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THE M_α-DELTA INTEGRAL ON TIME SCALES

Jae Myung Park*, Deok Ho Lee**, Ju Han Yoon***,
and Jong Tae Lim****

Abstract. In this paper, we define the M_α-delta integral and investigate the relation between the M_α and M_α-delta integrals on time scales.

1. Introduction and preliminaries

Throughout this paper, $I_0 = [a, b]$ is a compact interval in $\mathbb{R}$. Let D be a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0 and let δ be a positive function on I_0. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a $\delta-$fine McShane partition of I_0 if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_0$ for all $i = 1, 2, \cdots, n$ and $\cup_{i=1}^n I_i = I_0$,

(2) a $\delta-$fine $M_\alpha-$partition of I_0 for a constant $\alpha > 0$ if it is a $\delta-$fine McShane partition of I_0 and satisfying the condition

$$\sum_{i=1}^{n} dist(\xi_i, I_i) < \alpha,$$

where $dist(\xi_i, I_i) = \inf\{|t - \xi_i| : t \in I_i\}$,

(3) a $\delta-$fine Henstock partition of I_o if $\xi_i \in I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, \cdots, n$ and $\cup_{i=1}^n I_i = I_0$.

We introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z \in \mathbb{T} : z > t\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z \in \mathbb{T} : z < t\}$$

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Correspondence should be addressed to Deok Ho Lee, dhlee2@kongju.ac.kr.

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

A pair $\delta = (\delta_L, \delta_R)$ of two real-valued functions on $[a, b]_{\mathbb{T}}$ is a $\triangle$-gauge on $[a, b]_{\mathbb{T}}$ if $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a $\delta-$fine $M_\alpha-$partition of $[a, b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$, $\xi_i \in [a, b]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$, and $\sum_{i=1}^n dist(\xi_i, I_i) < \alpha$.

For a $M_\alpha-$partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$, we write

$$f(\mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

2. The M_α and $M_\alpha-$delta integrals

Definition 2.1. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is $M_\alpha-$delta integrable (or M_α^Δ-integrable) on $[a, b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a Δ-gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\Big| f(\mathcal{P}) - A \Big| < \epsilon$$

for every δ-fine $M_\alpha-$partition $\mathcal{P}$ of $[a, b]_{\mathbb{T}}$. The number A is called the M_α^Δ-integral of f on $[a, b]_{\mathbb{T}}$, and we write $A = (M_\alpha^\Delta)\int_a^b f$.

Recall that $f : [a, b] \to \mathbb{R}$ is M_α-integrable on $[a, b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ on $[a, b]$ such that

$$\Big| f(\mathcal{P}) - A \Big| < \epsilon$$

for every δ-fine $M_\alpha-$partition $\mathcal{P}$ of $[a, b]$.

Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a, b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a, b]_{\mathbb{T}}$ in $[a, b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for some } k. \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}}. \end{cases}$$

It is well-known [9] that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is McShane delta(or M_Δ)-integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is McShane(or M)-integrable, and $(M_\Delta)\int_a^b f = (M)\int_a^b f^*$.

LEMMA 2.2. *Suppose that f is $M_\alpha^\Delta-$integrable on $[a,b]_{\mathbb{T}}$. Let $\epsilon > 0$ and let $\delta = (\delta_L, \delta_R)$ be a gauge on $[a,b]_{\mathbb{T}}$ such that*

$$\Big| f(\mathcal{P}) - (M_\alpha^\Delta)\int_a^b f \Big| < \epsilon$$

for each $\delta-$fine $M_\alpha-$partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$.

Assume that $\mathcal{D} = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is a $\delta-$fine partial M_α-partition of $[a,b]_{\mathbb{T}}$, $u_i \le t_i \le v_i$, $t_i \in [a,b]_{\mathbb{T}}$, $i = 1, 2, \cdots, n$ and $\mathcal{D}_1 = \{(\xi_i, [u_i, t_i])\}_{i=1}^n$, $\mathcal{D}_2 = \{(\xi_i, [t_i, v_i])\}_{i=1}^n$.

