$||I(U+V)-I(U)-DI(U)\cdot V|| = O(\|V\|^2_{R^n}).$$

Thus $I \in C^1$. □

Lemma 2.3. *Assume that F satisfies the conditions $(F1)-(F4)$. Then the functional I satisfies $(P.S.)_c$ condition for every $c \in R$.*

Proof. Let $(U_n)_n$ be a sequence in E such that $I(U_n) \to c$ and $DI(U_n) \to 0$. We shall show that $(U_n)_n$ has a convergent subsequence. We claim that $(U_n)_n$ is bounded. By contradiction, we suppose that $\|U_n\| \to +\infty$ and set $W_n = \frac{U_n}{\|U_n\|}$. Up to a subsequence $W_n \rightharpoonup W_0$ weakly for some $W_0 \in E$. By the asymptotically linearity of $DI(U_n)$ we have

$$\begin{aligned}&\langle DI(U_n), \frac{U_n}{\|U_n\|}\rangle \\ &= \frac{2I(U_n)}{\|U_n\|} + \int_\Omega [\frac{2F(x,U_n)}{\|U_n\|} - \frac{(F_{u_1}(x,U_n),\cdots,F_{u_n}(x,U_n))\cdot U_n}{\|U_n\|}]dx,\end{aligned}$$

where $U_n = (U_{n_1}, \cdots, U_{n_n})$. Passing to the limit we get

$$\lim_{n\to\infty} \int_\Omega [\frac{2F(x,U_n)}{\|U_n\|} - \frac{(F_{u_1}(x,U_n), \cdots, F_{u_n}(x,U_n)) \cdot U_n}{\|U_n\|}]dx = 0.$$

Since F, F_{u_i} are bounded and $\|U_n\| \to \infty$ in Ω, $W_0 = 0$. Moreover we have

$$\begin{aligned}\langle \frac{DI(U_n)}{\|U_n\|}, W_n\rangle &= \int_\Omega [\frac{-\Delta U_n}{\|U_n\|} \cdot W_n - \frac{(F_{u_1}(x,U_n), \cdots, F_{u_n}(x,U_n)) \cdot W_n}{\|U_n\|}]dx \\ &= \int_\Omega [-\Delta W_n \cdot W_n - \frac{(F_{u_1}(x,U_n), \cdots, F_{u_n}(x,U_n)) \cdot W_n}{\|U_n\|}]dx.\end{aligned}$$

Since W_n converges to 0 weakly and $F_{u_i}(x, U_n)$, $i = 1, \cdots, n$ are bounded, $\int_\Omega -\Delta W_n \cdot W_n dx = \|W_n\|^2 \to 0$. Thus W_n converges to 0 strongly, which is a contradiction. Thus (U_n) is bounded. Up to a subsequence, U_n converges weakly to U for some $U \in E$. We claim that U_n converges to U strongly. We have

(2.5)
$$\langle DI(U_n), U_n\rangle = \int_\Omega [-\Delta U_n \cdot U_n - (F_{u_1}(x,U_n), \cdots, F_{u_n}(x,U_n)) \cdot U_n]dx \longrightarrow 0.$$

By the boundedness of $F_{u_i}(x, U_n)$, $i = 1, \cdots, n$,

$$\begin{aligned}\lim_{n\infty} \int_\Omega (-\Delta U_n \cdot U_n)dx &= \lim_{n\infty} \int_\Omega (-\Delta U \cdot U)dx \\ &= \lim_{n\infty} \int_\Omega (F_{u_1}(x,U_n), \cdots, F_{u_n}(x,U_n)) \cdot U_n dx\end{aligned}$$

Thus we have that U_n converges to U strongly. Thus we have

$$DI(U) = \lim_{n\to\infty} DI(U_n) = 0.$$

Thus we prove the lemma. □

Let V be $m-$dimensional subspace of E spanned by eigenfunctions corresponding to the eigenvalues λ_k, $k = h+1, \cdots, h+m$ of the eigenvalue problem $-\Delta U = \lambda_k U$ with $U|_{\partial\Omega} = (0, \cdots, 0)$. Let W be the orthogonal complement of V in E. Let $P : E \to V$ be the orthogonal projection of E onto V and $I - P : E \to W$ denote that of E onto W. Then every element $U \in L^2(\Omega)$ is expressed by $U = Y + Z$, $Y = PU$, $Z = (I - P)U$. Then (1.2) is equivalent to the two systems in the two unknowns Y and Z:

