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IMPULSIVE INTEGRAL INEQUALITIES WITH A NON-SEPARABLE KERNEL

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Abstract. In this paper we present some Gronwall-tpye inequalities with a non-separable kernel and obtain the explicit estimate for solutions of impulsive differential equations. Furthermore, we give an example to illustrate our results.

1. Introduction

Integral inequalities of various Gronwall types play important roles in the study of stability of solutions of differential and integral equations, as well as in the modeling of engineering and science problems. Gronwall-type inequalities can be provided explicit bound for solutions of differential and integral equations as well as difference equations. For a detailed discussion of impulsive integral inequalities and some basic concepts concerning about the impulsive differential equations, we refer the reader to [1, 2, 4].

Differential equations with impulse effect describe evolution process which at certain moments change their state rapidly. In the mathematical simulation of such processes it is convenient to assume that this change takes place momentarily and the process changes its state by jump. Thus, the impulsive differential equations are adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jump change of their states. The theory of impulsive differential equations has found its extensive applications in realistic mathematical modeling of many real world phenomena.

Choi et al. [3] studied h-stability for linear impulsive equations using the notion of t_∞-similarity and an impulsive integral inequality.

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In this paper we present some Gronwall-tpye inequalities with a non-separable kernel and obtain the explicit estimate for solutions of impulsive differential equations. Furthermore, we give an example to illustrate our results.

2. Main results

Let $PC(\mathbb{R}_+,\mathbb{R})$ denote the class of piecewise continuous functions with discontinuities of the first kind only at $t=\tau_k, k\in\mathbb{N}$.

We need the following conditions:

(H_1) the sequence $\{\tau_k\}$ satisfies $0\le t_0<\tau_1<\tau_2<\cdots$, and $\lim_{k\to\infty}\tau_k=\infty$;

(H_2) $m\in PC(\mathbb{R}_+,\mathbb{R})$ and $m(t)$ is left-continuous at $\tau_k,\ k\in\mathbb{N}$.

We need the Gronwall-type integral inequality with a non-separable kernel to prove our results.

Lemma 2.1. [5, Theorem 1.1.4] *Suppose that $h,m\in C[\mathbb{R}_+,\mathbb{R}_+], K\in C[\mathbb{R}_+^2,\mathbb{R}_+], K_t(t,s)$ exists, is continuous and nonnegative. Let*

$$m(t)\le h(t)+\int_{t_0}^{t}K(t,s)m(s)ds,\ t\ge t_0.$$

Then,

$$m(t)\le h(t)+\int_{t_0}^{t}B(s,t_0)\exp\Big(\int_{s}^{t}A(\xi,t_0)d\xi\Big)ds,\ t\ge t_0, \tag{2.1}$$

where

$$A(t,t_0)=K(t,t)+\int_{t_0}^{t}K_t(t,s)ds$$

and

$$B(t,t_0)=K(t,t)h(t)+\int_{t_0}^{t}K_t(t,s)h(s)ds.$$

If we set $h(t)=c$ in Lemma 2.1, then we obtain the following explicit estimate.

Theorem 2.2. *Suppose that $m,b\in C(\mathbb{R}_+,\mathbb{R}_+), K\in C(\mathbb{R}_+^2,\mathbb{R}_+), K_t(t,s)$ exists, is nonnegative and continuous and for $t\ge t_0$,*

$$m(t)\le c+\int_{t_0}^{t}K(t,s)b(s)m(s)ds,$$

where c is a nonnegative constant. Then, we have

$$m(t) \quad \leq \quad c\exp\Big[\int_{t_0}^{t}(K(s,s)b(s)+\int_{t_0}^{s}K_s(s,\sigma)b(\sigma)d\sigma)ds\Big],\ t\geq t_0.$$