Then

$$\big| f(\mathcal{D}_1) - \sum_{i=1}^n (M_\alpha^\Delta)\int_{u_i}^{t_i} f \big| \le 3\epsilon$$

and

$$\big| f(\mathcal{D}_2) - \sum_{i=1}^n (M_\alpha^\Delta)\int_{t_i}^{v_i} f \big| \le 3\epsilon.$$

Proof. Denote $I = \{1, 2, \cdots, n\}$, $I_1 = \{i \in I | \xi_i \le t_i\}$, $I_2 = \{i \in I | \xi_i > t_i\}$,

$$\mathcal{D}_1^1 = \{(\xi_i, [u_i, t_i])\}_{i \in I_1}, \quad \mathcal{D}_1^2 = \{(\xi_i, [u_i, t_i])\}_{i \in I_2},$$
$$\mathcal{D}_2^1 = \{(\xi_i, [t_i, v_i])\}_{i \in I_1}, \quad \mathcal{D}_2^2 = \{(\xi_i, [t_i, v_i])\}_{i \in I_2},$$

Then $\mathcal{D}_1^1$, $\mathcal{D}_2^2$ are $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$. By Saks-Henstock Lemma, we have

$$\Big| f(\mathcal{D}_1^1) - \sum_{i \in I_1} (M_\alpha)\int_{u_i}^{t_i} f \Big| \le \epsilon \cdots\cdots (1)$$

$$\Big| f(\mathcal{D}_2^2) - \sum_{i \in I_2} (M_\alpha)\int_{t_i}^{v_i} f \Big| \le \epsilon \cdots\cdots (2)$$

Since $\{(\xi_i, [u_i, v_i])\}_{i \in I_2}$ is a $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$, we have

$$\Big| \sum_{i \in I_2} f(\xi_i)(v_i - u_i) - \sum_{i \in I_2} (M_\alpha^\Delta)\int_{u_i}^{v_i} f \Big| \le \epsilon \cdots\cdots (3)$$

From (2) and (3), we have

$$\Big|f(\mathcal{D}_1^2) - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big| = \Big|\sum_{i\in I_2} f(\xi_i)(t_i-u_i) - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big|$$

$$= \Big|\Big(\sum_{i\in I_2} f(\xi_i)(v_i-u_i) - \sum_{i\in I_2} f(\xi_i)(v_i-t_i)\Big)$$
$$- \Big(\sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{v_i} f - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{t_i}^{v_i} f\Big)\Big|$$

$$\leq \Big|\sum_{i\in I_2} f(\xi_i)(v_i-u_i) - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{v_i} f\Big|$$
$$+ \Big|\sum_{i\in I_2} f(\xi_i)(v_i-t_i) - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{t_i}^{v_i} f\Big| \leq 2\epsilon.$$

Hence, we have

$$\big|f(\mathcal{D}_1) - \sum_{i=1}^{n}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\big|$$
$$= \Big|f(\mathcal{D}_1^1) + f(\mathcal{D}_1^2) - \sum_{i\in I_1}(M_\alpha^\Delta)\int_{u_i}^{t_i} f - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big|$$
$$\leq \Big|f(\mathcal{D}_1^1) - \sum_{i\in I_1}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big| + \Big|f(\mathcal{D}_1^2) - \sum_{i\in I_2}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big| \leq 3\epsilon.$$

Similarly, we have

$$\Big|f(\mathcal{D}_2) - \sum_{i=1}^{n}(M_\alpha^\Delta)\int_{t_i}^{v_i} f\Big| \leq 3\epsilon.$$

□

LEMMA 2.3. *Let* f, $\epsilon > 0$ *and* $\delta = (\delta_L, \delta_R)$ *be as in the lemma 2.2.*
Assume that $\mathcal{D} = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ *is a* $\delta-$*fine partial* $M_\alpha-$*partition of* $[a,b]_{\mathbb{T}}$, *and* $u_i \leq s_i < t_i \leq v_i$, s_i, $t_i \in [a,b]_{\mathbb{T}}$, $i = 1, 2, \cdots, n$. *Let* $\mathcal{D}' = \{(\xi_i, [s_i, t_i])\}_{i=1}^n$.
Then

$$\Big|f(\mathcal{D}') - (M_\alpha^\Delta)\int_{\mathcal{D}'} f\Big| \leq 6\epsilon$$

where $(M_\alpha^\Delta)\int_{\mathcal{D}'} f = \sum_{i=1}^n (M_\alpha^\Delta)\int_{s_i}^{t_i} f$.

