JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.635

REMARK ON PARTICLE TRAJECTORY FLOWS WITH UNBOUNDED VORTICITY

Hee Chul Pak*

Abstract. The existence and the regularity of the particle trajectory flow $X(x,t)$ along a velocity field u on $\mathbb{R}^n$ are discussed under the BMO-blow-up condition:

$$\int_0^T \|\omega(\tau)\|_{BMO} d\tau < \infty$$

of the vorticity $\omega \equiv \nabla \times u$. A comment on our result related with the mystery of turbulence is presented.

1. Background and the main assertion

The non-stationary Euler equations of a perfect incompressible fluid are

$$\frac{\partial}{\partial t} u + (u, \nabla)u = -\nabla p, \quad \text{div } u = 0. \tag{1.1}$$

Here $u(x,t) = (u_1, u_2, \cdots, u_n)$ represents the velocity of a fluid flow, and $p(x,t)$ is the scalar pressure.

One of the most outstanding open problems in the mathematical theory of fluid mechanics is, for dimension 3, either to prove the global-in-time continuation of local solution, or to find an initial datum for which the associated local solution blows up in finite time. A significant achievement in this direction is the Beale-Kato-Majda(BKM) criterion[1] for the finite time blow-up of solution u, which states as follows: Suppose the initial velocity $u(0)$ locates in Sobolev space $H^s(\mathbb{R}^3)$

Received August 25, 2014; Accepted October 10, 2014.
2010 Mathematics Subject Classification: Primary 35Q31; Secondary 35B44, 34A12, 35Q35, 76Dxx.
Key words and phrases: particle trajectory, incompressible fluid, flow, blowing-up, BMO, BKM-criterion.

with $s > \frac{5}{2}$, then

$$u \in L^\infty([0,T]; H^s(\mathbb{R}^3))$$

if and only if

$$\omega \in L^1([0,T]; L^\infty(\mathbb{R}^3)) \tag{1.2}$$

where $\omega = \nabla \times u$ is the vorticity. Some numerical observations and computations have indicated that the velocity u may not blow up even if the corresponding vorticity ω blows up as an unbounded function of space variables[13]. Even though this observation does not violate the BKM-criterion, it insists that some velocities staying in a Sobolev space may allow unbounded vorticities. It was recently noted by Kozono-Taniuchi[7] that one could replace L^∞-norm in (1.2) by BMO-norm;

$$\omega \in L^1([0,T]; BMO(\mathbb{R}^3)). \tag{1.3}$$

(For other generalizations and variants of the BKM theorem, see [2, 5, 6, 8, 12].) According to these updates, we can safely exclude such possibilities because those spaces (including BMO) contain singular (unbounded) functions such as logarithmic functions. In this context, it is worth paying attention to the regularity problem corresponding to unbounded vorticity.

Associated with the Euler equations, we have a system of ordinary differential equations

$$\begin{cases} \dfrac{\partial}{\partial t} X(a,t) &= u(X(a,t),t), \\ X(a,0) &= a, \end{cases} \tag{1.4}$$

which defines *particle trajectory flows* $X(a,t)$ along the velocity u, starting from initial position a. By turning our attention into the regularity of the flow $X(a,t)$, we expect a better understanding in the regularity of velocity. In this short communication, we deal with the existence and the regularity of the particle trajectory flows with unbounded vorticity.

It is well known that the Lipschitz condition:

$$|u(x,t) - u(y,t)| \le L|x-y|$$

(L is a constant) is sufficient to ensure unique existence of solution to (1.4). This condition however is not necessary. One of the necessary and sufficient conditions is known as

$$\int_0^\varepsilon \frac{dr}{|u(x,t) - u(y,t)|} = \infty, \tag{1.5}$$

where $r := |x-y|$(see, [4, 9]). For example, the latter condition is satisfied with $|u(x,t) - u(y,t)| \propto r \log r$, referred to as *log-Lipschitz* *condition*. Therefore, if we suppose vorticity $\omega \approx \frac{|u(x,t)-u(y,t)|}{r}$ in some sense, then the log-Lipschitz case corresponds to vorticity

$$\omega \approx \frac{|u(x,t)-u(y,t)|}{r} \propto \log r.$$

This goes along with the BMO-criterion (1.3). Therefore the regularity condition of the Euler equations (1.1) seems to be in parallel with that of the equations of trajectory flows (1.4)[10]. This turns the PDE-problem into an ODE-problem.

In this paper, we ask a question of the existence and the regularity of the trajectory flow (1.4) under the condition (1.3). Here is our main result:

THEOREM 1.1. *Let n be the spacial dimension and $T > 0$. For any vector field (of distribution) u, suppose $\omega = \nabla \times u$ is the vorticity field belonging to $L^1([0,T]; BMO(\mathbb{R}^n))$. Then there exists a unique flow X, continuous in $[0,T] \times \mathbb{R}^n$, with values in $\mathbb{R}^n$, satisfying the system (1.4).*

We have no information about the regularity of the initial velocities in the hypothesis of the theorem, and so BKM-criterion (1.3) might not work here. However, the theorem asserts that, at least, the continuity of the flows should be allowed.