(2.6) $$-\Delta Y = P(\text{grad}_U F(x, Y + Z)), \qquad \text{in } \Omega,$$

(2.7) $$-\Delta Z = (I - P)(\text{grad}_U F(x, Y + Z)), \qquad \text{in } \Omega,$$

$$Y = (0, \cdots, 0), \qquad Z = (0, \cdots, 0) \qquad \text{on } \partial\Omega.$$

Let W_1 be a subspace of W spanned by eigenfunctions corresponding to the eigenvalues $\lambda_k \leq \lambda_h$, $1 \leq k \leq h$ and W_2 be a subspace of W spanned by eigenfunctions corresponding to the eigenvalues $\lambda_k \geq \lambda_{h+m+1}$, $k \geq h+m+1$. Let $Y \in V$ be fixed and consider the function $h : W_1 \times W_2 \to R$ defined by

$$h(Z_1, Z_2) = I(Y + Z_1 + Z_2).$$

The function h has continuous partial *Fréchet* derivatives D_1h and D_2h with respect to its first and second variables given by

$$D_ih(Z_1, Z_2)(X_i) = DI(Y + Z_1 + Z_2)(X_i) \tag{2.8}$$

for $X_i \in W_i$, $i = 1, 2$. By Lemma 2.2, I is a functional of class C^1.

By the following Lemma 2.4, we can get the critical points of the functional $I(U)$ on the infinite dimensional space E from that of the functional on the finite dimensional subspace V.

LEMMA 2.4. *(Reduction lemma) Assume that F satisfies the conditions $(F1)$-$(F4)$. Then*

(i) *there exists a unique solution $Z \in W$ of the equation*

$$-\Delta Z = (I - P)(grad_U F(x, Y + Z)) \qquad \text{in } \Omega,$$

$$Z = (0, \cdots, 0) \qquad \text{on } \partial\Omega.$$

If we put $Z = \Theta(Y)$, then Θ is continuous on V and satisfies a uniform Lipschitz condition in V with respect to the L^2 norm(also norm $\|\cdot\|$). Moreover

$$DI(Y + \Theta(Y)) \cdot X = 0 \qquad \text{for all } X \in W.$$

(ii) *There exists $m_1 < 0$ such that if Z_1 and X_1 are in W_1 and $Z_2 \in W_2$, then*

$$(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2))(Z_1 - X_1) \leq m_1\|Z_1 - X_1\|^2.$$

(iii) *There exists $m_2 > 0$ such that if Z_2 and X_2 are in W_2 and $Z_1 \in W_1$, then*

$$(D_2h(Z_1, Z_2) - D_2h(Z_1, X_2)) \cdot (Z_2 - X_2) \geq m_2\|Z_2 - X_2\|^2.$$

(iv) *If $\tilde{I} : V \to R$ is defined by $\tilde{I}(Y) = I(Y + \Theta(Y))$, then $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to Y, and*

$$D\tilde{I}(Y) \cdot B = DI(Y + \Theta(Y)) \cdot B \qquad \text{for all } Y, B \in V. \tag{2.9}$$

(v) *If $Y_0 \in V$ is a critical point of $\tilde{I}$ if and only if $Y_0 + \Theta(Y_0)$ is a critical point of I.*

Proof. (i) Let $\delta = \frac{\alpha+\beta}{2}$. The equation (2.7) is equivalent to

$$Z = (-\Delta - \delta)^{-1}(I-P)(\mathrm{grad}_U F(x, Y+Z) - \delta(Y+Z)) \tag{2.10}$$