Proof. By lemma 2.2, we have

$$\Big|\sum_{i=1}^{n} f(\xi_i)(t_i-u_i)-\sum_{i=1}^{n}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big|\le 3\epsilon$$

and

$$\Big|\sum_{i=1}^{n} f(\xi_i)(s_i-u_i)-\sum_{i=1}^{n}(M_\alpha^\Delta)\int_{u_i}^{s_i} f\Big|\le 3\epsilon.$$

Hence, we have

$$\begin{aligned}&\Big|\sum_{i=1}^{n} f(\xi_i)(t_i-s_i)-\sum_{i=1}^{n}(M_\alpha^\Delta)\int_{s_i}^{t_i} f\Big|\\ &\le \Big|\sum_{i=1}^{n} f(\xi_i)(t_i-u_i)-\sum_{i=1}^{n}(M_\alpha^\Delta)\int_{u_i}^{t_i} f\Big|\\ &\qquad+\Big|\sum_{i=1}^{n} f(\xi_i)(s_i-u_i)-\sum_{i=1}^{n}(M_\alpha^\Delta)\int_{u_i}^{s_i} f\Big|\le 6\epsilon.\end{aligned}$$

□

THEOREM 2.4. *If $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is $M_\alpha^\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then $f^*:[a,b]\to\mathbb{R}$ is $M_\alpha-$integrable on $[a,b]$. In that case*

$$(M_\alpha)\int_a^b f^*=(M_\alpha^\Delta)\int_a^b f.$$

Proof. Let $[a,b]_{\mathbb{T}}^{rs}=\{w_k\}_{k\in\mathbb{N}}$ be the set of all right-scattered points of $[a,b]_{\mathbb{T}}$, and let $\epsilon>0$. Then there is a $\Delta-$gauge $\delta=(\delta_L,\delta_R)$ on $[a,b]_{\mathbb{T}}$ such that

(1) $\big|f(\mathcal{P})-(M_\alpha^\Delta)\int_a^b f\big|<\epsilon$ for each $\delta-$fine $M_\alpha-$partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$,
(2) $w_k+\delta_R(w_k)=\sigma(w_k)$, for $k=1,2,3,\cdots.$,
(3) $t+\delta_R(t)\in[a,b]_{\mathbb{T}}$ for $t\in[a,b]_{\mathbb{T}}^{rd}$, where $[a,b]_{\mathbb{T}}^{rd}$ is the set of all right-dense points of $[a,b]_{\mathbb{T}}$,
(4) $\delta_L(\sigma(w_k))<\min\left\{\frac{\epsilon}{2^k(|f(w_k)|+|f(\sigma(w_k))|+1)},\frac{\sigma(w_k)-w_k}{2}\right\}$, $k=1,2,\cdots.$,
(5) $t-\delta_L(t)\in[a,b]_{\mathbb{T}}$ for $t\in[a,b]_{\mathbb{T}}^{ld}$, where $[a,b]_{\mathbb{T}}^{ld}$ is the set of all left-dense points of $[a,b]_{\mathbb{T}}$.

Define a gauge $\widetilde{\delta}=(\widetilde{\delta}_L,\widetilde{\delta}_R)$ on $[a,b]$ by

(6) $\widetilde{\delta}_L(t)=\delta_L(t)$, $\widetilde{\delta}_R(t)=\delta_R(t)$ if $t\in[a,b]_{\mathbb{T}}$ and $\widetilde{\delta}_L(t)=\frac{t-w_k}{2}$, $\widetilde{\delta}_R(t)=\sigma(t)-t$ if $t\in(w_k,\sigma(w_k))$, $k=1,2,\cdots.$

Suppose that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a $\widetilde{\delta}-$fine $M_\alpha-$partition of $[a, b]$.

Put $A_1 = \{i | \xi_i = \sigma(w_k)$ and $(w_k, \sigma(w_k)) \cap [t_{i-1}, t_i] \neq \emptyset$ for some $k \in \mathbb{N}\}$, $A_2 = \{1, 2, \cdots, n\} \setminus A_1$.