Proof. If we set $h(t)=c$ in Lemma 2.1, then we have

$$\begin{aligned} m(t) \quad &\leq \quad c+c\int_{t_0}^{t}A(s,t_0)e^{\int_s^t A(\sigma,t_0)d\sigma}ds \\ &= \quad c\Big(1+\exp\Big[\int_{t_0}^{t}A(s,t_0)ds\Big]\int_{t_0}^{t}A(s,t_0)e^{-\int_{t_0}^{s}A(\sigma,t_0)d\sigma}ds\Big) \\ &= \quad c\Big(1+e^{\int_{t_0}^{t}A(s,t_0)ds}\Big(1-e^{-\int_{t_0}^{t}A(s,t_0)ds}\Big)\Big) \\ &= \quad c\exp\Big(\int_{t_0}^{t}A(s,t_0)ds\Big),\ t\geq t_0, \end{aligned}$$

where

$$A(t,t_0)=K(t,t)b(t)+\int_{t_0}^{t}K_t(t,s)b(s)ds.$$

Hence, we obtain

$$m(t) \quad \leq \quad c\exp\Big[\int_{t_0}^{t}(K(s,s)b(s)+\int_{t_0}^{s}K_s(s,\sigma)b(\sigma)d\sigma)ds\Big],\ t\geq t_0.$$

This completes the proof. □

Corollary 2.3. *If, in Lemma 2.1, h is assumed to be nondecreasing and positive, then the estimate (2.1) reduces to*

$$m(t) \quad \leq \quad h(t)\exp\Big[\int_{t_0}^{t}(K(s,s)+\int_{t_0}^{s}K_s(s,\sigma)d\sigma)ds\Big],\ t\geq t_0. \tag{2.2}$$

Proof. Setting $w(t)=\frac{m(t)}{h(t)}$, we get

$$w(t)\leq 1+\int_{t_0}^{t}K(t,s)w(s)ds,\ t\geq t_0.$$

Hence, we have

$$w(t) \quad \leq \quad \exp\Big[\int_{t_0}^{t}(K(s,s)+\int_{t_0}^{s}K_s(s,\sigma)d\sigma)ds\Big],\ t\geq t_0,$$

by Theorem 2.2, and which yields (2.2). This completes the proof. □

An impulsive integral inequality with a nonseparable kernel can also be reduced to an impulsive differential inequality if the kernel is differentiable. We need the following impulsive integral inequality of Gronwall-type to prove our main results.

LEMMA 2.4. [4, Theorem 1.5.4] *Assume that* (H_1) *and* (H_2) *hold and* $m(t) \geq 0$. *Suppose that* $h \in PC(\mathbb{R}_+, \mathbb{R}), K \in C(\mathbb{R}_+^2, \mathbb{R}_+), K_t(t,s)$ *exists, is nonnegative and continuous and for each* $k \in \mathbb{N}, t \geq t_0$,

$$m(t) \leq h(t) + \int_{t_0}^{t} K(t,s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k), \tag{2.3}$$

where $\beta_k \geq 0$ *are constants. Then, for* $t \geq t_0$,

$$\begin{aligned} m(t) \quad \leq \quad & h(t) + \sum_{t_0<t_k<t} \Big(\prod_{t_k<t_j<t} (1+\beta_j) \exp\Big(\int_{t_k}^{t} A(\sigma, t_0) d\sigma \Big) \beta_k h(t_k) \Big) \\ & + \int_{t_0}^{t} \prod_{s<t_k<t} (1+\beta_k) \exp\Big(\int_{s}^{t} A(\sigma, t_0) d\sigma \Big) B(s, t_0) ds, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} A(t,t_0) &= K(t,t) + \int_{t_0}^{t} K_t(t,s)ds, \\ B(t,t_0) &= K(t,t)h(t) + \int_{t_0}^{t} K_t(t,s)h(s)ds. \end{aligned}$$

If we set $h(t) = c$ in Lemma 2.4, then we obtain the following explicit estimate for impulsive integral inequality with a non-separable kernel.