We recall that the space BMO consists of all locally integrable functions such that

$$\|f\|_{BMO} := \sup_{r>0} \frac{1}{|Q(x;r)|} \int_{Q(x;r)} |f(x) - f_Q| dx$$

is finite, where $Q(x;r)$ is a cube centered at x and of diameter r and $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$.

The main difficulty of the proof is that no direct embedding of the velocity u into the space of log-Lipschitzian vector fields can be possible. However, it is possible to decompose u into a smooth part and a log-Lipschitzian part, so that we can regard it as log-Lipschitzian locally. For the decomposition, we note that the space $BMO(\mathbb{R}^n)$ is equivalent to the homogeneous Triebel-Lizorkin space $\dot{F}^0_{\infty,2}(\mathbb{R}^n)$ [15]. That is, we turn BMO-norm into $\dot{F}^0_{\infty,2}$-norm via the Littlewood-Paley decomposition. For it, we consider a smooth nonnegative radial function χ satisfying $\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^n : |\xi| \le \frac{5}{6}\}$, and $\chi = 1$ for $|\xi| \le \frac{3}{5}$. Set $h_j(\xi) := \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and we notice that

$$\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1, \text{ for } \xi \in \mathbb{R}^n.$$

Let φ_j and Φ be defined by $\varphi_j := \mathcal{F}^{-1}(h_j)$, $j \in \mathbb{Z}$ and $\Phi := \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^n$. For $f \in \mathcal{S}'$, we denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ $(j \in \mathbb{Z})$. Then there exists a constant C_0 such that for all $f \in BMO(\mathbb{R}^n)$ we have

$$\frac{1}{C_0}\left\|\sum_{j\in\mathbb{Z}}\Delta_j f\right\|_{BMO} \le \left\|\left(\sum_{j\in\mathbb{Z}}|\Delta_j f|^2\right)^{1/2}\right\|_{L^\infty} \le C_0 \|f\|_{BMO}.$$

2. The argument

For any $x_0 \in \mathbb{R}^n$, we will show that there exists a unique continuous solution $x(\cdot)$ on $[0,T]$ satisfying

$$x(t) = x_0 + \int_0^t u(x(s),s)ds. \tag{2.1}$$

Let B be an open ball centered at x_0. We choose $x, y \in B$ with $|x-y| < 1$. Then for an arbitrary nonnegative integer N, we have

$$\begin{aligned} |u(x,t)-u(y,t)| &= \left|\sum_{j=-\infty}^{\infty}\Delta_j u(x,t) - \Delta_j u(y,t)\right| \\ &\le \left\|\sum_{j>N}|\Delta_j u|\right\|_{L^\infty(\mathbb{R}^n)} + \left\|\sum_{j=0}^{N}|\Delta_j \nabla u|\right\|_{L^\infty(\mathbb{R}^n)} |x-y| \\ &\quad + 2\|\Phi * \nabla u\|_{L^\infty(B)}|x-y|. \end{aligned} \tag{2.2}$$

We estimate the first term of (2.2):

$$\begin{aligned} \left\|\sum_{j>N}|\Delta_j u|\right\|_{L^\infty(\mathbb{R}^n)} &\le \left\|\left(\sum_{j\ge N}2^{2j}|\Delta_j u|^2\right)^{1/2}\right\|_{L^\infty} \frac{2}{\sqrt{3}}\left(\frac{1}{2}\right)^{N+1} \\ &\le C2^{-N-1}\|\nabla u\|_{BMO} \\ &\le C2^{-N-1}\|\omega\|_{BMO}. \end{aligned} \tag{2.3}$$

For the second term, we have

$$\left\|\sum_{j=0}^{N}|\Delta_j \nabla u|\right\|_{L^\infty(\mathbb{R}^n)} \le C(N+1)\|\nabla u\|_{BMO} \le C(N+1)\|\omega\|_{BMO}. \tag{2.4}$$

Since $\nabla u(x)$ exists for almost every x, and so $\Phi * \nabla u$ is smooth, the last term of (2.2) is bounded on B:

$$\|\Phi * \nabla u\|_{L^\infty(B)} \leq M_{B,u}, \tag{2.5}$$

for some $M_{B,u} > 0$. Choosing $N = [-\log_2 |x-y|]$ and collecting terms (2.3), (2.4), (2.5), we have

$$|u(x) - u(y)| \leq C_{B,u}(1 - \log_2 |x-y|)|x-y|,$$

where $C_{B,u}$ is a constant depending only on B and u. This implies that u satisfies the condition (1.5) on B. So, for sufficiently small time interval $[0,t]$, we can localize to find a unique continuous flow $x(\cdot)$ starting from position x_0. (The temporal localization is needed for the flow to stay within the ball B.) Continuing this process, we obtain an existence of flow $X(x_0, \cdot) = x(\cdot)$ as long as the vorticity ω exists, that is to say, it exists on the time interval $[0,T]$.