From (4), we see that for each $i \in A_1$ there is a unique $k_i \in \mathbb{N}$ such that $\xi_i = \sigma(w_{k_i})$, and $(w_{k_i}, \sigma(w_{k_i})) \cap [t_{i-1}, t_i] \neq \emptyset$.

Put $A_{11} = \{i \in A_1 | [t_{i-1}, t_i] \subseteq [w_{k_i}, \sigma(w_{k_i})]\}$

$A_{12} = \{i \in A_1 | [t_{i-1}, t_i] \nsubseteq [w_{k_i}, \sigma(w_{k_i})]\}$.

Let

$$\begin{aligned}\mathcal{P}_1 = &\{(\xi, [t_{i-1}, t_i])\}_{i \in A_2} \cup \{(t_{i-1}, [t_{i-1}, t_i])\}_{i \in A_{11}} \\ &\cup \{(t_{i-1}, [t_{i-1}, \sigma(w_{k_i})])\}_{i \in A_{12}} \cup \{(\sigma(w_{k_i}), [\sigma(w_{k_i}), t_i])\}_{i \in A_{12}}.\end{aligned}$$

Then $\mathcal{P}_1$ is clearly a $\widetilde{\delta}-$fine $M_\alpha-$partition of $[a, b]$ and from (4) we have

$$\begin{aligned}|f^*(\mathcal{P}_1) - f^*(\mathcal{P})| \le &\sum_{i \in A_{11}} |f(w_{k_i}) - f(\sigma(w_{k_i}))|(t_i - t_{i-1}) \\ &+ \sum_{i \in A_{12}} |f(w_{k_i}) - f(\sigma(w_{k_i}))|(\sigma(w_{k_i}) - t_{i-1}) \\ \le &\sum_{k=1}^{\infty} \Big(\sum_{\substack{i \in A_{11} \\ k_i = k}} (|f(w_{k_i})| + |f(\sigma(w_{k_i}))|)(t_i - t_{i-1}) \\ &+ \sum_{\substack{i \in A_{12} \\ k_i = k}} (|f(w_{k_i})| + |f(\sigma(w_{k_i}))|)(\sigma(w_{k_i}) - t_{i-1}) \Big) \\ < &\ \epsilon \cdots\cdots (7)\end{aligned}$$

For the simplicity of notation , we rewrite $\mathcal{P}_1$ as $\mathcal{P}_1 = \{(\eta_i, [u_i, v_i])\}_{i=1}^m$.

Then (8) If $\eta_i = \sigma(w_k)$ for some $k \in \mathbb{N}$, then $[u_i, v_i] \cap (w_k, \sigma(w_k)) = \emptyset$.

Put $I = \{1, 2, \cdots, m\}$ and $W = \{w_k | (w_k, \sigma(w_k)) \cap [u_i, v_i] \neq \emptyset$ and $(w_k, \sigma(w_k)) \setminus [u_i, v_i] \neq \emptyset$ for some $i \in I\}$. Reorder $\{w_k\}_{i \in \mathbb{N}}$ so that $W = \{w_1, w_2, \cdots, w_l\}$ and $w_1 < w_2 < \cdots < w_l$. Put $J = \{1, 2, \cdots, l\}$.

If $u_i \notin [a, b]_{\mathbb{T}}$, then there is a unique $p_i \in J$ such that $u_i \in (w_{p_i}, \sigma(w_{p_i}))$. If $v_i \notin [a, b]_{\mathbb{T}}$, then there is a unique $q_i \in J$ such that $v_i \in (w_{q_i}, \sigma(w_{q_i}))$. Now separate I as follow;

$$I_1 = \{i \in I | \eta_i \in [a,b]_{\mathbb{T}}, u_i \in [a,b]_{\mathbb{T}}, v_i \in [a,b]_{\mathbb{T}}\}$$

$$I_2 = I \setminus I_1,\ \ I_2^j = \{i \in I_2 | [u_i, v_i] \cap (w_j, \sigma(w_j)) \neq \emptyset\}, j \in J.$$

$$I_{21} = \{i \in I_2 | \eta_i \in [a,b]_{\mathbb{T}}, u_i \in [a,b]_{\mathbb{T}}, v_i \notin [a,b]_{\mathbb{T}}\}, I_{21}^j = I_{21} \cap I_2^j, j \in J.$$