THEOREM 2.5. *Assume that* (H_1) *and* (H_2) *hold and* $m(t) \geq 0$. *Suppose that* $K \in C(\mathbb{R}_+^2, \mathbb{R}_+), K_t(t,s)$ *exists, is nonnegative and continuous and for each* $k \in \mathbb{N}, t \geq t_0$,

$$m(t) \leq c + \int_{t_0}^{t} K(t,s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k),$$

where c *and* β_k *are nonnegative constants. Then, for* $t \geq t_0$,

$$\begin{aligned} m(t) \quad \leq \quad & c \prod_{i=1}^{k} (1+\beta_i) \exp\Big[\int_{t_0}^{t} (K(s,s) \\ & + \int_{t_0}^{s} K_s(s,\sigma) d\sigma) ds \Big], t \in (t_k, t_{k+1}]. \end{aligned} \tag{2.5}$$

Proof. Let $t \in (t_1, t_2]$. Then, it follows from Lemma 2.4 that we have

$$\begin{aligned} m(t) &\le c\Big[1 + \sum_{t_0<t_1<t} \prod_{t_1<t_j<t} (1+\beta_j) e^{\int_{t_1}^{t} A(s,t_0)ds} \beta_1 \\ &\quad + e^{\int_{t_0}^{t} A(s,t_0)ds} \int_{t_0}^{t} \prod_{s<t_k<t} (1+\beta_k) A(s,t_0) e^{-\int_{t_0}^{s} A(\sigma,t_0)d\sigma} ds\Big] \\ &= c\Big[1 + e^{\int_{t_1}^{t} A(s,t_0)ds} \beta_1 \\ &\quad + e^{\int_{t_0}^{t} A(s,t_0)ds} \Big(\int_{t_0}^{t_1} (1+\beta_1) A(s,t_0) e^{-\int_{t_0}^{s} A(\sigma,t_0)d\sigma} ds \\ &\quad + \int_{t_1}^{t} A(s,t_0) e^{-\int_{t_0}^{s} A(\sigma,t_0)d\sigma} ds \Big)\Big] \\ &= c\Big[1 + e^{\int_{t_1}^{t} A(s,t_0)ds} \beta_1 \\ &\quad + e^{\int_{t_0}^{t} A(s,t_0)ds} \Big((1+\beta_1)\Big(1 - e^{-\int_{t_0}^{t_1} A(s,t_0)ds}\Big) \\ &\quad + (e^{-\int_{t_0}^{t_1} A(s,t_0)ds} - e^{-\int_{t_0}^{t} A(s,t_0)ds}) \Big)\Big] \\ &= c(1+\beta_1) e^{\int_{t_0}^{t} A(s,t_0)ds}, \ t \in (t_1, t_2], \end{aligned}$$

where $A(t, t_0) = K(t,t) + \int_{t_0}^{t} K_t(t,s)ds$.

Let $t \in (t_2, t_3]$. Then, by the same method, we also have

$$\begin{aligned} m(t) &\le c\Big[1 + (1+\beta_2) e^{\int_{t_1}^{t} A(s,t_0)ds} \beta_1 + e^{\int_{t_2}^{t} A(s,t_0)ds} \beta_2 \\ &\quad + e^{\int_{t_0}^{t} A(s,t_0)ds} \Big(\sum_{i=0}^{1} \int_{t_i}^{t_{i+1}} \prod_{s<t_k<t} (1+\beta_k) A(s,t_0) e^{-\int_{t_0}^{s} A(\sigma,t_0)d\sigma} ds \\ &\quad + \int_{t_2}^{t} \prod_{s<t_k<t} (1+\beta_k) A(s,t_0) e^{-\int_{t_0}^{s} A(\sigma,t_0)d\sigma} ds \Big)\Big] \\ &= c\Big[1 + (1+\beta_2) e^{\int_{t_1}^{t} A(s,t_0)ds} \beta_1 + e^{\int_{t_2}^{t} A(s,t_0)ds} \beta_2 \\ &\quad + e^{\int_{t_0}^{t} A(s,t_0)ds} \Big(\prod_{i=1}^{2} (1+\beta_i) \Big(1 - e^{-\int_{t_0}^{t_1} A(s,t_0)ds}\Big) \\ &\quad + (1+\beta_2)\Big(e^{-\int_{t_0}^{t_1} A(s,t_0)ds} - e^{-\int_{t_0}^{t_2} A(s,t_0)ds}\Big) \\ &\quad + \Big(e^{-\int_{t_0}^{t_2} A(s,t_0)ds} - e^{-\int_{t_0}^{t} A(s,t_0)ds}\Big)\Big)\Big] \end{aligned}$$

$$= c\prod_{i=1}^{2}(1+\beta_i)\exp\Big(\int_{t_0}^{t} A(s,t_0)ds\Big),\ t\in(t_2,t_3].$$