The operator $(-\Delta-\delta)^{-1}(I-P)$ is self adjoint, compact and linear map from $(I-P)L^2(\Omega)$ into itself and its L_2 norm is $(\min\{\lambda_{h+m+1}-\delta, \delta-\lambda_h\}^{-1}$. Let U_1, $U_2 \in E$. Since

$$\begin{aligned}&(\mathrm{grad}_U F(x, U_2) - \delta U_2) - (\mathrm{grad}_U F(x, U_1) - \delta U_1)\\ &\le \max\{|\alpha-\delta|, |\beta-\delta|\}\|U_2 - U_1\|_{R^n} = \frac{|\alpha+\beta|}{2}\|U_2-U_1\|_{R^n},\end{aligned}$$

it follows that the right-hand side of (2.10) defines, for fixed $Y \in V$, a Lipschitz mapping of $(I-P)L^2(\Omega)$ into itself with Lipschitz constant $r < 1$. Therefore, by the contraction mapping principle, for given $Y \in V$, there exists a unique $Z = (I-P)L^2(\Omega)$ which satisfies (2.10). If $\Theta(Y)$ denote the unique $Z \in (I-P)L^2(\Omega)$ which solves (2.10), then Θ is continuous and satisfies a uniform Lipschitz condition in Y with respect to the L^2 norm(also norm $\|\cdot\|$). In fact, if $Z_1 = \Theta(Y_1)$ and $Z_2 = \Theta(Y_2)$, then

$$\begin{aligned}&\|Z_1 - Z_2\|_{L^2(\Omega)}\\ &= \|(-\Delta-\delta)^{-1}(I-P)[(\mathrm{grad}_U F(x, Y_1+Z_1) - \delta(Y_1+Z_1))\\ &\quad - (\mathrm{grad}_U F(x, Y_2+Z_2) - \delta(Y_2+Z_2))\|_{L^2(\Omega)}\\ &\le r\|(Y_1+Z_1)-(Y_2+Z_2)\|_{L^2(\Omega)}\\ &\le r(\|Y_1-Y_2\|_{L^2(\Omega)} + \|Z_1-Z_2\|_{L^2(\Omega)}) \le r\|Y_1-Y_2\| + r\|Z_1-Z_2\|.\end{aligned}$$

Hence

$$\|Z_1 - Z_2\| \le C\|Y_1 - Y_2\|, \qquad C = \frac{r}{1-r}. \tag{2.11}$$

Let $U = Y + Z$, $Y \in V$ and $Z = \Theta(Y)$. If $X \in (I-P)L^2(\Omega) \cap E$,

$$\begin{aligned}&DI(Y+\Theta(Y))\cdot X\\ &= \int_\Omega [-\Delta(Y+\Theta(Y))\cdot X - P((\mathrm{grad}_U F(x, Y+Z) - \delta(Y+Z))\cdot X)\\ &\quad -(I-P)((\mathrm{grad}_U F(x, Y+Z) - \delta(Y+Z))\cdot X)]dx.\end{aligned}$$

It follows from (2.7) that

$$\int_\Omega [-\Delta Z(x)\cdot X(x) - \mathrm{grad}_U F(x, Y(x)+Z(x))\cdot X(x)]dx = 0.$$

Since

$$\int_\Omega -\Delta Y(x)\cdot X(x) = 0,$$

we have

$$DI(Y + \Theta(Y)) \cdot X = 0. \tag{2.12}$$

(ii) If Z_1 and X_1 are in W_1 and $Z_2 \in W_2$, then

$$\begin{aligned}&(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2))(Z_1 - X_1)\\&= \int_\Omega [|\nabla(Z_1 - X_1)|^2 - (\mathrm{grad}_U F(x, Y + Z_1 + Z_2)\\&\quad -\mathrm{grad}_U F(x, Y + X_1 + Z_2)) \cdot (Z_1 - X_1)]dx.\end{aligned}$$

Since $\int_\Omega |\nabla(Z_1 - X_1)|^2 = \|Z_1 - X_1\|^2 \leq \lambda_h \|Z_1 - X_1\|^2_{L^2(\Omega)}$ and

$$\begin{aligned}&\int_\Omega (\mathrm{grad}_U F(x, Y + Z_1 + Z_2) - \mathrm{grad}_U F(x, Y + X_1 + Z_2)) \cdot (Z_1 - X_1)\\&\geq \alpha \|Z_1 - X_1\|_{L^2(\Omega)} \geq \frac{\alpha}{\lambda_h}\|Z_1 - X_1\|,\end{aligned}$$

$$(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2))(Z_1 - X_1) \leq (1 - \frac{\alpha}{\lambda_h})\|Z_1 - X_1\|^2$$

where $1 - \frac{\alpha}{\lambda_h} < 0$.