$$I_{22} = \{i \in I_2 | \eta_i \in [a,b]_{\mathbb{T}}, u_i \notin [a,b]_{\mathbb{T}}, v_i \in [a,b]_{\mathbb{T}}\}, I_{22}^j = I_{22} \cap I_2^j, j \in J.$$

$$I_{23} = \{i \in I_2 | \eta_i \in [a,b]_{\mathbb{T}}, u_i \notin [a,b]_{\mathbb{T}}, v_i \notin [a,b]_{\mathbb{T}}, p_i = q_i\}, I_{23}^j = I_{23} \cap I_2^j, \ j \in J.$$

$$I_{24} = \{i \in I_2 | \eta_i \in [a,b]_{\mathbb{T}}, u_i \notin [a,b]_{\mathbb{T}}, v_i \notin [a,b]_{\mathbb{T}}, p_i \neq q_i\}, I_{24}^j = I_{24} \cap I_2^j, \ j \in J.$$

From the choice of $\widetilde{\delta}$, we have that if $\eta_i \notin [a,b]_{\mathbb{T}}$, then $u_i \notin [a,b]_{\mathbb{T}}$, and $\eta_i \in (w_{p_i}, \sigma(w_{p_i}))$.

Put $I_{25} = \{i \in I_2 | \eta_i \notin [a,b]_{\mathbb{T}}\}$, $I_{25}^j = I_{25} \cap I_2^j$, $j \in J$.

Since $\{(\eta_i, [u_i, v_i])\}_{i \in I_1}$ is a $\delta-$fine partition $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$, we have by Saks-Henstock lemma,

$$\Big| \sum_{i \in I_1} f^*(\eta_i)(u_i - v_i) - \sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{v_i} f \Big| \leq \epsilon \cdots\cdots (9)$$

From (2) and (3), we see that $\{(\eta_i, [u_i, \sigma(w_{q_i})])\}_{i \in B}$ is a $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$ for each $B \subseteq I_{21}$. Hence by lemma 2.2,

$$\Big| \sum_{i \in I_{21}} f^*(\eta_i)(w_{q_i} - u_i) - \sum_{i \in I_{21}} (M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f \Big| \leq 3\epsilon \cdots\cdots (10)$$

where we use the convention $(M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f = 0$ if $w_{q_i} = u_i$.

$$\Big| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{q_i}}^{\sigma(w_{q_i})} f \Big| \leq 3\epsilon \cdots\cdots (11)$$

for each $B \subseteq I_{21}$.

Similarly, we have

$$\Big| \sum_{i \in I_{22}} f^*(\eta_i)(v_i - \sigma(w_{p_i})) - \sum_{i \in I_{22}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{v_i} f \Big| \leq 3\epsilon \cdots\cdots (12)$$

$$\Big| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{p_i}}^{\sigma(w_{p_i})} f \Big| \leq 3\epsilon \cdots\cdots (13)$$

for each $B \subseteq I_{22}$.

If $i \in I_{23}$, then $[u_i, v_i] \subseteq [w_{p_i}, \sigma(w_{p_i})]$.

If $i \in I_{24}$, then $p_i+1 = q_i$ and $w_{p_i} < u_i < \sigma(w_{p_i}) \leq w_{q_i} < v_i < \sigma(w_{q_i})$.

For $B \subseteq I_{24}$, put $B_0 = \{i \in B | p_i \text{ is odd}\}$, $B_e = \{i \in B | p_i \text{ is even}\}$. From the choice of δ, $\{(\eta_i, [w_{p_i}, \sigma(w_{p_i})])\}_{i \in B_0}$ is a $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$. Hence by lemma 2.2 and lemma 2.3,

$$\Big|\sum_{i\in B_0} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i\in B_0}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{w_{q_i}} f\Big| \leq 6\epsilon \cdots\cdots (14)$$

$$\Big|\sum_{i\in B_0} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i\in B_0}(M_\alpha^\Delta)\int_{w_{p_i}}^{\sigma(w_{p_i})} f\Big| \leq 3\epsilon \cdots\cdots (15)$$

$$\Big|\sum_{i\in B_0} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i\in B_0}(M_\alpha^\Delta)\int_{w_{q_i}}^{\sigma(w_{q_i})} f\Big| \leq 3\epsilon \cdots\cdots (16)$$