Then it follows from mathematical induction that (2.5) holds for $k \in \mathbb{N}$. This completes the proof. □

Corollary 2.6. *If h is nondecreasing and positive in Lemma 2.4, then the estimate (2.4) reduces to*

$$\begin{aligned} m(t) \;\le\; & h(t)\prod_{i=1}^{k}(1+\beta_i)\exp\Big[\int_{t_0}^{t}(K(s,s) \\ & + \int_{t_0}^{s} K_s(s,\sigma)d\sigma)ds\Big],\ t\in(t_k,t_{k+1}]. \end{aligned} \tag{2.6}$$

Proof. Setting $w(t) = \frac{m(t)}{h(t)}$, we get

$$w(t) \le 1 + \int_{t_0}^{t} K(t,s)w(s)ds + \sum_{t_0<t_k<t}\beta_k w(t_k),\ t\ge t_0.$$

Hence, we have

$$\begin{aligned} w(t) \;\le\; & \prod_{i=1}^{k}(1+\beta_i)\exp\Big[\int_{t_0}^{t}(K(s,s) \\ & + \int_{t_0}^{s} K_s(s,\sigma)d\sigma)ds\Big],\ t\in(t_k,t_{k+1}], \end{aligned} \tag{2.7}$$

by Theorem 2.5, and which yields (2.6). This completes the proof. □

3. Example

In this section we give an example which illustrates our results from the previous section.

Example 3.1. [5, Lemma 8.1] *We consider the scalar differential equation with impulse effect*

$$\begin{cases} x'(t) = a(t)x + g(t,x), & t\neq t_k, t\in\mathbb{R}_+; \\ \Delta x(t) := x(t^+) - x(t) = b_k + I_k(x(t)), & t = t_k, k\in\mathbb{N}; \\ x(t_0) = x_0, & \end{cases} \tag{3.1}$$

with the following conditions:

(i) *The sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty, k\in\mathbb{N}$.*

(ii) *The function* $a \in PC(\mathbb{R}_+, \mathbb{R})$ *and* $b_k \in \mathbb{R}, k \in \mathbb{N}$, *and* $\prod_{k=1}^{\infty}(1 + |b_k|) < \infty$.

(iii) *Each function* $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is continuous and locally Lipschitz continuous with respect to* x *in the sets* $(t_{k-1}, t_k] \times \mathbb{R}, k \in \mathbb{N}$, *and for each* $k \in \mathbb{N}$ *and* $y \in \mathbb{R}$, *and the limit of* $g(t,x)$ *exists as* $(t,x) \to (t_k, y), t > t_k$. *Furthermore, the following inequality holds:*

$$|g(t,x) \leq b(t)|x|,$$

where the function $b \in PC(\mathbb{R}_+, \mathbb{R})$ *is nonnegative.*

(iv) *The function* $I_k : \mathbb{R} \to \mathbb{R}$ *is continuous and*

$$|I_k(x)| \leq \rho_k |x|, k \in \mathbb{N},$$

where each ρ_k *is a nonnegative constant.*

(v) $x(t_k^+)$ *represents the right limit of* $x(t)$ *at* $t = t_k$.