(iii) Similarly, using the fact that $\int_\Omega |\nabla(Z_2 - X_2)|^2 dx = \|Z_2 - X_2\|^2 \geq \lambda_{h+m+1}\|Z_2 - X_2\|^2_{L^2(\Omega)}$ and

$$\begin{aligned}&\int_\Omega (\mathrm{grad}_U F(x, Y + Z_1 + Z_2) - \mathrm{grad}_U F(x, Y + Z_1 + X_2)) \cdot (Z_2 - X_2)\\&\leq \beta \|Z_2 - X_2\|_{L^2(\Omega)} \leq \frac{\beta}{\lambda_{h+m+1}}\|Z_2 - X_2\|^2,\end{aligned}$$

we see that if Z_2 and X_2 are in W_2 and $Z_1 \in W_1$, then

$$(D_2h(Z_1, Z_2) - D_2h(Z_1, X_2))(Z_2 - X_2) \geq (1 - \frac{\beta}{\lambda_{h+m+1}})\|Z_2 - X_2\|^2$$

where $(1 - \frac{\beta}{\lambda_{h+m+1}}) > 0$.

(iv) Since the functional I has a continuous *Fréchet* derivative DI, $\tilde{I}$ has a continuous *Fréchet* derivative $D\tilde{I}$ with respect to Y.

(v) Suppose that there exists $Y_0 \in V$ such that $D\tilde{I}(Y_0) = 0$. From $D\tilde{I}(Y) \cdot B = DI(Y + \Theta(Y)) \cdot B$ for all $Y, B \in V$, $DI(Y_0 + \Theta(Y_0))(B) = D\tilde{I}(Y_0)(B) = 0$ for all $B \in V$. Since $DI(Y + \Theta(Y)) \cdot B = 0$ for all $B \in W$ and E is the direct sum of V and W, it follows that $DI(Y_0 + \Theta(Y_0)) = 0$. Thus $Y_0 + \Theta(Y_0)$ is a solution of (1.1). Conversely if U is a solution of (1.1) and $Y = PU$, then $D\tilde{I}(Y) = 0$. □

Remark 2.5. We note that if $Y \in V$, then $\Theta(Y) = 0$.

3. Proof of Theorem 1.1

Lemma 3.1. *Assume that F satisfies the conditions $(F1)$-$(F4)$. Then $Y=\theta$, $\theta=(0,\cdots,0)$, is neither a minimum nor degenerate.*

Proof. We have

$$\begin{aligned}\tilde{I}(Y) &= I(Y+\Theta_1(Y)+\Theta_2(Y))\\ &= \frac{1}{2}\int_\Omega(-\Delta(Y+\Theta_1(Y)+\Theta_2(Y))\cdot(Y+\Theta_1(Y)+\Theta_2(Y)))dx\\ &\quad -\int_\Omega F(x,Y+\Theta_1(Y)+\Theta_2(Y))dx\\ &= \frac{1}{2}\int_\Omega(-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx\\ &\quad -\int_\Omega F(x,Y+\Theta_1(Y))dx+\frac{1}{2}\int_\Omega(-\Delta\Theta_2(Y))\cdot\Theta_2(Y)dx\\ &\quad -\int_\Omega[F(x,Y+\Theta_1(Y)+\Theta_2(Y))-F(x,Y+\Theta_1(Y))]dx.\end{aligned}$$

We claim that

$$\begin{aligned}&\frac{1}{2}\int_\Omega(-\Delta\Theta_2(Y))\cdot\Theta_2(Y)dx\\ &-\int_\Omega[F(x,Y+\Theta_1(Y)+\Theta_2(Y))-F(x,Y+\Theta_1(Y))]dx\le 0.\end{aligned}$$