Similarly, we have

$$\Big|\sum_{i\in B_e} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i\in B_0}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{w_{q_i}} f\Big| \leq 6\epsilon \cdots\cdots (17)$$

$$\Big|\sum_{i\in B_e} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i\in B_e}(M_\alpha^\Delta)\int_{w_{p_i}}^{\sigma(w_{p_i})} f\Big| \leq 3\epsilon \cdots\cdots (18)$$

$$\Big|\sum_{i\in B_e} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i\in B_e}(M_\alpha^\Delta)\int_{w_{q_i}}^{\sigma(w_{q_i})} f\Big| \leq 3\epsilon \cdots\cdots (19)$$

From (14) to (19), we have

$$\Big|\sum_{i\in I_{24}} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i\in I_{24}}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{w_{q_i}} f\Big| \leq 12\epsilon \cdots\cdots (20)$$

$$\Big|\sum_{i\in B} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i\in B}(M_\alpha^\Delta)\int_{w_{p_i}}^{\sigma(w_{p_i})} f\Big| \leq 6\epsilon \quad \text{for } B \subseteq I_{24} \cdots (21)$$

$$\Big|\sum_{i\in B} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i\in B}(M_\alpha^\Delta)\int_{w_{q_i}}^{\sigma(w_{q_i})} f\Big| \leq 6\epsilon \quad \text{for } B \subseteq I_{24} \cdots (22)$$

If $i \in I_{25}$, then $u_i \notin [a,b]_{\mathbb{T}}$, $\eta_i \in (w_{p_i}, \sigma(w_{p_i}))$ and $[u_i, v_i] \subseteq [w_{p_i}, \sigma(w_{p_i})]$. Put $I_{241}^j = \{i \in I_{24}^j | p_i = j\}$, $I_{242}^j = \{i \in I_{24}^j | q_i = j\}$, $j \in J$, and put $I_{241} = \cup_{j\in J} I_{241}^j$, $I_{242} = \cup_{j\in J} I_{242}^j$.

From (9), (10), (12) and (20), we have

$$
\begin{aligned}
f^*(\mathcal{P}_1) &= \sum_{i\in I} f^*(\eta_i)(v_i-u_i) \\
&\le \sum_{i\in I_1}(M_\alpha^\Delta)\int_{u_i}^{v_i} f+\epsilon+\sum_{i\in I_{21}}(M_\alpha^\Delta)\int_{u_i}^{w_{q_i}} f+3\epsilon \\
&\quad + \sum_{i\in I_{22}}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{v_i} f+3\epsilon+\sum_{i\in I_{24}}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{w_{q_i}} f \\
&\quad + 12\epsilon+\sum_{i\in I_{21}} f^*(\eta_i)(v_i-w_{q_i})+\sum_{i\in I_{22}} f^*(\eta_i)\big(\sigma(w_{p_i})-u_i\big) \\
&\quad + \sum_{i\in I_{23}} f^*(\eta_i)(v_i-u_i)+\sum_{i\in I_{24}} f^*(\eta_i)(\sigma(w_{p_i})-u_i) \\
&\quad + \sum_{i\in I_{24}} f^*(\eta_i)(v_i-w_{q_i})+\sum_{i\in I_{25}} f^*(\eta_i)(v_i-u_i) \\
&= \Big[\sum_{i\in I_1}(M_\alpha^\Delta)\int_{u_i}^{v_i} f+\sum_{i\in I_{21}}(M_\alpha^\Delta)\int_{u_i}^{w_{q_i}} f+\sum_{i\in I_{22}}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{v_i} f \\
&\qquad + \sum_{i\in I_{24}}(M_\alpha^\Delta)\int_{\sigma(w_{p_i})}^{w_{q_i}} f+19\epsilon\Big]+\Big[\sum_{j\in J}\Big(\sum_{i\in I_{21}^j} f^*(\eta_i)(v_i-w_{q_i}) \\
&\qquad + \sum_{i\in I_{22}^j} f^*(\eta_i)(\sigma(w_{p_i})-u_i)+\sum_{i\in I_{23}^j} f^*(\eta_i)(v_i-u_i) \\
&\qquad + \sum_{i\in I_{241}^j} f^*(\eta_i)(\sigma(w_j)-u_i)+\sum_{i\in I_{242}^j} f^*(\eta_i)(v_i-w_j) \\
&\qquad + \sum_{i\in I_{25}^j} f^*(\eta_i)(v_i-u_i)\Big] = (I)+(II)\cdots\cdots(23)
\end{aligned}
$$

For each $j\in J$, choose $r_j\in I_2^j$ such that $f^*(\eta_{r_j})=\max\{f^*(\eta_i)|i\in I_2^j\}$. Then $r_j\in I_2$ for each $j\in J$.