Together with equation (3.1), we consider the impulsive differential equation

$$\begin{cases} x'(t) &= a(t)x, \quad t \neq t_k, t \in \mathbb{R}_+; \\ \Delta x(t) &= b_k, \qquad t = t_k, k \in \mathbb{N}; \\ x(t_0) &= x_0, \end{cases} \tag{3.2}$$

and the differential equation

$$\begin{cases} x'(t) &= a(t)x, \\ x(t_0) &= x_0. \end{cases} \tag{3.3}$$

Let $u(t,s)$ *and* $w(t,s)$ *be the fundamental solutions of (3.3) and (3.2), respectively. Furthermore, we assume that* $u(t,s)$ *satisfies the following condition:* $|u(t,s)| \leq \varphi(t)\psi(s)$ *for* $t_{k-1} < s \leq t \leq t_k, k \in \mathbb{N}$, *where* $\varphi, \psi \in PC(\mathbb{R}_+, \mathbb{R})$ *and* $\varphi(t) > 0, \psi(t) > 0, \varphi(t_k^+) > 0, \psi(t_k^+) > 0, \varphi(t_k)\psi(t_k^+) = 1$ *for* $t \in \mathbb{R}_+, k \in \mathbb{N}$. *Then the solution* $x(t) = x(t, t_0, x_0)$ *of (3.1) satisfies the following estimate*

$$\begin{aligned} |x(t)| \;&\leq\; c|x_0|\varphi(t)\psi(t_0^+) \prod_{t_0 < t_k < t} (1 + c\rho_k) \exp\Big[\int_{t_0}^{t} (b(s) \\ &\quad + \varphi'(s) \int_{t_0}^{s} \psi(\sigma) b(\sigma) d\sigma) ds\Big], \; t \geq t_0. \end{aligned} \tag{3.4}$$

Proof. It follows from [1, p. 45] that we have

$$\begin{aligned} |w(t,s)| \;&\leq\; \varphi(t)\psi(s) \prod_{s \leq t_k < t} (1 + |b_k|)\varphi(t_k)\psi(t_k^+) \\ &\leq\; c\varphi(t)\psi(s), \end{aligned}$$

where $c = \prod_{k=1}^{\infty}(1 + |b_k|)$. In view of the variation in parameters, the solution $x(t) = x(t, t_0, x_0)$ of (3.1) satisfies

$$\begin{aligned} x(t) &= w(t, t_0^+)x_0 + \int_{t_0}^{t} w(t,s)g(s,x(s))ds \\ &\quad + \sum_{t_0<t_k<t} w(t, t_k^+)I_k(x(t_k)). \end{aligned} \tag{3.5}$$

From conditions (iii), (iv) and (3.5), we obtain

$$\begin{aligned} |x(t)| &\le c|x_0|\varphi(t)\psi(t_0^+) + c\int_{t_0}^{t}\varphi(t)\psi(s)b(s)|x(s)|ds \\ &\quad + c\sum_{t_0<t_k<t}\varphi(t)\psi(t_k^+)\rho_k|x(t_k^+)|,\ t \ge s. \end{aligned} \tag{3.6}$$

Setting $m(t) = \frac{|x(t)|}{\varphi(t)}$ and $K(t,s) = \varphi(t)\psi(s)b(s)$, then it follows from Lemma 2.5 that

$$\begin{aligned} |x(t)| &\le c|x_0|\varphi(t)\psi(t_0^+)\prod_{t_0<t_k<t}(1 + c\rho_k)\exp\Big[\int_{t_0}^{t}(b(s) \\ &\quad + \varphi'(s)\int_{t_0}^{s}\psi(\sigma)b(\sigma)d\sigma)ds\Big],\ t \ge t_0. \end{aligned} \tag{3.7}$$

This completes the proof. □

Remark 3.2. If we apply well-known impulsive integral inequality of Gronwall-type in [4, Theorem 1.5.1.] to (3.6), then we also obtain the following estimate:

$$\begin{aligned} |x(t)| &\le c|x_0|\varphi(t)\psi(t_0^+)\prod_{t_0<t_k<t}(1 + c\rho_k)\exp\Big[\int_{t_0}^{t}c\varphi(s)\psi(s)b(s)ds\Big] \\ &= c|x_0|\varphi(t)\psi(t_0^+)\prod_{t_0<t_k<t}(1 + c\rho_k)\exp\Big[\int_{t_0}^{t}cb(s)ds\Big], t \ge t_0. \end{aligned} \tag{3.8}$$

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