It remains to show the spatial regularity of the flow $X(\cdot,t)$. To accomplish this, consider two integral curves $X(x,t)$ and $X(y,t)$, starting from two distinct points x and y, respectively, inside an open ball B of radius less than $1/2$. So we have $|x-y| < 1$. We observe

$$\begin{aligned}
&|X(x,t) - X(y,t)| \\
&\leq |x-y| + \int_0^t |u(X(x,\tau),\tau) - u(X(y,\tau),\tau)| d\tau \\
&\leq |x-y| + C_{B,u} \int_0^t (1 - \log_2 |X(x,\tau) - X(y,\tau)|)|X(x,\tau) - X(y,\tau)| d\tau.
\end{aligned}$$

We now let $Y(t) \equiv |X(x,t) - X(y,t)|$, and so $Y(0) \equiv |x-y|$. Then this inequality is equivalent to

$$\frac{d}{dt} Y(t) \leq C_{B,u}(1 - \log_2 Y(t))Y(t). \tag{2.6}$$

We solve a *separable* ordinary differential inequality (2.6). Indeed, integrating both sides of $\frac{\dot{Y}}{(1-\log_2 Y)Y} \leq C_{B,u}$ on $[0,t]$, we get

$$Y(t) \leq Y(0)^{\exp(-C_{B,u}t)} 2^{1-\exp(-C_{B,u}t)}.$$

That is,

$$|X(x,t) - X(y,t)| \leq \alpha |x-y|^\beta,$$

for some positive constants α, β depending only on B, u and t. This implies the spacial continuity of $X(\cdot,t)$. □

3. Remarks connected with turbulence

1. There may exist discontinuous flows which are corresponding to the vorticities outside BMO.
2. Turbulence is the last mystery of the classical mechanics. There is no valid physical and mathematical theories for occurrence and no foresight on turbulence yet. So it is openly said that 'turbulence is a riddle wrapped in a mystery inside an enigma'.

T. Kato himself described the BKM-criterion as "the solution does not blow up unless the vorticity does". Our result can be retold as "the solution does blow up when the continuity of the flow breaks down". Therefore, supposing that the turbulence occurs exactly at the discontinuous points of the flow, this explains a direct relationship among the turbulence, blowing up of a vorticity and the regularity of velocity field.

References

[1] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations,* Comm. Math. Phys. **94** (1984), 61-66.

[2] D. Chae, *On the Well-Posedness of the Euler Equations in the Triebel-Lizorkin Spaces,* Comm. Pure Appl. Math. **55** (2002), no. 5, 654-678.

[3] J.-Y. Chemin, *Perfect incompressible fluids,* Clarendon Press, Oxford, 1981.

[4] E. A. Coddington and N. Levinson, *Theory Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[5] P. Constantin, *Geometric statistics in turbulence,* SIAM Rev. **36** (1994), 73-98.

[6] P. Constantin, C. Fefferman, and A. Majda, *Geometric constraints on potentially singular solutions for the 3D Euler equations,* Commun. Part. Diff. Eq. **21** (1996), 559-571.

[7] H. Kozono and Y. Taniuchi, *Limiting case of Sobolev inequality in BMO,* Commun. Math. Phys. **214** (2000), 191-200.

[8] H. Kozono, T. Ogawa, and Y. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations,* Math. Z. **242** (2002), 251-278.

[9] P. Hartman, *Ordinary Differential Equations 2nd Ed.,* Society for Industrial and Applied Mathematics, Philadelphia, 2002.

[10] K. Ohkitani, *A miscellany of basic issues on incompressible fluid equations,* Nonlinearity **21** (2008), 255-271.

[11] H. Pak and Y. Park, *Existence of solution for the Euler equations in a critical Besov space $\mathbf{B}^1_{\infty,1}(\mathbb{R}^n)$,* Commun. Part. Differ. Eq. **29** (2004), 1149-1166.

[12] G. Ponce, *Remarks on a paper by J. T. Beale, T. Kato and A. Majda,* Comm. Math. Phys. **98** (1985), 349-353.

[13] A. Pumir and E. D. Siggia, *Collapsing solutions to the 3-D Euler equations,* Phys. Fluids A **2** (1990), 220-241.

[14] E. M. Stein, *Harmonic analysis; Real-variable methods, orthogonality, and oscillatory integrals,* Princeton Mathematical Series, 43, 1993.

[15] H. Triebel, *Theory of Function spaces,* Birkhäuser, 1983.
[16] M. Vishik, *Hydrodynamics in Besov spaces,* Arch. Rational Mech. Anal. **145** (1998), 197-214.
[17] M. Vishik, *Incompressible flows of an ideal fluid with unbounded vorticity,* Comm. Math. Phys. **213** (2000), 697-731.
[18] V. I. Yudovich, *Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid,* Math Res. Lett. **2** (1995), 27-38.

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Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: hpak@dankook.ac.kr