If $r_j\in I_{23}$, then $[u_{r_j},v_{r_j}]\subseteq[w_{p_{r_j}},\sigma(w_{p_{r_j}})]$ and $[u_{r_j},v_{r_j}]\cap[w_j,\sigma(w_j)]\neq\emptyset$. Hence, $p_{r_j}=j$ and $\{(\eta_{r_j},[w_j,\sigma(w_j)])\}_{r_j\in I_{23}}$ is a $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$. Hence, we have

$$
\Big|\sum_{r_j\in I_{23}} f^*(\eta_{r_j})(\sigma(w_j)-w_j)-\sum_{r_j\in I_{23}}(M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f\Big|\le\epsilon\cdots\cdots(24)
$$

If $r_j \in I_{25}$, then $\eta_{r_j} \in (w_j, \sigma(w_j))$, $[u_{r_j}, v_{r_j}] \subseteq [w_j, \sigma(w_j)]$ and $f^*(\eta_{r_j}) = f(w_j)$. Since $\{(w_j, [w_j, \sigma(w_j)])\}_{r_j \in I_{25}}$ is a $\delta-$fine partial $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$, we have

$$\Big| \sum_{r_j \in I_{25}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) - \sum_{r_j \in I_{25}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f \Big| \le \epsilon \cdots\cdots (25)$$

By (11), (13), (21), (22), (24) and (25), we have

$$\begin{aligned}
(II) \le & \sum_{r_j \in I_{21}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{22}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\
& + \sum_{r_j \in I_{23}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{241}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\
& + \sum_{r_j \in I_{242}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{25}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\
\le & \sum_{r_j \in I_{21}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{22}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{23}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f \\
& + \sum_{r_j \in I_{241}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{242}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f \\
& + \sum_{r_j \in I_{25}} (M_\alpha^\Delta)\int_{w_j}^{\sigma(w_j)} f + 20\epsilon \cdots\cdots (26)
\end{aligned}$$

From (7), (23) and (26), we have

$$f^*(\mathcal{P}) < f^*(\mathcal{P}_1) + \epsilon \le (M_\alpha^\Delta)\int_a^b f + 40\epsilon \cdots\cdots (27)$$

Similarly, we can show that

$$f^*(\mathcal{P}) > (M_\alpha^\Delta)\int_a^b f - 40\epsilon \cdots\cdots (28)$$

Hence, we have $|f^*(\mathcal{P}) - (M_\alpha^\Delta)\int_a^b f| < 40\epsilon$. Therefore f^* is $M_\alpha-$integrable on $[a,b]$ and

$$(M_\alpha)\int_a^b f^* = (M_\alpha^\Delta)\int_a^b f.$$

□

Theorem 2.5. *If $f^* : [a,b] \to \mathbb{R}$ is $M_\alpha-$integrable on $[a,b]$, then $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $M_\alpha^\Delta-$integrable on $[a,b]_{\mathbb{T}}$.*

Proof. Suppose that f^* is $M_\alpha-$integrable on $[a,b]$. Define a function $g:[a,b]\to\mathbb{R}$ by

$$g(t)=\begin{cases} |f(\rho(t))|+|f(t)| & \textit{if} \quad t\in[a,b]_{\mathbb{T}}^{ls} \\ 0 & \textit{if} \quad t\in[a,b]-[a,b]_{\mathbb{T}}^{ls}. \end{cases}$$

Then since $[a,b]_{\mathbb{T}}^{ls}$ is a countable set, $g=0$ a.e. on $[a,b]$. Hence, g is $M_\alpha-$integrable on $[a,b]$ and $(M_\alpha)\int_a^b g=0$. Let $\epsilon>0$. Then there exists a gauge $\delta=(\delta_L,\delta_R)$ on $[a,b]$ such that

$$\left|f^*(D)-(M_\alpha)\int_a^b f^*\right|<\epsilon \qquad \text{and} \qquad |g(D)|<\epsilon$$

for each $\delta-$fine $M_\alpha-$partition D of $[a,b]$.