In fact,

$$\begin{aligned}&\frac{1}{2}\int_\Omega(-\Delta\Theta_2(Y))\cdot\Theta_2(Y)dx\\ &\quad -\int_\Omega[F(x,Y+\Theta_1(Y)+\Theta_2(Y))-F(x,Y+\Theta_1(Y))]dx\\ &=-\frac{1}{2}\int_\Omega[-\Delta\Theta_2(Y)\cdot\Theta_2(Y)]dx\\ &\quad -\int_0^1\int_\Omega[F_U(x,Y+\Theta_1(Y)+t\Theta_2(Y))-F_U(x,Y+\Theta_1(Y)+\Theta_2(Y))]\cdot\Theta_2(Y)dxdt\\ &=-\frac{1}{2}\int_\Omega[-\Delta\Theta_2(Y)\cdot\Theta_2(Y)]dx\\ &\quad -\int_0^1\int_\Omega[(d_U^2F(x,Y+\Theta_1(Y)+t\Theta_2(Y))]\cdot t\Theta_2(Y))\cdot\Theta_2(Y)dxdt\le 0\end{aligned}$$

by condition $(F2)$. Thus we have
(3.1)

$$\tilde{I}(Y) \leq \frac{1}{2}\int_\Omega (-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx - \int_\Omega F(x, Y+\Theta_1(Y))dx.$$

We have that

$$|\int_\Omega F(x, Y+\Theta_1(Y)) - \frac{1}{2}\int_\Omega (d_U^2F(x,\theta)\cdot(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx|$$
$$= |\int_0^1\int_\Omega [F_U(x, t(Y+\Theta_1(Y))) - (d_U^2F(x,\theta)\cdot t(Y+\Theta_1(Y)))\cdot(Y+\Theta_1(Y))]dxdt|$$
$$\leq \frac{1}{2}\sup_{0<t<1}\|d_U^2F(x,t(Y+\Theta_1(Y))) - d_U^2F(x,\theta)|_{\mathcal{L}(V,V)}\|Y+\Theta_1(Y)\|^2.$$

Thus we have

$$-\int_\Omega F(x, Y+\Theta_1(Y))$$
$$\leq -\frac{1}{2}\int_\Omega (d_U^2F(x,\theta)\cdot(Y+\Theta_1(Y)))\cdot(Y+\Theta_1(Y)) + o(\|Y+\Theta_1(Y)\|^2).$$

Since $\Theta_1 \in C^1(V, W_1)$, it follows that if $\|Y\| \to 0$, then $\|\Theta_1(Y)\| = O(\|Y\|)$ because $\Theta_1(\theta) = 0$. Thus

$$\|Y+\Theta_1(Y)\| = O(\|Y\|).$$

Since $F_U(x,\theta) = \theta$, there exists a bounded self adjoint operator $A \in \mathcal{L}(E,E)$ which commutes with P_o and P_- such that

$$\lambda_{h+1}I \leq A \leq d_U^2F(x,\theta).$$

Thus we have

$$\begin{aligned}
\tilde{I}(Y) \;\leq\; & \frac{1}{2}\int_\Omega (-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx \\
& -\frac{1}{2}\int_\Omega (d_U^2F(x,\theta)\cdot(Y+\Theta_1(Y)))\cdot(Y+\Theta_1(Y))dx + o(\|Y\|^2) \\
\leq\; & \frac{1}{2}\int_\Omega (-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx \\
& -\frac{1}{2}\int_\Omega A(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y)) + o(\|Y\|^2) \\
=\; & \frac{1}{2}\int_\Omega (-\Delta\Theta_1(Y)\cdot\Theta_1(Y)dx - \frac{1}{2}\int_\Omega A(\Theta_1(Y))\cdot\Theta_1(Y) \\
& +\frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)dx - \frac{1}{2}\int_\Omega A(Y)\cdot Y + o(\|Y\|^2)
\end{aligned}$$

as $\|Y\| \to 0$. Since $\lambda_{h+1} I \le A$, it follows from that

$$\frac{1}{2}\int_\Omega (-\Delta\Theta_1(Y)\cdot\Theta_1(Y)dx - \frac{1}{2}\int_\Omega A(\Theta_1(Y))\cdot\Theta_1(Y)$$
$$\le \frac{1}{2}\int_\Omega (-\Delta\Theta_1(Y)\cdot\Theta_1(Y)dx - \frac{1}{2}\int_\Omega \lambda_{h+1}\Theta_1(Y)\cdot\Theta_1(Y) \le 0.$$

Therefore we have

$$\begin{aligned}\tilde{I}(Y) &\le \frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)dx - \frac{1}{2}\int_\Omega A(Y)\cdot Y + o(\|Y\|^2)\\ &\le \frac{1}{2}\int_\Omega [(-\Delta Y)\cdot Y - \lambda_{h+1}Y^2]dx + o(\|Y\|^2).\end{aligned}$$

as $\|Y\| \to 0$. Similarly we can choose a bounded self adjoint operator $B \in \mathcal{L}(E,E)$ which commutes with P_o and P_- such that

$$d_U^2 F(x,\theta) \le B \le \lambda_{h+m+1} I$$

It follows from that

$$\begin{aligned}\tilde{I}(Y) &\ge \frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)dx - \frac{1}{2}\int_\Omega B(Y)\cdot Y + o(\|Y\|^2)\\ &\ge \frac{1}{2}\int_\Omega [(-\Delta Y)\cdot Y - \lambda_{h+m+1}Y^2]dx + o(\|Y\|^2)\end{aligned}$$

as $\|Y\| \to 0$. Thus $Y = \theta$, $\theta = (0,\cdots,0)$, is neither a minimum nor degenerate. □

We shall show that $-\tilde{I}(Y)$ is bounded from below and $\tilde{I}(Y)$ satisfies the (P.S.) condition.

Lemma 3.2. *Assume that F satisfies the conditions $(F1)$-$(F4)$. Then*

$$\tilde{I}(Y) \to -\infty as \;\; \|Y\| \to \infty.$$

Proof. By (3.1), we have

$$\tilde{I}(Y) \le \frac{1}{2}\int_\Omega (-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx - \int_\Omega F(x, Y+\Theta_1(Y))dx.$$

Thus by $(F4)$, we have

$$\begin{aligned}\tilde{I}(Y) &\le \frac{1}{2}\int_\Omega (-\Delta(Y+\Theta_1(Y))\cdot(Y+\Theta_1(Y))dx - \int_\Omega F(x, Y+\Theta_1(Y))dx\\ &\le \frac{1}{2}(\lambda_{h+m+1}-\gamma)\|Y+\Theta_1(Y)\|^2_{L^\Omega} + C \longrightarrow -\infty \quad \text{as} \quad \|Y\|_{L^2(\Omega)} \to \infty.\end{aligned}$$

Thus the lemma is proved. □

The following result come from Lemma 3.2.

LEMMA 3.3. *Assume that F satisfies the conditions $(F1)$-$(F4)$. Then $-\tilde{I}(v)$ is bounded from below and $\tilde{I}(v)$ satisfies the Palais-Smale condition.*

Proof of Theorem 1.1 By Lemma 2.2, $\tilde{I}(Y)$ is continuous and *Fréchet* differentiable in V. By Lemma 3.3, $\tilde{I}(v)$ is bounded above, satisfies the (P.S.) condition and $\tilde{I}(Y) \to -\infty$ as $\|Y\| \to \infty$. By Lemma 3.1, $Y = \theta$ is neither a minimum nor degenerate. By Lemma 3.2, $\tilde{I}(Y) \to -\infty$ as $\|Y\| \to \infty$. We note that $\max_{Y \in V} \tilde{I}(Y) > 0$ is another critical value of $\tilde{I}$. By the shape of the graph of the functional $\tilde{I}$ on the m-dimensional subspace V, there exist the third critical point of $\tilde{I}(Y)$. Thus (1.1) has at least three nontrivial solutions. □

References

[1] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser, 1993.

[2] Q. H. Choi and T. Jung, *Multiple periodic solutions of a semilinear wave equation at double external resonances,* Communications in Applied Analysis **3** (1999), no. 1, 73-84.

[3] D. R. Dunninger and H. Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus domain,* Nonlinear Analysis TMA **42** (2000), no. 5.

[4] K. S. Ha and Y. H. Lee, *Existence of multiple positive solutions of singular boundary value problems,* Nonlinear Analysis, TMA, **28** (1997), 1429-1438.

[5] K. Lan and R. L. Webb, *Positive solutions of semilinear equation with singularities,* J. Differential Equations **148** (1998), 407-421.

[6] Y. H. Lee, *Existence of multiple positive radial solutions for a semilinear elliptic system on an unbounded domain,* Nonlinear Analysis, TMA **47** (2001), 3649-3660.

[7] Y. H. Lee, *A multiplicity result of positive radial solutions for a multiparameter elliptic system on an exterior domain,* Nonlinear Analysis, TMA **45** (2001), 597-611.

[8] Y. H. Lee, *Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus,* Journal of Differential Equations **174** (2001), 420-441.

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