Define a $\Delta-$gauge $\delta^1=(\delta_L^1,\delta_R^1)$ on $[a,b]_{\mathbb{T}}$ by

$$\delta_L^1(t)=\delta_L(t) \qquad \textit{if} \quad t\in[a,b]_{\mathbb{T}}$$

$$\delta_R(t)=\begin{cases} \delta_R(t) & \textit{if} \quad t\in[a,b]_{\mathbb{T}}^{rd} \\ \sigma(t)-t & \textit{if} \quad t\in[a,b]_{\mathbb{T}}^{rs}. \end{cases}$$

Assume that $D=\{(\xi_i,[u_i,v_i])\}_{i=1}^n$ is a δ^1-fine $M_\alpha-$partition of $[a,b]_{\mathbb{T}}$.

Let $A=\{i\le n|\xi_i\in[a,b]_{\mathbb{T}}^{rs}$ and $[\xi_i,\sigma(\xi_i)]\subset[u_i,v_i]\}$,

$$B=\{1,2,\cdots,n\}-A.$$

For each $i\in A$, choose a $\delta-$fine Henstock partition $D_i=\{(\xi_{ij},[u_{ij},v_{ij}])\}_{j=1}^{p_i}$ of $[\xi_i,\sigma(\xi_i)]$.

Let $D^*=\{(\xi_i,[u_i,v_i])\}_{i\in B}\cup\{(\xi_i,[u_i,\xi_i])|i\in A,u_i<\xi_i\}\cup\left[\bigcup_{i\in A}D_i\right]$. Then D^* is a $\delta-$fine $M_\alpha-$partition of $[a,b]$, and

$$\begin{aligned}&|f(D)-f^*(D^*)|\\ &=\Big|\sum_{i\in A}f(\xi_i)(\sigma(\xi_i)-\xi_i)-\sum_{i\in A}\sum_{j\le p_i}f^*(\xi_{ij})(v_{ij}-u_{ij})\Big|\end{aligned}$$

$$= \Big| \sum_{i \in A} \Big[f(\xi_i)(\sigma(\xi_i) - \xi_i) - \sum_{\substack{j \le p_i \\ \xi_{ij} < \sigma(\xi_i)}} f(\xi_i)(v_{ij} - u_{ij})$$

$$- \sum_{\substack{i \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} f(\sigma(\xi_i))(v_{ij} - u_{ij}) \Big] \Big|$$

$$= \Big| \sum_{i \in A} \sum_{\substack{j \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} \Big[f(\xi_i) - f(\sigma(\xi_i)) \Big] (v_{ij} - u_{ij}) \Big|$$

$$\le \sum_{i \in A} \sum_{\substack{j \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} \Big[|f(\xi_i)| + |f(\sigma(\xi_i))| \Big] (v_{ij} - u_{ij}) = g(D^*) < \epsilon.$$

Hence,

$$\Big| f(D) - (M_\alpha) \int_a^b f^* \Big|$$
$$\le |f(D) - f^*(D^*)| + \Big| f^*(D^*) - (M_\alpha) \int_a^b f^* \Big| < 2\epsilon.$$

Thus, f is $M_\alpha^\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(M_\alpha^\Delta) \int_a^b f = (M_\alpha) \int_a^b f^*.$$

□

From Theorem 2.4 and 2.5, we get the following theorem.

THEOREM 2.6. *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $M_\alpha^\Delta-$ integrable $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is M_α-integrable on $[a,b]$. In this case,*

$$(M_\alpha^\Delta) \int_a^b f = (M_\alpha) \int_a^b f^*.$$

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*
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: parkjm@cnu.ac.kr

**
Department of Mathematics Education
KongJu National University
Kongju 314-701, Republic of Korea
E-mail: dhlee2@kongju.ac.kr

***
Department of Mathematics Education
Chungbuk National University
Chungju 360-763, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr

****
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: shiniljt@gmail.com